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FUNDAMENTAL HIGHER MATHEMATICS

PART 1

LINEAR ALGEBRA AND ANALYTICAL GEOMETRY

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The feature of the given textbook is unified methodical approach to explaining the course of higher mathematics in future. The explanation of theoretical `material is accompanied by illustrations and solution of typical tasks. For the purpose of consolidating the material we offer practical tasks for independent work.

The textbook is designed for students in an university who study computer sciences and aim at fast assimilating the course of linear algebra and analytic geometry whose volume resembles university one

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TABLE OF CONTENTS

INTRODUCTION

The modern level of development of a science results in that more and more specialties which had before the applied (technical) character are included in sphere of university education. First of all to such specialties we should refer specialties in the field of computer sciences. Features of training of students on these specialties at university generate a need of the accelerated studying of a course of higher mathematics, which has the volume coming nearer to the university course. Such challenge is issued by this text-book on higher mathematics issues which is intended for students of the universities specializing in the field of computer sciences. Here the reader will find many perfectly developed pages as the course of the general mathematics cannot be original work. The reason of it that a course carries out the first contact to new knowledge and it is intended for the persons finished the school education and having only principles of elementary mathematics knowledge. Feature of the given text-book is also the uniform methodical approach to a statement of the entire higher mathematics course, consisting that the basic mathematical concepts follow from the general concepts and from logic concepts with the following distribution of a material.

The course is divided into five books.

The book 1 contains some logic concepts, the elementary concepts concerning to sets and operations on them (union, intersection, difference, product), and also the basic mathematical concepts, namely: concept of function or mapping; concept of n – dimensional arithmetic space.

The book 2 is dedicated to the linear algebra. From fundamental concept of mapping, concepts of internal and external laws of a composition are introduced. Conditions at which operations of these laws on a set transform them into groups, rings, fields and vector spaces are considered. It is investigated: a field of complex numbers; a ring of multinomials; vector space of multinomials; vector space of free vectors in geometrical space; vectors in n – dimension arithmetic space. Concepts of matrixes, determinants and system of the linear equations result from concepts of vector space and linear mapping of one vector space to another one. In the separate chapter it is considered reduction of matrixes by changing of basis to more simple form. Rather in detail, it is shown for reduction of the square matrix to the diagonal type, and the square-law form to the canonic type.

The book 3 contains a number of concepts of analytical geometry required by the program: the equations of a straight line on the plane and in the space; the equations of a plane; curves and surfaces of the second order, the equation of curves and surfaces of the second order are reduced to the canonical type with use of square-law forms. These geometrical concepts act as the direct appendix of the book 2 or as transferring of results of this book on language of geometry as it is made in it for free vectors in geometrical space.

The book 4 is dedicated to the mathematical analysis. Numerical functions of one and many real variables are considered. Concepts of limit and continuity are introduced for these functions. The book comes to an end with the statement of differential and integral calculus.

In the book 5 the chapters are collected which are concerning to the concepts, having technical character at a level of the general mathematics course; these are differential equations and lines.

The statement of a theoretical material is accompanied by the illustrative examples and the solutions of typical problems. With the purpose of reinforcement of educational material, here the exercises for independent work are offered.

BOOK 1
GENERAL CONCEPTS
CHAPTER 1
SETS
§ 1. DEFINITIONS AND LOGIC SYMBOLS

Many objects by some certain attribute, for example - objects of one nature, can be combined in a set, which is conceivable as the whole. The objects making a set, we shall name ***set members***. The set is usually designated with capital letters A, B, X, and its members are designated by small letters a, $в$, x. Belonging of the member $\boldsymbol{x}$ to the set A is written down $x \in A$.

If the set contains finite number of members such set is referred to as ***finite set.***

If for any beforehand given number β, what big it would not be, in set there will be the quantity of members which exceeds this number β it is said that such set is ***indefinite set***. More strict definition of infinite set will be given below.

1.1. Number sets

Sets which members are numbers refer to as ***number sets.***

Number set P can put in conformity a variable x which possesses all number values of this set i.e. which domain of variability are all number values of the set P. Such conformity is written down as follows $P = \{x\}$.

A number of number sets have standard designations:

1. Set of all natural numbers

$N = \{ n \}$, where $n = 1, 2, 3 \ldots;$

2. Set of all integers

$Z = \{ x \}$, where $x = 0, \pm1, \pm2, \pm3, \ldots;$

Set of all non-negative integers

$Z_0 = \{ x \}$, where $x = 0, 1, 2, 3, \ldots;$

3. Set of all rational numbers

$Q = \left(\frac{m}{n}\right)$, где $m \in Z$, $n \in N$.;

4. Set of all real numbers

$R = \{ x \}$, where $x = \pm\beta, \alpha_1, \alpha_2, \ldots \alpha_n \ldots$ - infinite decimal fraction or periodic one (set of rational numbers), or nonperiodic one (set of irrational numbers), here is $\beta \in Z_0$ and $\alpha_i \in Z_0$..

The set of all positive real numbers is designated R^+, and all negative ones -R^-. If these sets are added the number zero we shall write accordingly R_0^+ и R_0^-

1.2. Point sets of geometrical space

The least and indivisible structure of geometrical space is the point. All other geometrical figures and bodies of geometrical space are considered as set of points. Therefore geometrical figures on a plane, such as a segment, a line, a polygon, etc. and also bodies in geometrical space, for example, a sphere, the polyhedron, a cone, etc., represent point sets which members are points.

1.3. Set assignment

To assign a set, means, to specify that general features, that separates its members from other objects. In most cases set is assigned with the help of characteristic property of its members. Characteristic property of the set *A* is understood as such property which all members of the given set have and only they have it. If characteristic property of the set *A*, which member is *x*, we designate through *G (x)*, the set is written down:

$A = \{x | G(x)\}$

For example, if *A* is the set of all even natural numbers, it is written down:

$A = \{ x \,|x=2n.\ n \in N \}$

If two sets *A* and *B* consist of the same members such sets refer to as ***equal sets***. Equality of two sets is written down $A=B$.

1.4. Inclusion. Empty set

The set A which all members belong to some set B, is called ***a subset*** or a part of set *of* B. It is written down as $A \subset B$ or $B \supset A$ and it is read as: A is included into B or B contains A. . Symbol $\subset$ is called ***inclusion symbol***.

The subset which does not contain any members, is referred to as ***empty set*** and it is designated with symbol $\varnothing$.

By definition it is accepted, that for any set $A: \varnothing \subset A;\ A \subset A$.

If $A \subset B$ and $B \subset E$, then $A \subset E$ – is the property of transitivity. For example, $N \subset Z \subset Q \subset R$, то $N \subset R$.

If $A \subset B$ and $B \subset A$, then $A = B$.

1.5. Propositional logic. The theorem Necessary and sufficient conditions

Implication. We shall speak, that proposition W implies or attracts, and also has as consequence the proposition Q, if Q is valid every time as it is valid W**,** and we shall write down $W \Rightarrow Q$. If, in its turn, Q attracts W then propositions W and Q refer to as equivalent ones; it is written down $W \Rightarrow Q$. Then in any reasoning it is possible to replace one of these two propositions by another one.

Quantifiers. For designation of expressions “for all”, “for everyone”, “however that may be”, "exists", “there will be even one”, symbols which refer to as quantifiers are used:

Universal quantifier $\forall$**:** “for all”, “for everyone”, “however that may be”.

Quantifier of existence $\exists$:: "exists", “there will be even one”.

For example, the statement, that $A \subset B$ it is possible to write down as follows - $\forall a \in A \Rightarrow a \in B$**.** The opposite is incorrect. That fact, that $a \in B$ does not attract, that $a \in A$**.** Propositions are not equivalent.

Negation. Negation of the given property is represented by a symbol ⁄ of the given property crossed out with line $\not\subset, \notin, \not\Rightarrow$.

For example, the statement, that the set *F* is not a part of the set *B*, is equivalent to the following: there is such member *a* from *F*, that *a* does not belong *B*.

$F \not\subset B \Leftrightarrow (\exists a \in F \Rightarrow a \notin B)$. Propositions are equivalent.

The theorem. The mathematical proposition, which validity is defined by the proving (by the reasoning), is referred to as ***the theorem.*** The auxiliary theorem is referred to as ***the lemma.***

The formulation of any theorem consists of two parts: conditions and conclusion which follows from the given condition. The condition and the conclusion can interchange the position: a condition can become the conclusion, and the conclusion – can become a condition. Then one of these theorems is referred to as ***direct theorem***, and another to ***inverse theorem***.

In mathematics there are theorems with three various conditions; necessary, sufficient and both necessary and sufficient.

The necessary condition is a condition without fulfillment of which the given statement is incorrect.

The sufficient condition is a condition from which follows, that the given statement is true.

For example: 1. For the quadrangle to be a square, it is necessary, that its diagonals are mutually perpendicular.

This condition is necessary, but there is not enough. Actually, if diagonals are not perpendicular, a quadrangle is not a square but if diagonals are perpendicular, it does not mean still, that a quadrangle is a square.

2. If the sides of a quadrangle are equal, such quadrangle – is a parallelogram. This condition is sufficient, but it is not necessary since and without its fulfillment (the sides are not equal) the quadrangle can be a

parallelogram. The same condition can be both necessary, and sufficient at the same time.

For example, if in a triangle two angles are equal, such triangle is isosceles. The given condition is ***sufficient***, since the theorem is true and it is ***necessary***. Actually, if in a triangle two angles are not equal, such triangle cannot be isosceles - the condition is necessary.

Necessity and sufficiency of a condition can be written down, using implication. If the theorem is considered as set of two propositions W and Q and if the theorem is true, i. e. implication is true $W \Rightarrow Q$, then Q is a necessary condition for W, and W is a sufficient condition for Q. If the propositions are equivalent $W \Leftrightarrow Q$, then W is a necessary and sufficient condition for Q, on the contrary Q is a necessary and sufficient condition for W.

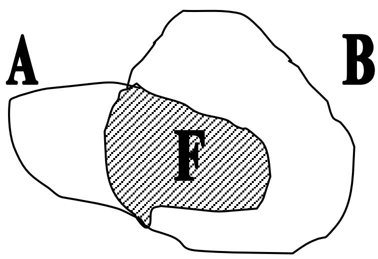
§ 2. OPERATIONS ON SETS

2.1. Intersection of sets

Let there are two sets A and B. The set of all members x, belonging at the same time *to* A and B, makes new set F which is referred to as ***intersection of A and B*** and it is written down:

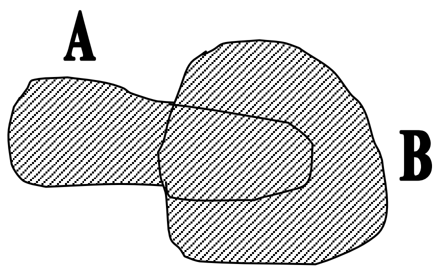
$$F = A \cap B = \{x \mid x \in A \text{ u } x \in B\} \quad (\text{fig. } 1.1)$$

Sign $\cap$ – is a symbol of intersection.



– $F = A \cap B$

Fig. 1.1



$C = A \cup B$

Fig.1.2

Operation of crossing possesses the following properties:

1. $A \cap B = B \cap A$ – operation $\cap$ is commutative;
2. $(A \cap B) \cap C = A \cap (B \cap C)$ – it is associative;
3. $A \cap A = A$, $A \cap \varnothing = \varnothing$;
4. If $A \subset B$, then $A \cap B = A$.

If sets have no common members, i.e. they are not intersected, then $A \cap B = \varnothing$.

2.2. Sum of sets

Let there are two sets *A* and *B*. The set *C*, consisting of members belonging to *A* or *B*, i.e. belonging or *A* or *B*, or *A* and *B* at the same time, is referred to as ***sum A and B*** and it is designated

$$C = A \cup B = \{x \mid x \in A, \textit{ or } x \in B, \text{ or } x \in A \text{ and } x \in B\}. \text{ (fig.1.2).}$$

Sign $\cup$ – a symbol of sum.

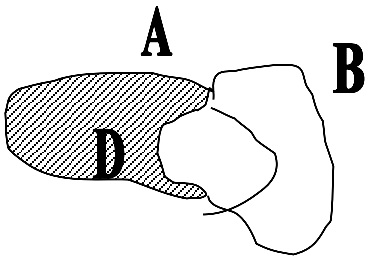
The basic properties of summing operation are as follows:

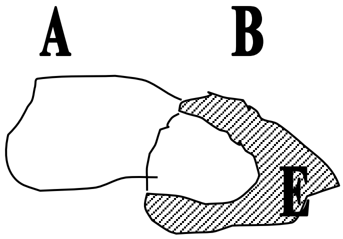
1. $A \cup B = B \cup A$ – operation is commutative;
2. $(A \cup B) \cup C = A \cup (B \cup C)$ – it is associative;
3. $A \cup A = A$, $A \cup \varnothing = A$;
4. If $A \subset B$, then $A \cup B = B$.

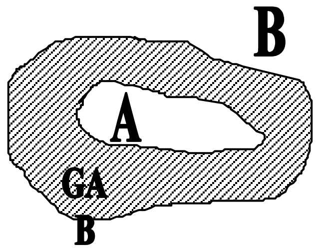
2.3. Set difference

Let there are two sets *A* and *B*. The set *D* consisting of members *x* of the set *A* and not belonging to the set *B*, is referred to as ***set difference of A and B*** and it is designated:

$$D = A \backslash B = \{x \mid x \in A \text{ и } x \notin B\} \text{ (fig.1.3).}$$







– D = A\B **– E = B\A** **– GA= B\A**
B

Fig. 1.3 **Fig. 1.4** **Fig. 1.5**

The basic properties

1. $A \backslash B \neq B \backslash A$ – operation is not commutative (fig.1.3 and 1.4);
2. $(A \backslash B) \backslash C \neq A \backslash (B \backslash C)$ – it is not associative ;
3. If $A \subset B$, then $A \backslash B = \varnothing$, but $B \backslash A$ makes the set named ***complement of set A relative to B*** and it is designated $\underset{B}{GA} = B \backslash A = \{x$ / $x \in B$ *and* $x \notin A,\ A \subset B\}$ (fig. 1.5).

We have: $A \cup \left(\underset{B}{GA} \right) = B$ *u* $A \cap \left(\underset{B}{GA} \right) = \varnothing$.

2.4. Product of sets

Let there are two sets *A* and *B*. And let $a \in A,\ b \in B$. Let's consider the ordered couple (a, b), and couples (a, b) and (b, a) are considered to be distinct, even if $A=B$. Set of the every possible ordered couples (*a*, *b*) makes the new set named ***product A and B*** and is designated $A \times B$. Elements *a* and *b* refer to as ***components***, or ***coordinates*** of the couple (*a*, *b*).

As an example product of two point sets *A* and *B* of the geometrical spaces is considered on fig. 1.6.

From fig. 1.6 we can see, that $A\times B \neq B\times A$ and, hence, product of sets is not commutative.

When set B is identical to set A($B = A$), then $A\times A$ represents set of the ordered couples **(**a, a'), where a and a′ belong to the same set A ($a\in A$ *and* $a'\in A$). Such set is referred to as ***the Cartesian square***. But also in this case $(a, a') \neq (a', a)$. Let’s illustrate it by the example of point sets (fig. 1.7).

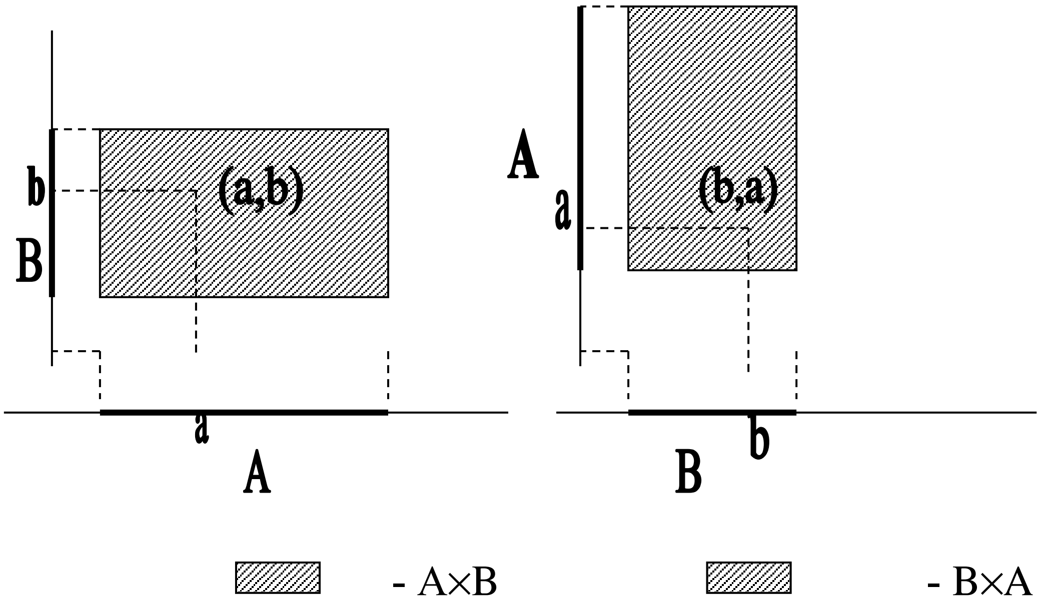


Fig. 1.6

The set of points of the shaded part of the plane makes the set $A\times A$ - the Cartesian square.

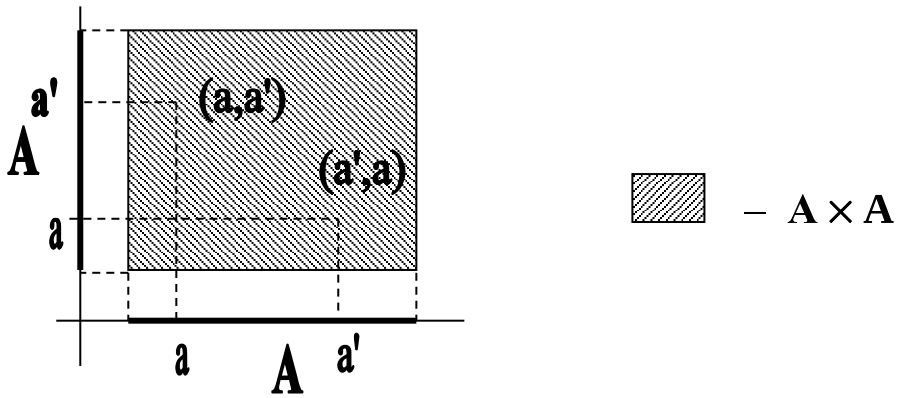


Fig. 1.7

In general let there is aggregate of sets A_1, A_2, A_3 . . . A_n, not necessarily distinct, we shall name as product and designate through

$$\prod_{i=1}^{n} A_i = A_1 \times A_2 \times A_3 \times \ldots \times A_n$$

the set of the ordered systems$(a_1, a_2, a_3 \ldots a_n$ where ***i-*** member belongs to set A_i. Symbol **Π** signify a sign of product:

$$\prod_{i=1}^{n} \alpha_i = \alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n .$$

The index *i* is referred to as an operational index. It can be replaced with any other letter:

$$\prod_{i=1}^{n} \alpha_i = \prod_{k=1}^{n} \alpha_k$$

Definition. The element of product of infinite number of the sets which is equal to the set *R* of real numbers is referred to as ***number sequence.***

$$(\beta_1, \beta_2, \beta_3, \ldots, \beta_n, \ldots) \in R \times R \times R \times \ldots \times R \times \ldots$$

CHAPTER 2. FUNCTIONS, MAPPINGS

§ 1. FUNCTIONS

Let there is the set D named ***a definitional domain***. And let there is the set E named ***range of values***.

Definition. Conformity which refers each element $x \in D$ to some element $y \in E$**,** is called ***a mapping*** D into E**.**

The element $x \in D$ (a prototype *of y*) is referred to as ***variables*** or ***argument***, the element $y \in E$ is referred to as ***value*** or ***direct image***.

Mapping is called also ***a function***, it is usually designated by letters f, ψ, φ and it is written down $y = f(x)$. Designation $x \rightarrow f(x)$ also is used, which is read as: the element x corresponds to the element $f(x)$. There is also a designation $f:D \rightarrow E$**,** which is read: f is a mapping of the set D into the set E**.** Also we can say, that f is a function of variable x with values in E or that $y=f(x)$ is a direct image of the element x at mapping f (or by means of f).

It is necessary to distinguish precisely the variable x which is a member of the set D, value of function $f(x)$ which is a member of the set E, and operation f which represents a category which is distinct from two previous ones. In the given definition of a function, two aspects are essential: first, indication of the set D for members x (i.e. a function domain) and, second, an establishment of a rule or the law of correspondence f between members $x \in D$ и $y \in E$. The range of values possessed by function $f(x)$, which is usually a subset of the set E of function domain, usually is not indicated, as the law of correspondence already defines this subset. The range of values possessed by function or $f(D)$, or $E(f)$ is designated;

$$f(D) = E(f)=\{ f(x) / x \in D \} \subset E$$

and it is referred to as ***image of set*** D at the mapping f or simply ***image of mapping*** f. So, at the mapping $f: D \rightarrow E$ not all members $y \in E$ should be images of any $x \in D$.

1.1. Identical mapping

If *E* = *D,* then *f* defines the mapping *D* into (or onto) itself. ***Definition.*** Mapping which puts any member $x\in D$ in conformity with the same member, is a mapping *D* onto *D,* named ***identical mapping,*** and designated *e***,** i.e. e: $D\rightarrow D$ and $e(x) = x, \forall x\in D$.

1.2. Function (mapping) graph

Definition. Let *f* there is a mapping of the set *D* into the set *E.* The set of the ordered couples $(x, f(x))$, where $x\in D$, and $f(x)\in E$, which are a subset of the product $D\times E$ is referred to as ***the graph of function*** *f.*

Let's consider it by the example of point sets (fig. 1.8)

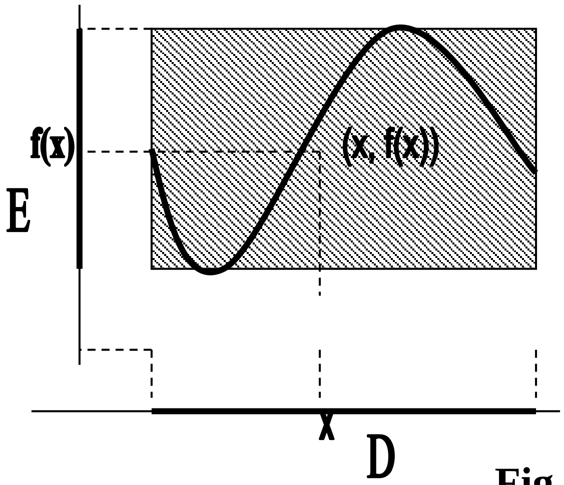


Fig. 1.8

The set of points of the shaded part of a plane makes the set $D\times E$. A line $\sim$ is a subset of couples $(x, f(x))$ - the graph of function *f.* Let's note, that each value of argument *x* corresponds only one point $(x, f(x))$, belonging to the graph of function *f.*

1.3. Sequence of set members

Let' take as *D* the set *N* of natural numbers, and as *E* - any set.

Definition 1. Mapping *f* of the set *N* into the set *E* is referred to as ***sequence*** of members from *E***.**

Thus, the sequence *f* connects each natural number *n* with some member *y* from *E,* which is usually designated y_n or f_n, $_{yn}$, instead of $f(n)$, and n is called an index. The sequence will be frequently designated $f = \{ f_1, f_2, \dots f_n, \dots$ or in abridged form $f = \{f_n\}$, , and the member $f_n = y_n$ from E we shall name a member with an index n (or *n-th* member) of the sequence f.

The mapping (sequence) f can not be unequivocal: the same member from E can serve as image of many various numbers from N. Therefore we should not confuse expression "sequence $f = \{f_n\}$" with expression "range of sequence f ". The range of the sequence $\{f_n\}$ can consist only of one member $y = a$ from E *at = and*, such sequences refer to as ***constant sequences*** and these are designated $\{a\}$, i.e. $f_n = a, \ \forall n \in N$.

Definition 2. Two sequences $\{f_n\}$ and $\{\Psi_n\}$ from E $\{f_n\}$ are equal, if $f_n = \Psi_n$ at all $n \in N$.

We should not confuse equality of two sequences with equality of ranges of these sequences. So, we shall consider sequence $\{f_n\}$, determined by means of $f_{2p} = 0$, $f_{2p+1} = 1$, where $p \in N$, i.e. $f_n = 0$,, if n –is even, and $f_n = 1$, if n – is odd, and sequence $\{\Psi_n\}$, determined as $\Psi_{2p} = 1$, $\Psi_{2p+1} = 0$. These sequences represent mapping of the set N into $E = Z_0$; range of these two sequences is the same; it consists of two members - 0 and 1; the sequences $\{f_n\}$ and $\{\Psi_n\}$ are not equal.

Using concept of function for numerical sequence (Chapter 1, § 2, item 2.4.), we can give following definition.

Definition 3. Mapping f of the set N of natural numbers into the set R of real numbers is referred to as ***numerical sequence***.

For example, the mapping

$$f : n \to f_n = \frac{2n^2 - 17}{\sqrt{n} + 3},$$

where $n \in N$ is a numerical sequence, and it is written down $\{f_n\} = \dfrac{2n^2 - 17}{\sqrt{n} + 3}$

The given numerical sequence is sequenced in such a manner that with index of the n-th member of sequence$\{f_n\}$ we can define also numerical value f_n of this member. For example, 9-th member of the specified sequence is equal to

$$f_9 = \frac{2\times 9^2 - 17}{2\sqrt{9}+3} = \frac{162-17}{9} = \frac{145}{9}$$

We also shall consider such ***preset numerical sequences*** below.

Definition 4. Numerical sequence $f : n \to a_n = a_1 + (n-1)d$, where $a_1 \in R$ and is referred to as ***an arithmetical progression***. The number d is referred to as ***a difference*** of an arithmetical progression.

Definition 5. Numerical sequence $f : n \to a_n = a_1\, g^{n-1}$**,** where $a_1 \in R$ and $g \in R$ is referred to as ***a geometrical progression***. The number g is referred to as ***a denominator*** of a geometrical progression.

§ 2. TYPES OF MAPPINGS

Lets' consider the mapping f of the set D into the set E. set of all images $f(x)$, where $x \in D$ at the mapping $f : D \to E$ forms a subset in the set E and as noted above, this subset is designated $f(D)$. Then $f(D)=\{f(x)/\ x \in D\} \subset E$.

Definition 1. If $f(D) = E$ i.e. when any member from E serves as image even of one member from D, mapping is referred to as ***superposition*** **(surjective)** , and we can say that f is the mapping D onto E.

And so, if $\forall\, y \in E \Rightarrow y = f(x)$, where $x \in D$ then f – is superposition, and $E = f(D)$.

Definition 2. Mapping at which different members of set D have various images, is referred to as ***a nesting (injective)***, i.e. if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

2.1. Biunique mapping

Definition. Mapping which is surjective and injective is referred to as ***biunique mapping***. In other words: any member $x \in D$ has as the image some unique member $y = f(x) \in E$, and any member $y \in E$ has a prototype some unique member $x \in D$.

For biunique mapping the operation, which is inverse to ***f*,** is mapping E oto D as for $\forall y \in E$ the image is the unique member $x \in D$. Such mapping is referred to as ***an inverse mapping*** to f and it is designated f^{-1}.

Thus, the distinctive feature of biunique mapping is existence of an inverse mapping for it.

For example, mapping f: $f: x \to y = x^3$, where $x \in R$ is mapping R onto R and it is biunique mapping. Inverse mapping for it will be $f^{-1}: y \to x = \sqrt[3]{y}$, where $y \in R$. Mapping $x \to x^2$ is mapping R into R and it is not biunique. As not any member $y \in R$ is an image of some member $x \in R$, and that member $y \in R$ which is an image, is the image of not a unique member $x \in R$**:** $y = -5$ is not an image $\forall x \in R$, and $y = 4$ is an image for $x = 2$ and $x = -2$. Therefore operation $y \to x = \pm\sqrt{y}$, which is inverse to mapping $x \to y = x^2$, is not a mapping.

2.2. Countable sets

Definition 1. If for sets D and E there is even one biunique mapping D onto E so we can say, that D and E have ***identical potency*** and also, that such sets ***are equivalent***.

The potency concept serves as generalization of usual concept of the counting. Actually, the counting consists in an establishment of biunique conformity between set of objects and some finite set of successive integers, starting with one.

The potency concept allows giving the exact meaning for the concept of the set having ***infinite*** number of members. Such set will be determined by means of the following property: ***there is even one subset distinct from all set and having with it identical potency***. So, let N there is a set of natural numbers; the set of even numbers constitute a part of the set N which is distinct from N. But conformity $n \to 2n$ is biunique; so, these two sets have identical potency, so N is infinite.

Definition 2. Set E is referred to as ***countable set*** if it has the same potency as the set N has.

It means, that there is a biunique mapping f of the set N onto E, i.e. anyone $n \in N$ can be put in conformity with one and only one such member $x \in E$, that $x = f(n)$, and $n = f^{-1}(x)$. Usually the member from E, corresponding *to* n, is designated through x_n, and n is referred as an index. So, the countable set is the set all members of which can be given natural indexes. We shall notice, however, that the opposite is not true; the member set of the sequence can not be countable, but it can be finite. So, the sequence determined by means $x_n = 1$ at any n, forms the set consisting of a unique member 1 so, this set is finite and thereby it cannot be set of the same potency with N.

An example of countable set. Set N' of even numbers is countable: actually, mapping $n \rightarrow 2n$ is biunique mapping N onto N'.

The theorem. Product of finite number of finite or countable sets is finite or countable.

Let's accept this theorem without the proving.

Corollary fact. Set Q of all rational numbers is countable. Set R of all real numbers - is uncountable.

2.3. Finite set permutation

Definition 1. Any biunique mapping of the set D onto itself is referred to as ***permutation*** of the set D.

Let D be a finite set from n members $D = \{a_1 ... a_n\} = \{a_i\}$, where $i = 1, 2, ... n$. Mapping f is permutation for the set D, if $f(a_i) = a_j$, where $a_i \in D$ and $a_j \in D$, $i = 1, 2, ... n, j = 1, 2, ... n$. If $i = j$, then $f(a_i) = a_i$ and there is identical mapping. $(f = e)$. Thus, identical mapping is always permutation.

Number of various permutations of the set D from n members is equal to $n!$ (n factorial). $n! = 1 \cdot 2 \cdot 3 \cdot \ ... \ \cdot n$ - is the product of n successive natural numbers, starting with one.

Definition 2 .Permutation in which places only two members of set are changed, is referred to as ***transposition.***

$$\tau_{ij} = \begin{matrix} a_1, a_2, \ldots, a_i, \ldots\ a_j, \ldots, a_n \\ a_1, a_2, \ldots, a_j, \ldots\ a_i, \ldots, a_n \end{matrix} .$$

Any permutation can be obtained from the basic permutation by successive transpositions. The choice of the basic permutation is completely arbitrary. For definiteness we shall name the basic permutation $a_1, a_2, \ldots, a_n$ and we shall consider arbitrary permutation f this set $a_i' = f(a_i)$, $i = 1, 2, \ldots n$, but a_i' is one of members $a_1, a_2, \ldots, a_n$ and so $a_i' = a_{m_i}$, where $m_1, m_2, \ldots, m_n$ - values of some set permutation $1, 2 \ldots n$, the first n natural numbers. Thus, the following two permutations are equivalent to:

$$f = \begin{pmatrix} a_1, & a_2, \ldots & a_n \\ a_{m_1}, a_{m_2}, \ldots a_{m_n} \end{pmatrix} \sim \begin{pmatrix} 1, & 2, \ldots, n \\ m_1, m_2, \ldots m_n \end{pmatrix}.$$

If into the permutation $m_1, \ldots\ldots m_n$ there will be such couple (m_j, m_i),, that $i > j$, *and*$>j$, *a* $m_i < m_j$, we can say that such couple forms ***an inversion***

Definition 3. The general number of the inversions formed by every possible couple of permutation $m_1, m_2, \ldots m_n$, is referred to as ***number of inversions*** of this permutation.

Permutation f is referred to as ***even permutation,*** if number of its inversions $\nu(f)$ is even otherwise it is referred to as ***odd permutation***.

For example, in the permutation

$$f = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7 \\ 3\,1\,5\,6\,4\,7\,2 \end{pmatrix}$$

number of inversions is equal to: (2); (0) ; (2) ; (2) ; (1) ; (1) - general number of inversions $\nu(f) = 8$. The permutation f is even.

The theorem. At transposition the permutation evenness changes, i.e. transposition - is an odd permutation.

The proof

$$\tau(i,j) = \begin{pmatrix} 1,2,\dots,i,\dots,j,\cdots n \\ 1,2,\dots,j,\dots,i,\dots,n \end{pmatrix}$$

$\nu(\tau(i,j)) = (j - i) + (j - i - 1) = 2(j - i) - 1$ – odd number.

Here: $(j - i)$ –is number of inversions for number j after it was permuted; $(j - i - 1)$ - is number of inversions for all numbers located after j and before i. For all other numbers the number of inversions has not changed.

§ 3. COMPLEX FUNCTION. INVERSE MAPPING

Definition 1. Let f there is a mapping of the set D onto the set E (i. e. $f(D) = E$), and g - mapping of the set E into the set G. And let $x \in D$, then $y = f(x) \in E$, also it is possible to consider member $z = g(y)$ which belongs to G. Thus, each $x \in D$ corresponds with $z = g[f(x)]$ from G and thereby mapping of the set D into G is determined, which is named ***complex function***, or ***a composition (superposition)*** of mapping f onto g and it is designated $g_0 f$ (here it is read from right to left, instead of from left to right since $g_0 f$ is $g[f(x)]$), g – is referred to as ***external function***, and f – ***internal function***.

Example. Let there be $f: x \to y = f(x) = 2^x$, where $x \in R, y \in R^+$. In this case f is a mapping of the set R onto the set R^+ and let $g: y \to z = g(y) = 5 - \frac{3}{y}$, where $z \in R$, and, hence, g is mapping of R^+ into R. Then $g_0 f: x \to z = \; = g[f(x)] = 5 - \frac{3}{2^x} = 5 - 3 \cdot 2^{-x}$ и $g_0 f : R \to R$.

Operation of composition of mappings $(_0\ )$ is generally non-commutative: $g_0 f \neq f_0 g$ and $f_0 g$ can not make sense, as f is a mapping of D onto E, and g – is a mapping of E into G.

Contrariwise, it is associative: if h is a mapping of G into H, then $h_0(g_0 f) = (h_0 g)_0 f$. Let $f(x) = y$, $g(y) = z$, $h(z) = \omega$ then $(g_0 f)(x) = g(y) = \; = z$ and $[h_0(g_0 f)](x) = h(z) = \omega$; just as $(h_0 g)_0 f(x) = [(h_0 g)(y)] = h(z) = \omega$.

Now with the help of a composition of mappings we shall define inverse mapping f^{-1} to the mapping *f***.**

Definition 2. Let mappings f be given: $f : D \to E$ and $\psi : E \to D$.. Mapping ψ is referred to as the inverse mapping to f and it is designated $\psi = f^{-1}$**,** if $\psi_0 f = f_0 \psi = e$, where e **–** is identical mapping: $e(x) = x$.

As it was mentioned above, inverse mapping exists, if f **–** is a biunique mapping. The inverse proposition is true – if f has inverse mapping f^{-1}, so this is biunique mapping.

§ 4. MAPPINGS OF SETS R, R×R и RxRxR ON TO POINT SETS OF THE GEOMETRICAL SPACE

4.1. Biunique mapping of the set R of real numbers onto set of points of the coordinate axis

Let's take a straight line and set on it a positive direction (usually it is shown with an arrow). Then the opposite direction will be negative. Such directed straight line is referred to as ***an axis.*** If we chose on the axis any reference point O and scale segment OE, such axis is referred to as ***coordinate*** or ***number axis***. The point O is referred to as the origin of coordinates. Coordinate axes usually are designated as x, y, z or Ox, Oy**,** Oz.

Let's choose on an axis Ox a point M and define its position. For this purpose let's measure the length of segment OM by scale segment OE. The length of a scale segment is accepted as equal to one $OE=1$. We shall obtain an abstract number $\alpha \in R_0^+$ which will be rational if scale unit and the given segment are commensurable, andit will be irrational if they are incommensurable.

Definition. C*oordinate of a point *M*. *on a number axis is the number $x \in R$ and equal to length of a segment OM $x = \alpha$, if the point M is located in a positive direction from the origin of coordinates and negative $x = -\alpha$, if the point is located in a negative direction from the origin of coordinates. The coordinate of the origin of coordinates is considered to be zero. That fact, that *x* is coordinate of point *M*, *is* written down *M (x)*.

In this case between set *R* of real numbers and set of points of a coordinate axis *Ox* it is possible to establish conformity $f{:}x \rightarrow M(x)$ – it is a conformity *f* will be biunique mapping. Each point *M* of a coordinate axis *Ox* corresponds to a unique real number *x* from *R* and on the contrary, each real number *x* from *R* corresponds to only one certain point *M* on a coordinate axis *Oh.* Thus, the set *R* and point set of a straight line have identical potency and, hence, they are equivalent. Mapping *f* here is understood as a way of definition of coordinate of a point *of M* on a coordinate axis *Ox.*

4.2. Biunique mapping of set R×R onto set of points of the coordinate plane

Let two intersected coordinate axes be given on a plane and their sequence on a plane be specified, for example, the first axis *x*, and the second – y. Such axes refer to as ***ordered axes***. The intersection point of axes *O* is taken as origin of both axes of coordinates. Scale segments at these axes can be various.

An angle two ordered axes *x* and *y* is an angle at which it is necessary to turn an axis *x* to *y* so that directions of both axes coincide. If turn is made counter-clockwise the angle is considered to be positive and if turn is made clockwise – the angle is considered to be negative. The angler between axes is defined ambiguously. If we designate the least angle between axes through φ, then the angle $\varphi+2\pi\kappa$, where $\kappa\in Z$, also will be an angle between these axes. If it is necessary to determine and angle unambiguously we bring restrictions, considering, for example, $0 \le \varphi < 2\pi$ *or* $-\pi < \varphi \le \pi$.

Definition 1. Two ordered coordinate axes intersected with an angle φ in a point accepted as origin of both axes, make the general ***Cartesian coordinate system on a plane*** (fig. 1.9, a).

The first axis Ox is referred to as ***an abscissa axis***; the second Oy - ***an ordinate axis***. The plane is referred to as ***coordinate plane*** and it is designated xOy.

Definition 2. The ordered set of two mutually perpendicular axes of coordinates $(\ \varphi = \pm \pi/2\ )$ with equal scale segments pieces $OE_1 = OE_2 = OE$ and with the general origin of coordinates Ot on each axis is referred to as ***the Cartesian rectangular system of coordinates on a plane***.

If $\varphi = +\ \pi/2$, the system of coordinates is referred to as ***right*** (fig. 1.9, b) if $\varphi = -\pi/2$ the system is referred to as ***left*** (fig. 1.9, c).

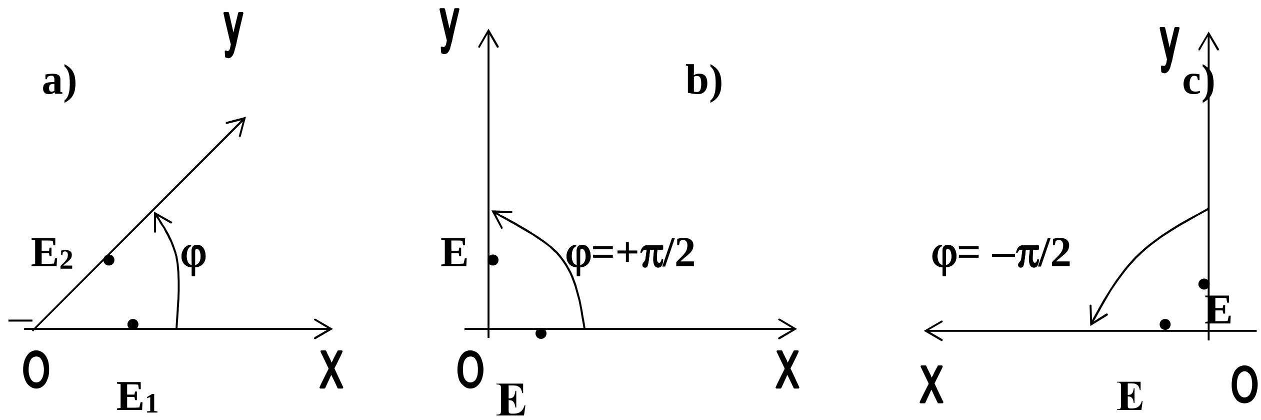


Fig. 1.9

Further we shall use only the right Cartesian rectangular system of coordinates.

We shall take in a coordinate plane xOy an any point M and we shall draw through at two straight lines parallel to axes Ox and Oy (fig. 1.10). Such operation is referred to as ***parallel projection***. Intersection points of these straight lines with coordinate axes we shall designate M_1 and M_2, and their coordinates - accordingly through x and y. Points $M_1(x)$ and $M_2(y)$ refer to ***projections*** of a point M to corresponding coordinate axes (fig. 1.10).

As a result of projection operation the point M is put into conformity with the ordered couple of numbers (x, y), where $x \in R$ and $y \in R$, hence, $(x,y) \in R \times R$.. These numbers are located in sequence of coordinate axes, and refer to as ***the Cartesian coordinates of point M on a plane*** and these are written down $M\ (x, y)$.

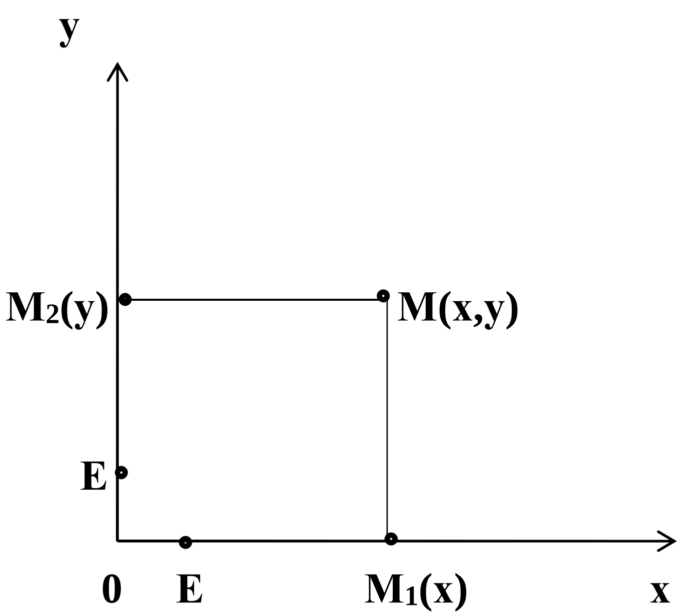


Fig. 1.10

It is easy to see, that each point *M* located in a coordinate plane *xOy*, corresponds to the unique ordered couple of numbers $(x,y)\in R\times R$.. On the contrary, each set ordered couple of numbers $(x,y)\in R\times R$ corresponds to unique point *M* in a coordinate plane *xOy*. To define it, it is necessary to draw straight lines through points $M_1(x)$ и $M_2(y)$ which are parallel to coordinate axes. The intersection point is the desired point $M(x, y)$**.**

Thus, between the set $R\times R$ of the ordered couples of real numbers and point set of the coordinate plane *xOy* it is set up a biunique conformity $(x,y) \rightarrow M(x,y)$, so, the set $R\times R$ and set of points of a plane are equivalent sets.

4.3. Biunique mapping of set R×R×R onto set of points of geometrical space in chosen system of coordinates

Let's take three ordered coordinate axes *x*, *y*, *z* which do not lay in one plane and are intersected in the point *O*. Lets take this point as the origin for all three coordinate axes. Such ordered set of coordinate axes is referred to as ***the general Cartesian system of coordinates in geometrical space***.

Definition. The ordered three in pairs perpendicular axes of coordinates with the general origin of coordinates *O* on each of them and with same scale segment *OE=1* for each coordinate axis, is referred to as ***The Cartesian rectangular system of coordinates in geometrical space*** (fig. 1.11).

The first axis is referred to as an axis *Ox*, or an **abscissa axis**, the second – axis *Oy*, or an ***ordinate axis***, the third – axis *Oz,* or an ***applicate axis***. The plane which is passing through two out of three axes *Oh, Oy, Oz* is referred to as ***a coordinate plane***; there are three coordinate planes; they are designated as: *xOy, yOz* and *zOx.*

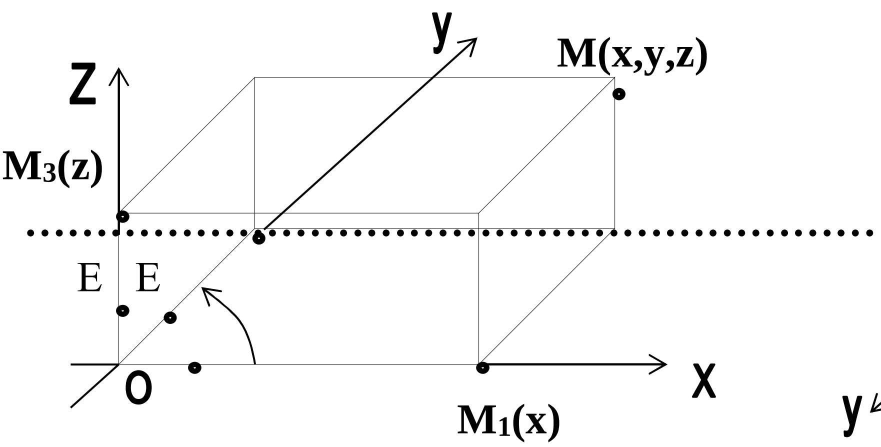


Fig.1.11,a

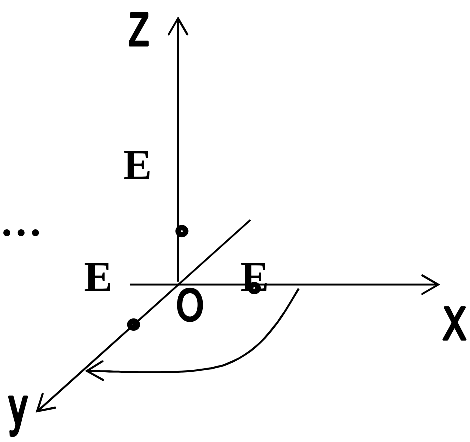


Fig.1.11,b

The ordered triple of coordinate axes which are not laying in one plane, is referred to ***right*** if from the end of a positive direction of axis O_Z the shortest turn from the axis O_X to the axis O_Y is seen counter-clockwise (fig. 1.11, a). Otherwise the system of coordinates is referred to ***left*** (fig. 1.11, b). We shall use only the right system of coordinates.

Let *M* – be any point of space. Let's draw through it the planes parallel to the coordinate planes (fig. 1.11, a). Intersection points of planes with corresponding coordinate axes we shall designate through M_1, M_2, M_3, and their coordinates - *x, y, z.* . Such ordered number triple $(x, y, z) \in R\times R\times R$ is named ***the Cartesian coordinates of a point M in geometrical***

space, and points $M_1(x)$, $M_2(y)$, $M_3(z)$ - are named the projections of a point M to coordinate axes and it is write down as $M\ (x, y, z)$.

It is obviously that each point of geometrical space is correspondent in the Cartesian system of coordinates to the unique ordered number triple. It is valid also the converse proposition: each ordered number triple in the Cartesian system of coordinates is correspondent to the unique point of space. To find it, we need to draw planes through points $M_1(x)$, $M_2(y)$, $M_3(z)$ which are parallel to corresponding coordinate planes. Straight intersection of these planes are intersected in a point which is the desired $M\ (x, y, z)$.

Thus, in the Cartesian system of coordinates it is established the biunique mapping of set $R\times R\times R$ of the ordered triple of real numbers onto the set of points of geometrical space: $(x, y, z) \to M(x,y,z)$, i.e. we can say, that the set $R\times R\times R$ and the set of points of geometrical space are equivalent. This mapping is made by means of the Cartesian system of coordinates and a way of definition of the point coordinates.

In case of product $R\times R\times R\times \ldots \times R$,, with number of factors $n > 3$, point sets in the geometrical space, which are equivalent to these sets, do not exist, in view of fact that we have no intuition of space with number of measurements, more than three. However, if we want to distribute geometrical methods also onto products of sets R, by number which is more than tree, we introduce the concept n - dimensional arithmetic space R^n and at $n > 3$.

CHAPTER 3
ARITHMETIC SPACE R^n

A point M of arithmetic space is the ordered set from n real numbers ($x_1, x_2, \ldots x_n$), which are called the coordinates of the point M, i.e. $M = M =$ **(**$x_1, x_2,$ **. . .** x_n) (или $M (x_1, x_2, \ldots x_n)$). The arithmetic space makes a set of all conceivable points M. The number n of coordinates of the point M, determined by quantity of factors in product $RxRxRx \ldots xR$, is referred to as ***dimension of arithmetic space***. It is designated as n - dimensional arithmetic space R^n.

For example: ***one-dimensional*** arithmetic space R^1. A point M of this space is the number $x \in R$, i.e. $M = (x)$. In geometrical space, the space $R.^1$ is mapped by a straight line; **bidimentional** space R^2. A point M of this space is the ordered couple of numbers $(x_1, x_2) \in R \times R$, i.e. $M = (x_1, x_2)$. In geometrical space, the space R^2 is mapped by a plane; ***three-dimensional*** space R^3 is mapped on all geometrical space and point $M = (x_1, x_2, x_3) \in RxRxR$**.** The further conformity of arithmetic space R^n, which can not have dimension $n > 3$ with geometrical space, also these spaces have no geometric vizualization.

§ 1. EUCLIDEAN SPACE

In arithmetic space R^n by analogy with geometrical space it is introduced the concept of "distance" between points $M_1 = (x_1, x_2, \ldots , x_n)$ and $M_2 = (y_1, y_2, \ldots , y_n)$, designated $d (M_1, M_2)$. If this "distance" is defined by the formula

$$d (M_1, M_2) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + \ldots (y_n - x_n)^2} \, , \qquad (3.1)$$

so such arithmetic space is referred to as ***Euclidean space***. In this case for $n \le 3$ *"distance"* between points in arithmetic space coincides with distance between points in geometrical space.

In n – dimensional Euclidean arithmetic space, as well as in geometrical space, we can introduce the concepts of "line", "figure", "body", etc.

For example. 1. Set of points $M =(x_1,x_2, \ldots , x_n)$, which coordinates independently one from another satisfy to inequalities

$$a_1 \le x_1 \le в_1,\ a_2 \le x_2 \le в_2 , \ldots, a_n \le x_n \le в_n,$$

is referred to ***as closed*** n - dimensional rectangular "parallelepiped" and it is designated as following:

$[\, a_1, в_1;\ a_2, в_2;\ \ldots ; a_n, в_n \,] = \{ M(x_1,x_2, \ldots , x_n)|\ a_i \le x_i \le в_i,\ i = 1, 2, \ldots n \}$

If there the strict inequality $a_i < x_i < в_i$,, "parallelepiped" is referred to ***open***.

At n ≤ 3 n-dimensional rectangular "parallelepiped" has real geometrical representations. If n = 1 and $a \le x \le в$, such closed one-dimensional rectangular "parallelepiped" is referred to as ***a segment***, it is designated $[\, a, в]$ and it is geometrically represented by a segment. Open one-dimensional "parallelepiped" ($a < x < в$), , is referred to as ***an interval*** and it is designated $(a, в)$.

In the case n = 2 closed bidimentional rectangular "parallelepiped" ($a \le x \le в,\ c \le y \le d$) it is geometrically represented by a rectangular with the sides $в – a$ and $d - c$.

Three-dimensional (n=3) closed rectangular "parallelepiped" $a \le x \le в,\ c \le y \le d,\ f \le z \le l$ is geometrically represented by an ordinary rectangular parallelepiped with the sides $в - a$, $d – c$ and $l - f$.

2. Set of points $M = (x_1,x_2, \ldots , x_n)$, determined by an inequality

$$(x_1 – y_1^0)^2 + (x_2 – y_2^0)^2 + \ldots + (x_n – y_n^0)^2 \le r^2 \ (\text{or} < r^2) ,$$

where $M_0 = (y_1^0, y_2^0, \ldots , y_n^0)$ is a constant point, and r is the positive constant number, forms ***closed*** (or ***opened***) n - dimensional "sphere" with radius r, with the center in point M_0**.** In other words, "sphere" is a set of points M, which distance from some constant point M_0 does not surpass (or less) r. It is clear, that this "sphere" if $n = 1$ is correspondent to segment, if $n = 2$ - a circle, and if $n = 3$ - an ordinary sphere.

Open "sphere" of any radius $r > 0$ with the center in point M_0 $(y_1^0, y_2^0, . . . , y_n^0)$ can be considered also as ***the vicinity*** of radius r or r - ***vicinity*** of this point. At n=1 the vicinity of a point x_0 of radius r represents an interval with the center in this point and it is designated (x_0-r, x_0+r).

All stated in this paragraph should be considered as an establishment only the certain geometrical language; it is not connected (at $n > 3$) with any real geometrical representations, therefore all geometrical terms which were used in the sense which is distinct from usual, we placed so-called: "distance", “ a rectangular parallelepiped ”, "sphere". Henceforth we will do it any more.

§ 2. THE BASIC PROPERTIES OF THE ARITHMETIC SPACE R^1

That fact, that between the set R of real numbers (space $R1$) and the point set of coordinate axis is established biunique conformity (Chapter.2, §4, the item 4.1.) enables with sufficient presentation to illustrate the basic properties of real number set.

2.1. Orderliness property

For any two real numbers x_1 and x_2 there is one, and only one of ratios:

а) $x_1 = x_2$ - points $M_1(x_1)$ and $M_2(x_2)$ coincide on a coordinate axis;

б) $x_1 > x_2$ - point $M_1(x_1)$ is located to the right of points $M_2(x_2)$ on a coordinate axis;

в) $x_1 < x_2$ – point $M_1(x_1)$ is located to the left of points $M_2(x_2)$ on a coordinate axis.

Signs **>** (greater than) and **<** (less than) have transitive property. It follows from $x_1 > x_2, x_2 > x_3$, that $x_1 > x_3$ and from $x_1 < x_2, x_2 < x_3 \Rightarrow x_1 < x_3$.

2.2. Density property

However what may be two real numbers x_1 and x_2, at that $x_2 > x_1$ there always will be a number x_3, , put between them: $x_2 > x_3 > x_1$.

There is an uncountable set of numbers x_3, moreover, among them there is also an uncountable set of rational numbers. Actually, points $M_1(x_1)$ and $M_2(x_2)$ are the segment ends $M_1 M_2$,, which length $d(M_1 M_2)$ is distinct from zero, and according to the formula (3.1.) it is equal to $x_2 - x_1$. Let's choose on a coordinate axis any point M_3 which coordinate we shall designate x_3. We shall demand the point $M_3(x_3)$ not to coincide with point $M_2(x_2)$ and we shall consider the ratio

$$\frac{d(M_1 M_3)}{d(M_3 M_2)} = \frac{x_3 - x_1}{x_2 - x_3} \qquad (3.2)$$

If this ratio equal to any positive number λ from R^+, then it follows from orderliness property of the set R, that point $M_3(x_3)$ is inside the segment M_1M_2 and, means, $x_1<x_3<x_2$ (provided that $x_2> x_1$).

Thus, for all $\lambda \in R^+$, the point M_3 with coordinate

$$x_3 = \frac{x_1 + \lambda x_2}{1+\lambda} \qquad (3.3)$$

is inside the segment M_1M_2, i.e. $x_1<x_3<x_2$ (if $x_2> x_1$) piece $M1M2$, i.e. $x1 < x3 < x2$ (if $x2 > x1$) and there is an uncountable set of such points, since λ - is any number from R^+.

The formula (3.3) which is obtained from (3.2.) provided that $\frac{x_3 - x_1}{x_2 - x_3} = \lambda$, is referred to **as *the formula of segment division in the given ratio.***

2.3. Continuity property

Let's partite the set R into two nonempty sets P , and P^1 and let the following conditions be satisfied:

1. Each real number gets into one and only in one of the sets P, P^1.
2. Each number α of the set P is less than each number α^1 of the set P^1.

Such partition is referred to as ***section***. Set P is referred to as ***the lower class*** of a section, set P^1 **- *the upper class*** of a section. The section is designated P/P^1. For section in the field of real numbers the following theorem is valid.

The theorem. For any section P/P^1 in the field of real numbers there is real number β, which makes this section. This number β, will be:

1) either the greatest in lower class P (and then there is no the least one in upper class P^1 there is no the least),
2) or the least in top class P^1 (then there is no the greatest one in bottom class P).

Really, since $x \rightarrow M(x)$ is a biunique mapping, and there is space between the points on a coordinate axis which are images of real numbers and thus the section always falls at a point of a coordinate axis which serves as image of real number β which is making a section of the set R.

2.4. Absolute value

Let x be some number from R. For it only one case exists from three cases $x < 0, x=0, x > 0$. Now let's define mapping $x \rightarrow f(x)$, as follows. We shall put $f(x)=x$, if $x \geq 0$ and $f(x)= -x$, if $x < 0$.. Then mapping (function) $x \rightarrow f(x)$, is referred to as ***absolute value*** or ***the module*** of number x and $f(x)$ it is designated $|x|$, i.e. $f(x) = | x |$.

Geometrically an absolute value of the real number x is equal to distance from the origin of coordinates 0 up to the point M mapping the given number x on a coordinate axis, i.e. $| x | = 0M$ (Chapter.2,§4.item.4.1).

Absolute value has the following three properties: however that numbers $\beta \in R$, $\gamma \in R$, may be, it always is

1. $|\beta| \geq 0$, и $|\beta| = 0 \Leftrightarrow \beta = 0;$
2. $|\beta \gamma| = |\beta| \, |\gamma|;$
3. $|\beta + \gamma| \leq |\beta| + |\gamma|$

Last inequality is referred to as an inequality of a triangle.

§ 3. MAPPING R^n INTO R; NUMERICAL FUNCTIONS OF REAL VARIABLES

Let’s consider the set *D* of points $M = (x_1, x_2, . . ., x_n)$ from R^n. If on this set *D* function *f* with value in *R* is determined, i.e. $\forall M \in D$ is put in conformity some number $y \in R$, such function is referred to as ***numerical function of real variables*** and it is designated $y = f(x_1, x_2, . . . x_n)$.

If $D \subset R$, function *f*, determined on *D*, is referred to as ***numerical function of one real variable***. In this case the variable *x* and the value $y = f(x)$ of function *f* belongs to same space R^1. The graph of such function – is a set of points in space R^2 with coordinates $(x, f(x))$. In geometrical space it is a line in coordinate plane *xOy*. Let's note also, that the sequence of real numbers (Chapter 2, §1.item.1.3) is a sequence of values of numerical function determined on the set *N*, and, hence, at which the role of performs a natural number *n*, taken increasing order.

When $D \subset R^2$, function *f*, determined on *D*, is referred to as ***numerical function of two real variables***. In this case variable is a point from R^2, i.e. the ordered couple (x, y), and value $z = f(x, y)$of functions *f* – is number from *R*. The graph of such function – is a set of points from space R^3 with coordinates $(x, y, f(x, y))$: in geometrical space - it is a surface. For example: $z = ax + by + c$ - a plane; $z = \frac{x^2}{a} + \frac{y^2}{b}$, where a >0 and b >0 - an elliptic paraboloid (fig. 1.12).

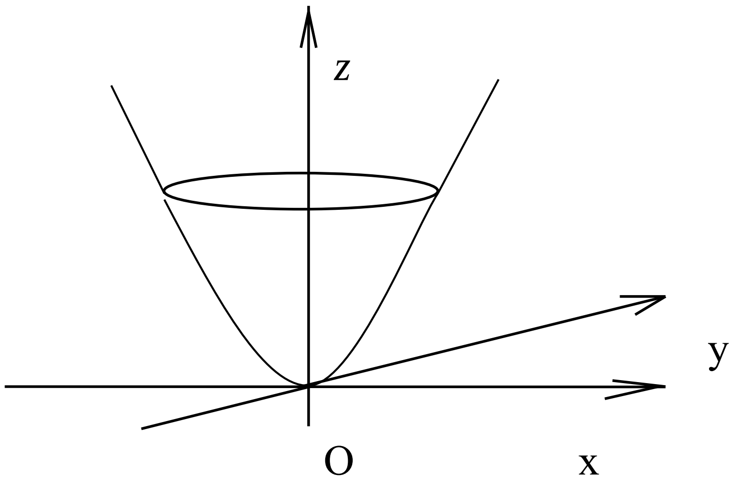


Fig.1.12

Function *f*, determined on $D \subset R^n$ (n ≥ 2), is referred to as ***numerical function of many real variables***. In this case value of function

y from R, it is designated: $y = f(x_1, x_2, \dots, x_n)$. For example, $f(x_1, x_2, \dots, x_n) = a_1x_1 + + a_2x_2 + \dots + a_nx_n$ – a linear function.

The descriptive graph in geometrical space at such functions (n> 2) does not exist.

EXERCISES

1. Represent on a plane $A \cap B$, if:

   a) $A = \{M(x,y) \mid x^2 + y^2 \le 1\}$, $B = \{M(x,y) \mid y \ge x^2\}$;

   b) $A = \{M(x,y) \mid x^2 + y^2 \le 25\}$, $B = \{M(x,y) \mid |x| \le 7, |y| \le 4\}$.

Prove that operation of intersection of sets is associative one.

2. Define all members of set $A \times B$, if $A = B = \{a, в\}$.
3. What from the following conformity are mappings $f: R \to R$?

   a) $x \to \sqrt{x}$; b) $x \to tg\, x$; c) $x \to \sin x$.
4. Define set $D \subset R$, so that the following conformity are mappings $f: D \to R$: a) $f(x) = \frac{1}{x}$; b) $f(x) = \ln x$; c) $f(x) = \beta^x$, $\beta > 0$ и $\beta \ne 1$.
5. Let's consider system of coordinates on a plane. Each point of a plane we shall put in conformity with its projection onto axis O_X. Say, whether this mapping is: a) mapping onto axis Ox; b) biunique mapping?
6. Define $f(R)$, if :

   a) $f(x) = x^2$, $\forall x \in R$; b) $f(x) = (0,3)^x$, $\forall x \in R$; c) $f(x) = \cos x$, $\forall x \in R$.
7. Make all mappings set $A = \{a, b, c\}$ into itself and choose among them permutations of the set
8. Define the length of a bisector of angle A in a triangle with vertexes $A(2, -1), B(5, 3), C(-6, 5)$
9. Define the number of inversions in the permutation

$$f = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 3\,5\,2\,1\,6\,4\,7\,8 \end{pmatrix}.$$

10. Say, whether the set $A = \{ \frac{m}{n} \mid m \in N, n \in N\}$ is countable?
11. Define all points on a number axis, coordinates x which satisfy to an inequality: a) $|2x-7| < 5$; b) $|x^2 - 4x - 5| > x^2 - 4x - 5$.

12. Under what condition can mappings $f: x \to y = \sqrt{x}$ and $g : y \to z = 5^y$ form a complex function $g_0 f : x \to z = 5^{\sqrt{x}}$.

13. Define the sets D and E for which the following numerical functions of one real variable have inverse functions: a) $y = x^2$; b) $y = a^x$, $a > 0$ и $a \neq 1$; c) $y = \sin x$.

BOOK 2
LINEAR ALGEBRA
CHAPTER 1
LAWS OF THE COMPOSITION

§ 1. INTERNAL LAWS OF THE COMPOSITION

Definition. The internal law of a composition or ***the algebraic operation*** given on the set *K*, is referred to as mapping of the product $K\times K$ (Cartesian square) into *K*. In other words, algebraic operation is a rule, according which, the ordered couple (x_1, x_2), where $x_1\in K$ (x_1, x_2) and $x_2\in K$, is compared to the member x_3 from the same set *K*.

Instead of writing down a rule, by means of a functional symbol $f:(x_1,x_2) \to x_3$ or $f(x_1,x_2) = x_3$, some special symbols are used, namely: + for addition $x_1+x_2 = x_3$,, symbol $\bullet$ for multiplication, $x_1 \cdot x_2 = x_3$, designation $x_1^{x_2} = x_3$ for power, etc. To have an opportunity to study the general properties inherent in all these laws, we shall use a uniform symbol $\top$, and we shall write $x_1 \top x_2 = x_3$, that verbally is expressed: x_1 in a composition with x_2 gives x_3.

1.1. Properties of internal laws of the composition

Commutativity. The internal law $\top$ is referred to as ***commutative*** if for any x_1 and x_2 the condition satisfies

$$x_1 \top x_2 = x_2 \top x_1 \qquad (1.1)$$

Examples. Let $K = \mathbb{Z}$. Operations of addition and multiplication of integers are commutative, and exponentiation and subtraction – are not commutative:

$$x_1^{x_2} \neq x_2^{x_1} \quad \text{и} \quad x_1 - x_2 \neq x_2 - x_1.$$

Associativity. The internal law $\top$ is referred to as ***associative*** if for any x_1, x_2, x_3 from *K* , the condition satisfies

$$(x_1 \top x_2) \top x_3 = x_1 \top (x_2 \top x_3) \qquad (1.2)$$

Here it is important to observe the order of members.

Examples. Addition and multiplication of integers are associative, and exponentiation and subtraction – are not associative: (3 - 5) - 2 ≠ 3 - (5 - 2); $(2^2)^3 = 64$, but $2^{(2^3)} = 256$..

Neutral element. If there is such element $e \in K$, that

$$e \top x = x \top e = x, \qquad (1.3)$$

whatever $x \in K$ may be, so e is referred to as ***a neutral element*** concerning operation $\top$.

If the neutral element e exists, it will be unique. Since, if we can have other element e' we would have $e' \top y = y \top e' = y$ if any y. Then, having taken $x \top e = x$ as x , an element e', we shall obtain $e' \top e = e'$. Having taken $e' \top y = y$ as y an element e, we shall also obtain $e' \top e = e$. Hence, $e = e'$.

Examples. If $K = N$, addition has no a neutral element, and 1- neutral element of multiplication. If $K = Z$, both addition and multiplication have neutral elements, accordingly 0 and 1. For the law of a composition of mappings $g \circ f$, the identical mapping $e \circ f = f \circ e = f$ serves as a neutral element.

Symmetric elements. Let $\top$ be an internal law of a composition on K, which has a neutral element. We can say, that the element $\bar{x}$ from K $\bar{x}$ ***is symmetric*** to an element x from K concerning operation $\top$, if

$$\bar{x} \top x = e . \qquad (1.4)$$

If $x = e$, it serves as a symmetric element of itself, since $e \top e = e$.

If the element x has the symmetric element $\bar{x}$, and the element $\bar{x}$, has the symmetric member x i.e. when the condition is satisfied,

$$\bar{x} \top x = x \top \bar{x} = e \qquad (1.5)$$

we can say, that the element x ***is reversible*** concerning operation $\top$.

If each element $x \in K$ **i**s convertible concerning operation $\top$ such operation on this set K is referred as to ***reversible.***

Examples. If x is a real number, so $-x$ is symmetric to it concerning addition, and operation of addition is reversible on the set R. If, besides x

$\neq 0$, then $\frac{1}{x}$ is symmetric to x concerning multiplication, and operation of multiplication also is reversible on the set R, but without $x = 0$.

Distributivity. If on the set K two laws of a composition is defined which are designated as $\top$ and $\top$, then the law will refer to as ***distributive*** concerning the law $\perp$, if for any x, y, z from K we have:

$$x \top (y \perp z) = (x \top y) \perp (x \top z) \qquad (1.6)$$

Examples. Multiplication of numbers is distributive concerning addition, since $x\cdot(y + z)=x\cdot y + xz$, but addition is not distributive concerning multiplication, as equality $x +(y\cdot z) = (x+y) \cdot(x + z)$ is not valid for all x , y, z from R.

Operations of association and intersection of sets also are the laws of a composition and as it is easy to show, for any A, B, C

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C);$$

Hence, each of these laws is distributive concerning another.

1.2. The basic algebraic formations: groups, rings, fields

Group. We can say, that the set K, which have the internal law $\top$, is a group if the law $\top$ possesses the following three properties:

a) the law is associative;

b) there is a neutral element;

c) any element $x \in K$ has symmetric.

If these three properties are added the fourth property of commutativity, then the group is referred to as ***commutative or Abelian group.***

Examples. If $K = N$, then addition does not transform N into group since the last two conditions are not satisfied. If $K = Z$, then addition transforms Z into Abelian group.

Ring. Nonempty set K , on which two algebraic operations $\perp$ and $\top$ are specified, is named a ring if the set K relative to $\perp$ forms the Abelian group, and the second law $\top$ is associative on K and distributive relative to $\perp$.

If the second law $\top$ is commutative, then a ring is called a commutative ring.

Example. The set *Z* is a commutative ring: the law of group (Abelian) – is addition, the second law – is multiplication.

Field. The ring *K*, possessing the same property, that the set of members from *K*, having no a neutral element of the first law, forms Abelian group concerning the second law, and it is referred to as ***a field.***

It follows from definition of a field that it contains, at least, two neutral elements (but they belong to the different laws).

Example. The set *R* of real numbers is a field (law $\perp$- is addition, $\perp$ –is multiplication).

§ 2. EXTERNAL LAWS OF THE COMPOSITION

Definition. Let there be two sets *K* and *L*; mapping of the product $K \times L$ into *K* is referred to as ***the external law of a composition*** on *K*.

An example of the set of such type is ***the vector space***, IV chapter of the given book is devoted to its study.

§ 3. ISOMORPHISM

Definition. Let there be two various or coinciding sets *K* and *L*; and let *K* be given the internal law $\top$, and *L* – the internal law$\perp$. Isomorphism of the set *K* onto *L* is referred to as such biunique mapping *f* of the set; we can say, that *K* and *L* ***are isomorphic*** concerning the laws $\top$ and $\perp$.

Examples. 1. $K = Z$, the law $\top$ is addition; *L* – is the set of numbers of the 2^m kind (where $m \in Z$), and the law $\perp$ -is multiplication. Mapping $f: m \to 2^m$ is an isomorphism since $m + m' \to 2^{m+m'} = 2^m \cdot 2^{m'}$, i.e. $f(m + m') = f(m) \cdot f(m')$, and the mapping is biunique, since $2^p = 2^g$ result in $p = g$.

2. Let $K = R^+$, and the law $\top$ is multiplication; let further $L = R$, and the law $\perp$ is addition. Mapping $x \to ln\ x$, i. e. $f(x) = ln\ x$, is

isomorphism, as *ln* $(x, y) = \ln x + \ln y$ and besides this, mapping is biunique since $\ln u = \ln \text{v} \Rightarrow u = \text{v}$.

Isomorphism allows to replace operation $a \top в$ in the set K with following operations: we form members $a' = f(a)$ and $в' = f(в)$ of the sets L, and in L it is applicable to them the operation $\perp$, i.e. we form member $a' \perp в' = c'$; at last, we shall obtain $a \top в = f^{-1}(c')$. This process is of interest in that case when operation $\perp$ in L is more simple, than operation $\top$ in K. We do so when replacing by means of logarithms multiplication by addition.

When there is an isomorphism between two sets, each of them is given one or the several internal laws corresponding to each other at this isomorphism, these sets ***are*** often ***identified,*** i.e. for a designation of their members and symbols of the internal laws corresponding to each other at isomorphism, the same symbols are used. We shall meet an example of such identification when studying complex numbers and vector spaces.

CHAPTER 2
COMPLEX NUMBERS

We shall consider the equation $x2 + 1 = 0$. It is obvious, that any real number $x \in R$ is not the solution of this equation. We shall draw such field C, containing R as a subfield ($R \subset C$),on which the given equation can be solved. This field is the field of complex numbers.

§ 1. THE FIELD *C* OF THE COMPLEX NUMBERS

Definition. The ordered couple (a, $в$) of two real numbers, $a \in R$ and $в \in R$ is referred to as ***complex number***. Hence, $z = (a, в)$ is a member of the product $R \times R$ or a point of arithmetic space R^2.

We shall define on set $R \times R$ two internal laws - addition and multiplication - by means of the following rules:

$$z_1 + z_2 = (a_1, в_1) + (a_2, в_2) = (a_1 + a_2, в_1 + в_2)$$

$$z_1 \cdot z_2 = (a_1, в_1) \cdot (a_2, в_2) = (a_1 a_2, - в_1 в_2,\ a_1 в_2 + a_2, в_1) \qquad (2.1)$$

For $z_1 = z_2$ it is necessary and sufficiently, that $a_1 = a_2$ and $в_1 = в_2$.

We shall show now, that the set of complex numbers on which these two operations are given, is the field C.

Addition on the set C:

1. is associative: $z_1 + (z_2 + z_3) = (z_1 + z_2)+ z_3;$
2. is commutative: $z_1 + z_2 = z_2 +z_1;$
3. has a neutral element $e = (0, 0);$
4. is invertible, i. e. each complex number $(a, в)$ has a symmetric element $(-a,-в)$

$$(a, в) + (-a, -в) = (0, 0) = e .$$

Hence, for addition the set C is Abelian group.

Multiplication on set C:

1. is associative $z_1 \cdot (z_2 \cdot z_3) = (z_1 \cdot z_2) \cdot z_3;$
2. is commutative $z_1 \cdot z_2 = z_2 \cdot z_1;$
3. has a neutral element $e = (1, 0)$

$$(a, в) \cdot (1, 0) = (a \cdot 1 - в \cdot 0, a \cdot 0 +1 \cdot в) = (a, в);$$

4. Without a neutral element $e = (0, 0)$ for addition - it is invertible

$$(a, в)\left(\frac{a}{a^2+в^2}, \frac{-в}{a^2+в^2}\right) = \left(\frac{a^2}{a^2+в^2}+\frac{в^2}{a^2+в^2}, -\frac{aв}{a^2+в^2}+\frac{aв}{a^2+в^2}\right) = (1,0).$$

Thus, the set *C* without $e = (0, 0)$ for operation of multiplication is ***Abelian group*.**

Multiplication is distributive concerning addition

$$[(a_1, b_1) + (a_2, b_2)] \bullet (a_3, b_3) = (a_1, b_1) \bullet (a_3, b_3) + (a_2, b_2) \bullet (a_3, b_3)$$

So, all conditions are satisfied, and the set of complex numbers makes field ***C*.**

We shall prove that the plotted field meets the desired requirements.

1. We shall designate through *D* the set of couples of $(a, 0)$ kind , where $a \in R$, and $D \subset C$. We shall define how the operations (2.1) determined on *C*, on the set *D* function.

$$(a_1, 0) + (a_2, 0) = (a_1 + a_2, 0),$$

$$(a_1, 0) \bullet (a_2, 0) = (a_1 \bullet a_2 - 0 \bullet 0, a_1 \bullet 0 + a_2 \bullet 0) = (a_1 \bullet a_2, 0).$$

Hence, if each number $a \in R$ is put in conformity with $(a, 0) \in D$, the set *D* of complex numbers of $(a, 0)$ kind is isomorphic concerning addition and multiplication of corresponding numbers *a* from *R*. Therefore sets *D* and *R* can be identified. Thus, the first condition is satisfied: $R \subset C$.

1. In field *R* the equation $x^2 + 1 = 0$ has no solution. We search for the solution of this equation in a field *C*. The real number $1 \rightarrow (1, 0);\ 0 \rightarrow (0, 0)\ ;\ x \rightarrow (u, v)$, and the equation in it become

$$(u, v)^2 + (1, 0) = (0, 0).$$

When we executed operation of multiplication $(u, v) \bullet (u, v)$ and addition with $(1, 0)$, , we obtain

$$(u^2 - v^2 + 1, 2uv) = (0, 0).$$

By definition of couple equality we have $u^2 - v^2 + 1 = 0$ and $2uv = 0$. From here $u=0$ (or $v=0$) and $v = \pm 1$ (or $u^2 = -1$, has no solution). Hence, we obtain two solutions

$$x_1 = (0, 1) \text{ and } x_2 = (0, -1).$$

Couples which are solutions of the equation $x^2 + 1 = 0$, we designate $(0, 1) = i$, a $(0, -1) = -i$, and *i* is called ***imaginary unit***.

In this case any complex number can be written down as

$$z = (a, в) = (a, 0) + (0, в) = a + (0, 1)(в, 0) = a + iв, \quad (2.\ 2)$$

Where a and b – are real numbers, and $i^2 = (-i)^2 = -1$. Such form of record of complex number is referred to as ***algebraic***.

The number a is referred to as ***valid***, and b – as ***imaginary*** part of number z. We designate $a = Re z$, $в = Im z$. If $a = 0$, number $0 + iв = iв$ is referred to as ***imaginary***. Hence, in any operation of addition and multiplication it is possible to replace complex numbers z with the sum $a + iв$ and to make operations as with real numbers; it is sufficient to replace i^2 with -1 $i2$ every time when i appears with a power not less than 2, for example $i^3 = i^2 \bullet i = -i$, $i^4 = 1$, $i^5 = i$ etc.

Example.

$(a + iв)^3 = a^3 + 3a^2iв + 3a(iв)^2 + (iв)^3 = a^3 + i3a^2в - 3aв^2 - iв^3 =$
$= (a^3 - 3aв^2) + i(3a^2в - в^3)$.

§ 2. COMPLEX CONJUGATE NUMBERS

Since $(-i)^2 = -1$, the number-i has property of number i, namely, its square is equal to -1.

Definition. The complex number $\bar{z} = a - iв$ is referred to as ***complex conjugate number*** with number $z = a + iв$, *i.e.* the number distinct from z only by sign of an imaginary part.

Mapping $z \to \bar{z}$ is biunique mapping of the set of complex numbers onto itself, i.e. ***permutation*** of this set, since if $z = a + iв$, $z' = a' + iв'$, the condition $\bar{z} = \bar{z}'$ results in $a = a'$ and $в = в'$, , and, hence, $z = z'$.

Let $z = a + iв$ and $z' = a' + iв'$; we have

$(\overline{z + z'}) = (a + a') - i(в + в') = \bar{z} + \bar{z}'$.

The same $z \bullet z' = (aa' - вв') - i(aв' + a'в) = \bar{z} \bullet \bar{z}'$.

So, mapping $z \to \bar{z}$ is isomorphism concerning addition and multiplication.

The following properties also occur:

1. $z + \bar{z} = 2Re\, z = 2a$.. Hence, the sum of complex number with its conjugate number is always a real number;

2. $z - \bar{z} = 2i\, Im\, z = 2iв$ Hence, the difference of complex number with its conjugate number is always an imaginary number;

3. $z\bar{z} = a^2 + в^2$. Hence, product of complex number and its conjugate number is always a real number, which is ≥ 0;

4. if $z = \bar{z}$, then z –is a real number.

Let's consider the equation $ax^2 + вx + c = 0$,
where $a \in R$, $в \in R$, and $c \in R$. (2.3)

Solutions of such equation are numbers:

$$x_1 = \frac{-в + \sqrt{в^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-в - \sqrt{в^2 - 4ac}}{2a}$$

$$x_1 = \frac{-в + \sqrt{в^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-в - \sqrt{в^2 - 4ac}}{2a}.$$

If discriminant is $D = в^2 - 4ac > 0$, the solutions of the equation (2.3) will be two various real numbers. Under condition that $D = 0$ $x_1 = x_2 = -\frac{в}{2a}$ it also belongs to R. If $D < 0$, the equation (2.3) has no solutions in the field R. We shall define them in the field C of complex numbers. With this purpose we shall transform discriminant $D = в^2 - 4ac = -(4ac - в^2) = i^2(4ac - в^2)$, where $4ac - в^2 > 0$; Then we have:

$$x_1 = -\frac{в}{2a} + i\frac{\sqrt{4ac - в^2}}{2a} = \alpha + i\beta \quad \text{and} \quad x_2 = -\frac{в}{2a} - i\frac{\sqrt{4ac - в^2}}{2a} = \alpha - i\beta;$$

Hence, the equation (2.3.) where $D < 0$, has two roots on the field C: complex number $x = \alpha + i\beta$ and its complex conjugate number $\bar{x} = \alpha - i\beta$.

§ 3. THE MODULE OF A COMPLEX NUMBER. DIVISION TWO COMPLEX NUMBERS

Definition. The module of complex number z is referred to and designated $|z|$ mapping $z \to |z|$ of the sets C into the set of non-negative numbers from R, determined as $|z| = \sqrt{z \cdot \bar{z}} = \sqrt{a^2 + в^2}$.

The module is an absolute value. Actually:

1. $|z| \geq 0$, a $|z| = 0$ result in $z = 0$ and vice versa.

2. $|z_1 z_2| = |z_1| \cdot |z_2|$. Indeed,

$|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = z_1 \cdot z_2 \cdot \bar{z}_1 \cdot \bar{z}_2 = (z_1\bar{z}_1) \cdot (z_2\bar{z}_2) = |z_1|^2 \cdot |z_2|^2$.

3. $|z_1 + z_2| \leq |z_1| + |z_2|$.

Except for these three properties one more property is added:

4.$|z| = |\bar{z}|$.

Introduction of the module allows immediately to write down the real and imaginary parts for quotient of two complex numbers z_1 and z_2.

$$\frac{z_1}{z_2} = \frac{z_i \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \frac{(a_1 + i\textit{в}_1)(a_2 - i\textit{в}_2)}{(a_2 + i\textit{в}_2)(a_2 - i\textit{в}_2)} = \frac{a_1 a_2 + \textit{в}_1 \textit{в}_2}{a_2^2 + \textit{в}_2^2} + i\frac{a_2 \textit{в}_1 - a_1 \textit{в}_2}{a_2^2 + \textit{в}_2^2} \quad (2.4)$$

§ 4. GEOMETRICAL INTERPRETATION OF COMPLEX NUMBERS

Geometrically complex number $z = a + i\textit{в}$ as the member of the set *RxR*, is represented by a point *M* on a coordinate plane *xOy* with coordinates *(a, в)*. And this mapping, as we saw (Book 1, Chapter. 2, § 4, item 4.2), is biunique.

We shall consider a segment *OM* and angle φ, which it forms with axis *Ox* (fig. 2.1). We shall define length of the segment *OM*. From rectangular triangle (fig. 2.1) by Pythagorean theorem

$$d(OM) = \sqrt{a^2 + \textit{в}^2}\ ,$$

hence, the length of the segment *OM* corresponds to the module of complex number *z*: $d\,(OM) = |z\,|$.

Angle φ, if *(OM)* is given, unambiguously defines the position of a point on the coordinate plane, and, hence, a complex number. This angle is called ***argument*** of complex number and it is designated *Arg z*. The argument of complex number is considered to be positive if it is counted from positive direction of axis *Ox* counter-clockwise, and negative – at the opposite direction of counting. It is obvious, that argument φ for given complex number is defined not unequivocally, but accurate within an item, which is divisible by 2π, i.e.

$$\varphi = Arg\, z = arg z + 2\pi m, \text{ where } m = 0, \pm 1, \pm 2, \dots ,$$

argz – values of argument of complex number determined by inequalities $0 \le arg z < 2\pi$ (or $-\frac{\pi}{2} \le arg z < \frac{3\pi}{2}$) and which is called ***principal argument*** of complex number.

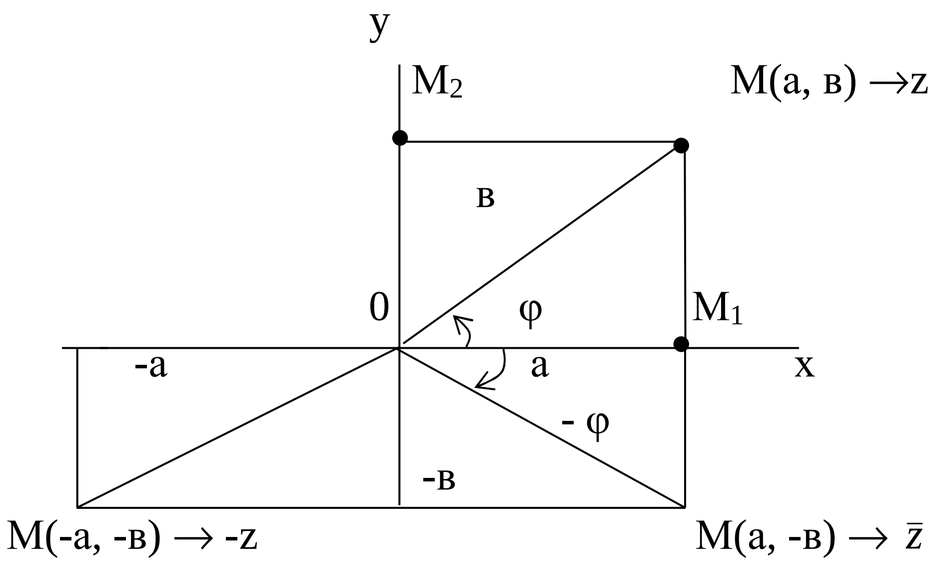


Fig 2.1

If $-\frac{\pi}{2} \leq arg\ z < \frac{3\pi}{2}$ and the point of M is in the coordinate half plane of the positive values of axis *Ox* *(a > 0)*,then $arg\ z = arc\ tg\frac{в}{a}$, if it is in the half plane of negative values *(a < 0)*, then $arg\ z = \pi + arc\ tg\frac{в}{a}$ If *a=0:* $arg\ z = \frac{\pi}{2}$, if *b>0*, and $arg\ z = -\frac{\pi}{2}$, if *b<0*. For definition of *arg z* we can to use also the following system of the equations

$$cos\varphi = \frac{a}{\sqrt{a^2+b^2}},\ sin\varphi = \frac{b}{\sqrt{a^2+b^2}}.$$

Whence, provided that $0 \leq \varphi < 2\pi$

$$\arg z = \begin{cases} \arccos\frac{a}{\sqrt{a^2+b^2}}, & \text{if } b \geq 0; \\ 2\pi - \arccos\frac{a}{\sqrt{a^2+b^2}}, & \text{if } b < 0. \end{cases}$$

Thus, we understand *Arg.z* as all set of the angles adequate to number *z* and, apparently from fig. 2.1, we have: $Arg\,\bar{z} = -Arg.z;\ Arg(-z) = \pi + Arg.z.$

The set of real numbers is characterized by condition *(a, 0)* and, hence, they lay on axis *Ox*. Set of imaginary numbers is characterized by condition *(0, в)* and they lay on axis *Oy*. Therefore axis *Ox* is referred as ***real***, and axis ***O – imaginary*** axis. Whole plane is referred to as ***a complex*** plane.

§ 5. THE TRIGONOMETRICAL FORM OF A COMPLEX NUMBER. MOIVRE FORMULA. EXTRACTION OF THE ROOT

We shall consider a complex number which is distinct from zero $z = a + ib$, and we shall write down it, using value $|z| = d\ (OM)$ and $\varphi = Argz$. Using fig. 2.1, we can write down $a = |z|\cos\varphi$ и $b = |z|\sin\varphi$. Then for complex number we obtain:

$$z = |z|(\cos\varphi + i\sin\varphi) \text{ or } z = r(\cos\varphi + i\sin\varphi), \text{ where } r = |z|. \qquad (2.5)$$

This record is referred to as ***the trigonometrical form*** of complex number. For $z = 0$ trigonometrical form is not determined, and for argument we can to take any real number.

Use of the trigonometrical form of complex number considerably simplifies operations of multiplication, division and extraction of a root.

Multiplication. Let $z_1 \cdot z_2 \neq 0$ and $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$, and $z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$. Then

$$z_1 \cdot z_2 = r_1 r_2(\cos\varphi_1 + i\sin\varphi_1)(\cos\varphi_2 + i\sin\varphi_2) =$$
$$= r_1 r_2[(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2)] =$$
$$= r_1 r_2[\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2)].$$

Thus, product of two complex numbers which are distinct from zero, is complex number which module is equal to product of modules of these numbers, and the argument is equal to the sum of arguments of the multiplied numbers. The obtained result is easy for transferring on product n of numbers $z_1, z_2, \ldots, z_n$. In particular if $z_1 = z_2 = \ldots z_n = z = r(\cos\varphi + i\sin\varphi)$, then

$$z^n = r^n(\cos n\varphi + i\sin n\varphi). \qquad (2.6.)$$

This equality is referred to as ***Moivre formula***. From here

$$|z^n| = |z|^n, \qquad Arg\, z^n = n\, Arg\, z.$$

Division.

$$\frac{z_1}{z_2} = r(\cos\varphi + i\sin\varphi) \Rightarrow z_1 = z_2 r(\cos\varphi + i\sin\varphi) = r_2 r[\cos(\varphi_2 + \varphi) + i\sin(\varphi_2 + \varphi)].$$

$$r_1(\cos\varphi_1 + i\sin\varphi_1) = r_2 r[\cos(\varphi_2 + \varphi) + i\sin(\varphi_2 + \varphi)]$$

Equality is possible, if

$$r_1 = r_2 r \Rightarrow r = \frac{r_1}{r_2}$$

$$\varphi_1 = \varphi_2 + \varphi \Rightarrow \varphi = \varphi_1 - \varphi_2$$

The quotient of two complex numbers which are distinct from zero, is a complex number which module is equal to the quotient of modules of the given numbers, and argument – is equal to a difference of numerator and denominator arguments.

Extraction of a root. The root of the *n-th* power of a complex number z is referred to as any number $z_k \in C$ which n-th power is equal to z. Thus $\sqrt[n]{z} = z_k \Rightarrow z_k^n = z$. From the last equation we have:

$\left|z_k^n\right| = \left|z_k\right|^n = |z|$ and $Argz_k^n = nArgz_k = Argz$. Thus, $\left|z_k\right| = \sqrt[n]{|z|}$ and $Argz_k = \dfrac{Argz}{n}$.

Hence, $\left|z_k\right| = \sqrt[n]{|z|}$ and $Argz_k = \dfrac{Argz}{n}$.

If $z = 0$, then it is indispensable that $z_\kappa = 0$ that is, zero has in C only one root of the n - th power, namely a zero. Now let's assume, that $z \neq 0$. As $Arg\ z$ it is determined accurate within 2π, and therefore the argument of number $z_{\kappa\kappa}$ can take n, and only n values determined accurate within 2π, namely:

$$\arg z_k = \frac{\arg z}{n} + \frac{2\pi k}{n}, \text{ where } \kappa = 0, 1, 2, \ldots n-1.$$

Hence, $\sqrt[n]{z}$ has on the set C n various values $z_0, z_1, \ldots, z_{n-1}$, which n-th power is equal *to z:* $z_k^n = z$, $\kappa = 0, 1, 2, \ldots n - 1$.

$$z_k = \sqrt[n]{|z|}\left[\cos\left(\frac{\arg z + 2\pi k}{n}\right) + i\sin\left(\frac{\arg z + 2\pi k}{n}\right)\right]. \qquad (2.7.)$$

It is obvious, that the points which are mapping the numbers z_κ on a complex plane, lay on a circle with the center O and radius $\sqrt[n]{|z|}$ and represent vertexes of regular n- square.

We shall consider a special case, when $z = 1$; then $|z| = 1$, $arg\ z = 0$, $Argz = 0 + 2\pi m$, $m = \pm1, \pm2, \ldots$ and, then , n-th roots of one have the module 1, and the argument $\left(\dfrac{2\pi k}{n} + 2\pi m\right)$, where $\kappa = 0, 1, 2, \ldots n - 1$. So,

roots of one on set C will be numbers: $z_k = \cos\left(\frac{2\pi k}{n} + 2\pi m\right) + i\sin\left(\frac{2\pi k}{n} + 2\pi m\right)$,
where k = 0, 1, 2... n - 1, m = 0, ±1, ±2....

Points, mapping the numbers z_k on a complex plane in the case if n = 6, are shown on fig. 2.2.

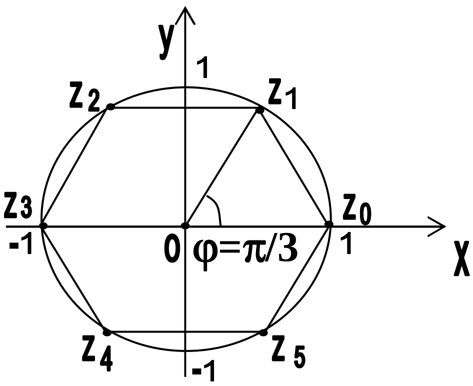


Fig.2.2

§ 6. COMPLEX FUNCTIONS

6.1. Complex functions of one real variable

Definition. Complex function of one real variable is referred to as mapping R (or some subset from R) into C.

Let x belongs to some set P from R, and F is a complex function from x, determined on P. Value of function F in the point x is a complex number $F(x)$, which real and imaginary parts are the essence real numbers which value depends on x, i.e. these are numerical functions of real variable. Thus, $F(x) = \psi(x) + ig(x)$, where ψ and g – are numerical functions of real variable, determined on $P \subset R$.

It follows from definition of set C, that it is identical to the set R^2. Therefore complex function F of one real variable can be considered as mapping of the set P into R^2 or if $P = R$, then $F: R \to R^2$, or as the ordered couple of two numerical functions of one real variable $F(x) = (\psi(x), g(x))$.

6.2. Complex functions of one complex variable

Definition. Complex function of one complex variable is referred to as mapping *C* (or some subset from *C*) into *C*.

Let P be some set from C. If each complex number $z \in P$ at mapping *F* is put in conformity with complex number *F (z)*, then real and imaginary parts *F (z)* are the essence real numbers which values depend in *z***,** so these will be values of two numerical functions of complex variable $z \in P$. Thus

$$F(z) = \psi(z) + ig(z).$$

But *C* is identified with R^2, i. e. each complex number $z = x + iy \in C$ is identified with point $(x, y) \in R^2$, therefore we can consider ψ and *g* to be numerical functions of two real variables *x* and *y*. Hence, we can write

$$F(z) = \psi(x, y) + ig(x, y) \text{ or } F = \psi + ig.$$

Then function *F* acts as mapping R^2 into R^2, or as the ordered couple of two numerical functions of two real variables:

$$F(z) = (\psi(x, y), g(x, y)).$$

6.3. Exponential function $z \to e^z$ with complex factor and its properties

Numerical exponential function $x \to a^x$ $(a > 0$ *u* $a \neq 1)$ of the real variable $x \in R$ makes the biunique mapping of the set *R* of real numbers onto the set R^+ of positive real numbers; this mapping transfers addition into multiplication, i.e. this function puts the sum $x_1 + x_2$ in conformity with the product $a^{x_1} \cdot a^{x_2}$ images of items: $a^{x_1} \cdot a^{x_2} = a^{x_1 + x_2}$. Is there a complex function *f* of complex variable *z*, determined on *C* and such, so that any $z_1 \in C$ and $z_2 \in C$,

$$f(z_1 + z_2) = f(z_1) \cdot f(z_2).$$

It is determined, that such function *f* exists also it is function $z \to e^z$, which values for any $z = x + iy \in C$ are defined as follows

$$f(z) = e^{x+iy} = e^x(\cos y + i \sin y).$$

Actually, it is not difficult to show, that for this function we have

$$e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}.$$

Except for this feature the exponent function $f(z) = e^z$ has as well the following features:

1. $e^{z_1 - z_2} = \dfrac{e^{z_1}}{e^{z_2}}$;

2. $(e^z)^m = e^{mz}$, where m – is an integer number;

3. $|e^z| = |e^{x+iy}| = e^x$, $(e^z \cdot e^{\bar{z}} = e^{2x})$

4. $e^{z+i2\pi m} = e^z$, where m – an integer number.

On the basis of feature 4 it follows, that exponential function e^z is a periodic function with the period $2\pi i$.

6.4. Euler's formulas. The exponential form of the complex number

If we put $x = 0$ into $z = x + iy$, then for e^z we shall obtain

$$e^{iy} = \cos y + i \sin y \tag{2.8}$$

It is Euler's formula expressing the exponential function with an imaginary parameter through trigonometrical functions. Replacing in Euler's formula y with $-y$, we shall obtain:

$$e^{-iy} = \cos y - i \sin y.$$

Now, combining e^{iy} and e^{-iy}, we have:

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}; \qquad \sin y = \frac{e^{iy} - e^{-iy}}{2i}.$$

These formulas also referred to as Euler's formulas.

We shall represent the complex number $z = a + iв$ in the trigonometrical form

$z = r(\cos\varphi + i\sin\varphi)$, where $r = |z| = \sqrt{a^2 + b^2}$; $\varphi = \arg z + 2\pi m$, $m = 0, \pm 1, \pm 2, \ldots$;

$\arg z = arctg\dfrac{b}{a}$, if $a>0$; $\arg z = \pi + arctg\dfrac{b}{a}$ if $a<0$; $\arg z = \pi/2$ or $-\pi/2$ $(3\pi/2)$ if $a = 0$.

By Euler's formula $\cos\varphi + i \sin\varphi = e^{i\varphi}$ and, hence, any complex number can be presented in the so-called ***exponential form:***

$$z = |z|\, e^{i\varphi} = r\, e^{i\varphi} = r\, e^{i(\arg z + 2\pi m)}$$

CHAPTER 3
MULTINOMIALS

Definition. Let P – be the given field (R or C), and x - some formal symbol. Expression of a kind:
$\alpha_\kappa x^\kappa + \alpha_{\kappa-1} x^{\kappa-1} + \ldots + \alpha_1 x + \alpha_0 x^0$, where an index $\kappa \in Z_0$: $\alpha_0, \alpha_1, \ldots, \alpha_\kappa \in P$, is referred to as ***a multinomial*** from variable or (unknown) x above the field P. Under the agreement it is written down as $x^0 = 1$, and a multinomial is written down as

$$\alpha_\kappa x^\kappa + \alpha_{\kappa-1} x^{\kappa-1} + \ldots + \alpha_1 x + \alpha_0 \qquad (3.1)$$

Members $\alpha_0, \alpha_1, \ldots, \alpha_\kappa \in P$, are referred to as ***factors of a multinomial;*** factor α_0, is referred to as ***a free term.*** If all factors are equal to zero the corresponding multinomial is referred to zero multinomial and it is designated with zero.

Maximum index k at which $\alpha_\kappa \neq 0$, is referred to as ***a degree* (*or order*) *of a multinomial***, and α_κ – is the leading coefficient of a multinomial. Zero multinomial has no a degree.

If $x \in R$ and $P = R$, the multinomial represents numerical function of one real variable. Such function is referred to as ***a polynomial*** or ***integer rational function.***

Multinomials of the variable x we shall designate as $f(x), g(x)$***, etc.***, and set of multinomials above the field P - $P[x]$.

Let's consider two multinomials from the set $P[x]$

$$f(x) = \alpha_\kappa x^\kappa + \ldots + \alpha_1 x + \alpha_0 \text{ and } g(x) = \beta_m x^m + \ldots + \beta_1 x + \beta_0$$

to be equal and we write down $f(x) = g(x)$, if $m = k$ (an identical degree) and $\alpha_i = \beta_i$, for $i = 0, 1, \ldots, \kappa$.

The multinomial can be written down also in the increasing order of indexes

$$\alpha_0 + \alpha_1 x + \ldots + \alpha_{\kappa-1} x^{\kappa-1} + \alpha_\kappa x^\kappa \qquad (3.2)$$

We shall note, that a multinomial $g(x)$ of the degree m always can be replaced with a multinomial which is equal to it with an index $\kappa > m$, adding to $g(x)$ a multinomial

$\beta_{m+(\kappa-m)} x^{m+(\kappa-m)} + \ldots + \beta_{m+1} x^{m+1}$, where $\beta_{m+1} = \beta_{m+2} = \ldots = \beta_{m+(\kappa-m)} = 0$, i.e.

$$g(x) = \beta_0 + \beta_1 x + \ldots + \beta_m x^m + 0\, x^{m+1} + 0\, x^{m+2} + \ldots 0\, x^{\kappa}.$$

So, any multinomial can be considered as sequence $\{\beta_0, \beta_1, \ldots \beta_m, 0, 0 \ldots\}$ from P which all members with some index are equal to zero.

§ 1. A RING OF MULTINOMIALS

Let's introduce on the set with multinomial $P[x]$ two internal laws of a composition - addition and multiplication of multinomials, distributive concerning addition of multinomials.

Addition. Sum of two multinomials *f (x) and g (x)* is referred to as multinomial

$h(x)=y_t x^t+\ldots+ y_1 x + y_0$, where $y_i = \alpha_i + \beta_i$, $i = 0, 1, 2, \ldots t$,

the degree of a multinomial t is equal to the greatest of two degrees if these degrees are not equal; if they are equal, it can occur, that the degree appears to be less (at$m = k$, $\alpha_k = - \beta_k$) and, hence, always we have

$$Cm\, h(x) \le max\, [Cm\, f(x), Cm\, g(x)].$$

It is clear, that operation of addition is associative and commutative.

There is a neutral member, namely a multinomial designated as $0 = 0\, x^{\kappa} + \ldots + 0\, x$, which all factors are zero.

At last any multinomial has symmetric, designated as

$-f(x) = -\alpha_{\kappa} x^{\kappa} - \alpha_{\kappa-1} x^{\kappa-1} - \ldots - \alpha_1 x_1 - \alpha_0$; it is a multinomial which all factors are opposite to factors of a multinomial *f (x).*

Hence, the set of multinomials provided with this law forms Abelian (commutative) group.

Multiplication. By virtue of distributivity of multiplication concerning addition it is sufficient to determine it for multinomials of a kind $\alpha_i x^i$. For $\alpha_i \in P$, $\beta_j \in P_{\circ\ i}\, x$ we shall suppose

$$(\alpha_i x^i)(\beta_j x^j) = \alpha_i \beta_j x^{i+j} \qquad (3.3)$$

In other words, we multiply variables as though their indexes were exponents of power. If

$$f(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_{\kappa} x^{\kappa}, \quad g(x) = \beta_0 + \beta_1 + \ldots + \beta_m x^m,$$

then by virtue of distributivity,

$f(x)\cdot g(x) = \alpha_0\beta_0 + (\alpha_0\beta_1 + \alpha_1\beta_0)\,x + \ldots + (\alpha_0\beta_i + \alpha_1\beta_{i-1} + \ldots + \alpha_i\beta_0)\,x^i + \ldots + \alpha_\kappa\beta_m\,x^{\kappa+m}$.

This operation is commutative and distributive concerning addition. With the help of rather long, but not complicated calculation we ascertain that it is associative.

We shall note the following important feature:

$$Cm[f(x)\cdot g(x)] = Cm\,f(x) + Cm\,g(x). \qquad (3.4)$$

Thus, the set $P\,[x]$ is a commutative ring. A multinomial $u(x) = \eta_0 + \eta_1 x + \ldots\ldots + \eta_t x^t$ is a neutral element concerning multiplication, if $u(x)\cdot f(x) = f(x)$ *for* any multinomial $f(x)$. In particular, it should be fulfilled $u(x) \bullet x^k = x^k$, and, then,

$$\eta_0 x^k + \eta_1 x^{k+1} + \ldots\ldots + \eta_t x^{k+t} = x^k,$$

that gives us $\eta_0 = 1$, $\eta_1 = \eta_2 = \ldots\ldots = \eta_t = 0$. So, $u(x) = x^0 = 1$; it enables us to identify a multinomial x^0 with number 1.

The multinomial $f(x)$ has no multinomial symmetric to it *concerning* multiplication.

Corollary fact. Equality $f(x)\cdot g(x) = f(x)\cdot\psi(x)$ at $f(x) \neq 0$ implyies $g(x) = \psi(x)$. Indeed, equality is written down also as

$f(x)\,[\,g(x) - \psi(x)] = 0$, and $f(x) \neq 0$, then , $g(x) - \psi(x) = 0$ and $g(x) = \psi(x)$.

§ 2. DIVISION OF MULTINOMIALS IN DECREASING DEGREES

If two multinomials $f(x)$ and $g(x)$ are given, we can not always define such multinomial h (x), that $f(x) = g(x)\,h(x)$. If $h(x)$ exists, we shall say, that $f(x)$ is divided by $g(x)$ *or* that $g(x)$ divides $f(x)$, and also, that the multinomial $f(x)$ is divisible by $g(x)$. So, the multinomial 0 is divisible by any multinomial: $0 = g(x)\cdot 0$.

The theorem (of division of a multinomial with a remainder). Let there be two multinomials $f(x)$ and $g(x)$ *of* the ring $P\,[x]$. There are such unique multinomials $h(x)$ *and* $r(x)$, that $f(x) = g(x)\,h(x) + r(x)$ where

$Cm\ r(x) < Cm\ g(x)$. $h(x)$ is called a quotient, and $r(x)$ - a remainder of division $f(x)$ by $g(x)$.

Remark. If $Cm\ g(x) > f(x)$, $h(x) = 0$, and $r(x) = f(x)$. Therefore $h(x) \neq 0$, when $Cm\ g(x) < Cm\ f(x)$.

The proof of the theorem is omitted.

Corollary fact. To divide the multinomial $f(x)$ by a multinomial $g(x)$, it is necessary and sufficient that the remainder of division $f(x)$ *by* $g(x)$ is equal to zero.

Practical calculation

Arrangement of operations is the same as at division of integers, and multinomials are written down in decreasing order of the variable degrees. Therefore such division also is referred to as division by decreasing degrees.

Example. $f(x) = 5x^6 + 1, \quad g(x) = x^2 + 2x + 1$

$$
\begin{array}{l|l}
f(x) = 5x^6 + 0x^5 + 0x^4 + 0x^3 + 0x^2 + 0x + 1 & x^2 + 2x + 1 = g(x) \\
\hline
y_4x^4 \cdot g(x) = 5x^6 + 10x^5 + 5x^4 & 5x^4 - 10x^3 + 15x^2 - 20x + 25 \\
f_5(x) = -10x^5 - 5x^4 \qquad + 1 & y_4x^4 + y_3x^3 + y_2x^2 + y_1x + y_0 \\
y_3x^3 \cdot g(x) = -10x^5 - 20x^4 - 10x^3 & \\
f_4(x) = 15x^4 + 10x^3 \qquad + 1 & \\
y_2x^2 \cdot g(x) = 15x^4 + 30x^3 + 15x^2 & \\
f_3(x) = -20x^3 - 15x^2 \qquad + 1 & \\
y_1x \cdot g(x) = -20x^3 - 40x^2 - 20x & \\
f_2(x) = 25x^2 + 20x + 1 & \\
y_0 \cdot g(x) = 25x^2 + 50x + 25 & \\
f_1(x) = -30x - 24 &
\end{array}
$$

Here we have $f(x) = g(x) \cdot h(x) + r(x)$, where $h(x) = 5x^4 - 10x^3 + 15x^2 - 20x + 25$,

$r(x) = 30x - 24$. $Cm\ r(x) = 1$; $Cm\ r(x) < Cm\ g(x) = 2$.

§ 3. MUTUALLY DISTINCT AND IRREDUCIBLE MULTINOMIALS. THE EUCLIDEAN THEOREM AND ALGORITHM

Let two fixed multinomials *f (x)* and *g (x)* be given, if even one of them is not equal to zero. The multinomial *t (x)* is referred to as ***the common divisor*** *f (x)* and *g (x)* if it divides these multinomials without the remainder. The multinomial of a degree zero, i.e. constants $\alpha_0 \neq 0$, is always the common divisors.

Definition 1. The multinomial of the greatest degree which is a common divisor of multinomials *f (x)* and *g (x),* is referred to as ***the greatest common divisor*** **(GCD)** of multinomials *f (x)* and *g (x).*

If *h (x)* is GCD of multinomials *f (x)* and *g (x),* then GCD of multinomials $f(x)\ \psi(x)$ and $g(x)\ \psi(x)$ is $h(x)\ \psi(x)$ for any multinomial $\psi(x)$. Besides GCD are accordingly multinomials $\lambda\, h(x)$ and $\lambda\, h(x)\ \psi(x)$, where $\lambda \in P†$ and it is not equal to zero. Therefore further we shall understand GCD as that GCD which highest coefficient is equal to 1.

Any common divisor *t (x)* of the multinomials *f (x)* and *g (x)* divides GCD *h (x)* and any GCD *h (x)* divides *f (x)* and *g (x);* so, the set of the common divisors of multinomials *f (x)* and *g (x)* coincides with the set of divisors of the multinomial *h (x).*

Definition 2. Two multinomials *f (x)* and *g (x) are* referred to as ***mutually distinct*** if their GCD has zero degree (i.e. is not a zero constant).

If *f (x)* and *g (x)* – are mutually distinct two multinomials from $P\ \{x\}$, there are unique multinomials *v (x) and w (x)* from $P\ \{x\}$, which have the following property $v(x)\, f(x) + w(x)\, g(x) = 1$, and $Cm\ v(x) < Cm\ g(x)$, $Cm\ w(x) < Cm\ f(x)$. This equality is referred to as ***Bezout identity equation***.

Euclidean theorem. If *f (x)* divides the product *g (x) c (x)* and if *f (x)* and *g (x)* are mutually distinct, *f (x)* divides *c (x).*

The proof. Indeed, GCD of multinomials *f (x)* and *g (x)* is a nonzero constant λ and, then, GCD of multinomials *f (x) c (x)* and *g(x) c(x)* is $\lambda\, c(x)$. But *f (x)* divides *f (x) c (x)* and, by the data, divides *g (x) c(x),* and, hence, divides them GCD which is equal to λ *c(x),* and, so, *f (x)* divides *c (x).*

Definition 3. The multinomial *p(x)* is referred to as ***distinct*** or ***irreducible*** if it has no other divisors, except for itself and nonzero constants.

We shall take now any multinomial *f (x)* and GCD *h (x)* of multinomials *f (x)* and *p (x)****;*** since *p (x)* is irreducible, then *h (x)* is equal to either *p (x),* or a constant; in the first case *f (x)* is divided by *p (x),* and in the second case *f (x)* is mutually distinct with *p(x).* Thus, any multinomial either is divided by *p (x),* or it is mutually distinct with it. It can be the proof of the following theorem for factorization of multinomials.

The theorem 2. Each multinomial *f (x)* from the ring *P [x]* of the degrees ≥1, is factorized in the product of irreducible multinomials *p (x)* and *c* accurate within the sequence order, this factorization is unique

$$f(x) = p_1(x) \cdot p_2(x) \cdot .... \cdot p_n(x) = \prod_{i=1}^{n} p_i(x) \, . \qquad (3.5)$$

It should be mentioned, that the multinomial irreducibility concept significantly depends on a field of factors *P*; so, the multinomial $x^2 - 4$ is not irreducible in the field *Q* of rational numbers as it is divided by $x - 2$ and by $x + 2$; a multinomial $x^2 - 2$ is irreducible in *Q*, but not in *R* since it is divided by $x + \sqrt{2}$ and by $x - \sqrt{2}$; the multinomial $x^2 + 1$ is irreducible in *R*, and, then, and in *Q*, but not in *C* as it is divided by $x + i$ and by $x - i$.

We should mention that the multinomial of the first degree is irreducible for any field *P* since its any divisor is either a constant, or itself and it is a unique irreducible multinomial above the field *C* of complex numbers. Above a field of real numbers, except of a multinomial of the first degree also all multinomials of the second degree which have negative discriminant, will be irreducible.

Determining GCD: Euclidean algorithm. Let *f (x)* and *g (x)* - two multinomials and *Cm f (x)> Cm g (x);* let's divide *f (x)* by *g (x) by* decreasing degrees:

$$f(x) = g(x)h_0(x) + r_0(x), \quad Cm\, r_0(x) < Cm\, g(x).$$

Then we shall divide *g(x)* by $r_0(x)$,

$$g(x) = r_0(x)h_1(x) + r_1(x), \quad Cm\, r_1(x) < Cm\, r_0(x).$$

Let's divide again $r_0(x)$ by $r_1(x)$, we obtain the remainder $r_2(x)$, which degree is less than degree of$r_1(x)$. Then we shall divide $r_1(x)$ by $r_2(x)$, etc.; degrees of the consecutive remainders strictly decrease; hence, there will come the moment when some remainder $r_{n-1}(x)$ will be divided by the remainder $r_n(x)$, and, so, we shall obtain

$$r_{n-2}(x) = r_{n-1}(x)\, h_n(x) - r_n(x), \ Cm\, r_n(x) < Cm\, r_{n-1}(x),$$

$$r_{n-1}(x) = r_n(x)\, h_{n+1}(x).$$

Any common divisor of multinomials $f(x)$ and $g(x)$ divides $r_0(x)$ and, then, it divides $r_1(x)$ etc., at last, it divides $r_n(x)$; inversely, any divisor of the remainder $r_n(x)$ divides $r_{n-1}(x)$, so,then $r_{n-2}(x)$, etc., and, hence, divides $f(x)$ and $g(x)$; thus, $r_n(x)$ is GCD of multinomials $f(x)$ and $g(x)$.

This method of determination of GCD has the name ***Euclidean algorithm*** where a word algorithm means process of calculation.

§ 4. ZERO (ROOTS) OF THE MULTINOMIAL. MULTIPLICITY OF ZERO. MULTINOMIAL EXPANSION IN THE PRODUCT OF IRREDUCIBLE MULTINOMIALS ABOVE FIELD C AND R

If we substitute variable x in a multinomial $f(x)\in P[x]$ for the number $\beta\in P$, we shall obtain the number which we refer to as value of a multinomial when $x = \beta$ and it is designated

$$f(\beta) = \alpha_\kappa \beta^\kappa + \alpha_{\kappa-1}\beta^{\kappa-1} + \ldots + \alpha_1\beta + \beta_0.$$

Definition. Number λ from field P is referred to *as **zero*** (or *a **root***) of the multinomial $f(x)\in P[x]$, if $f(\lambda) = 0$.

Bezout theorem. For $\lambda\in P$ to be a root of a multinomial $f(x)\in P[x]$, it is necessary and sufficient that the multinomial $f(x)$ is divided by a multinomial $x - \lambda$. Further we shall designate a multinomial $x - \lambda$ as $p_\lambda(x)$.

The proof. Necessity: λ - a root and $f(\lambda)) = 0$. We shall divide $f(x)$ *by* $p_\lambda(x)$ *in descending powers*. As $p_\lambda(x)$ has a power 1 the remainder has a degree equal to zero so, is a constant β, which can be equaled to zero, and we have $f(x) = p_\lambda(x)\, f_1(x) + \beta$. We shall take $f(\lambda)$. As $p_\lambda(\lambda) = 0$, then

$f(\lambda) = \beta$. Hence, if λ there is a root of a multinomial $f(x)$, *then* $f(\lambda) = 0$; so, $\beta = 0$, and $f(x)$ is divided by $p_\lambda(x)$.

Sufficiency. If $f(x)$ is divided by $p_\lambda(x)$, then the remainder is $\beta = 0$, and then

$f(\lambda) = P_\lambda(\lambda)\, f_1(x) = 0$, since $p_\lambda(\lambda) = 0$.

Multiplicity of zero. Let $\lambda \in P$ to be a zero of a multinomial $f(x) \in P[x]$; then, if $p_\lambda(x) = x - \lambda$, then $f(x) = p_\lambda(x)\, f_1(x)$. It may be, that $f_1(x)$ has λ as zero, and then $f_1\, f_1(x) = p_\lambda(x)\, f_2(x)$; and $f(x) = p^2{}_\lambda(x)\, f_2(x)$.

Definition 2. Multiplicity of zero (root) λ is referred to as the greatest integral exponent h for which $f(x) = p^h{}_\lambda(x)\, f_h(x)$, and $f\, f_h(x)$ has no λ as zero, i.e. $f\, f_h(\lambda) \neq 0$.

If $h = 1$, then λ is referred to as ***simple zero*** if $h = n$, *then* λ is called a zero of multiplicity n, or n - th (double, triple, etc.) zero.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ – be a various zero of a multinomial $f(x)$, and let $h_1, h_2, \ldots, h$ – be their multiplicity. Then, $f(x) = p_{\lambda_1}^{h_1}(x)\, b(x) = p_{\lambda_2}^{h_2}(x)\, c(x)$. Multinomial $p_{\lambda_1}(x)$ has a power 1 and therefore it is irreducible for any field P, and, hence, it is mutually simple with $p_{\lambda_2}(x)$, if $\lambda_2 \neq \lambda_1$.. So, $p_{\lambda_1}^{h_1}(x)$ and $p_{\lambda_2}^{h_2}(x)$ are also mutually simple. But $p_{\lambda_2}^{h_2}$ ***(x)*** divides the product $p_{\lambda_1}^{h_1}(x)b(x)$**,** and, hence, according to Euclidean theorem **,** $p_{\lambda_2}^{h_2}(x)$ divides b (x), and we have $b(x) = p_{\lambda_2}^{h_2}(x)b^*(x)$., so, $f(x) = p_{\lambda_1}^{h_1}(x) \cdot p_{\lambda_2}^{h_2} \cdot b^*(x)$. Continuing this reasoning in sequence for all multinomials $p_{\lambda_i}^{h_i}(x)$, $i = 1,2,\ldots,n$**,** we finally obtain the formula for multinomial expansion product of irreducible multinomials.

$$f(x) = p_{\lambda_1}^{h_1}(x) \cdot p_{\lambda_2}^{h_2}(x) \cdot \ldots \cdot p_{\lambda_n}^{h_n}(x)\psi(x) = (x - \lambda_1)^{h_1} \cdot \ldots \cdot (x - \lambda_n)^{h_n}\, \psi(x),$$

and this form of representation makes obvious that fact, that, λ_i is zero of a multinomial $f(x)$ and that its multiplicity is equal to h_i.

Let k be a power of a multinomial *f (x);* the last expression for *f (x)* shows, that $\kappa = Cm\ f(x) = h_1 + h_2 + \ldots + h_n + Cm\ \psi(x)$, whence

$$h_1 + h_2 + \ldots + h_n \le \kappa = Cm\ f(x).$$

It follows that the multinomial of a degree k cannot have more than k of various roots – ***Lagrange theorem.*** If they are equal to k, so all of them are simple.

Let's a situation when field P is a field C of complex numbers and $f(x) \in C[x]$. In this case the theorem which has the name ***D'Alembert theorem (**or **the fundamental theorem of algebra)*** is valid**.** Any multinomial *f (x)* of $C\ \{x\}$ power which is greater than or equal to one, has in a field C of complex numbers and, at least, one root.

Corollary fact

1. Any multinomial *f (x) of* $C\ \{x\}$ power k has all its roots in a field C of complex numbers and their quantity in accuracy is equal k if to count each root as many times as it has its multiplicity.

2. Thus, if $f(x) \in C[x]$ and $Cm\ f(x) = k$, *then* $h_1 + h_2 + \ldots + h_n = \kappa$, , and $Cm\ \psi(x) = 0$; hence, $\psi(x)$ *is* a constant which is distinct from zero and expansion *f (x)* is represented as

$$f(x) = \alpha_k (x - \lambda_1)^{h_1} \cdot (x - \lambda_2)^{h_2} \cdot \ldots \cdot (x - \lambda_n)^{h_n}, \tag{3.6}$$

where α_n – is the highest coefficient $f(x)$, $h_1, \ldots, h_n \in N$, $\lambda_1, \ldots, \lambda_n \in C$. the formula is referred to as **canonical expansion *f (x) above a field C*** of complex numbers.

3. Let *f (x)* – be a multinomial with a real coefficient from the field R, then if among zeros $\lambda_1, \ldots, \lambda_n$ there is zero $\lambda_i \in C$ of multiplicity μ**,** so there should be a complex-conjugate root $\overline{\lambda_i}$ of same multiplicity μ among roots. Real irreducible multinomials, above the field R of the power more than one, are multinomials $\alpha_2 x^2 + \alpha_1 x + \alpha_0$, which has negative discriminant; such multinomials in a field of complex numbers have as roots two complex-conjugate numbers λ and $\bar{\lambda}$**.** (Book 2, Chapter 2, § 2).

Now, after we combine in couples a multiplier $(x - \lambda_i)^{h_i} \cdot (x - \lambda_j)^{h_j}$, where $\lambda_j = \overline{\lambda_i}$, $h_i = h_j = \mu_i$ in canonical expansion of a multinomial *f (x)* ***under*** the field, C we shall obtain**:**

$$f(x)=\alpha_k\left[x^2+\alpha_{11}x+\alpha_{01}\right]^{\mu_1}\cdot\ldots\cdot\left[x^2+\alpha_{1m}+\alpha_{0m}\right]^{\mu_m}\cdot(x-\lambda_1)^{e_1}\cdot\ldots\cdot(x-\lambda_t)^{e_t}, \qquad (3.7)$$

$\lambda_1, \lambda_2, \ldots, \lambda_t$ – are real zeros $f(x)$, $\alpha_{1j}\in R$, $\lambda_{0j} \in R$, $j = 1, \ldots, m$, $f(x)\in R[x]$, $Cmf(x) = 2(\mu_1 + \mu_2 + \ldots + \mu_m) + e_1 + \ldots + e_t$ mulitnomails $(x^2 + \alpha_{1j}x + \alpha_{0j})$ conform to zero couples λ_i and$\lambda_j = \bar{\lambda}_i$.

The obtained expansionis referred to as ***canonical expansion of a multinomial f* (x) *above the field R*** of real.

EXERCISES

1. Prove that multinomial intersection operation is dictributive relative operation of set summing.
2. Is the set Q of rational numbers, on which operation of multiplication is assigned, a group?
3. Is set Q a field, if :

a) on this set the law of multiplication is assigned as the first law, and as the second – the law of addition?

b) the first law – is the addition, the second – is multiplication?

4. Calculate $z=\frac{\sqrt{3}+i}{2-i\sqrt{3}}$.
5. Define the real values of x and y from the equation

$$(1 + i)x^2 + (2 + i)x - (1 - i)y = 7(1 + i).$$

6. What geometrical sense has the difference magnitude of two complex numbers? To define this magnitude for $z = 3 + i2$ and $\bar{z}=3-i2$. Represent these points on a complex plane.
7. Define all roots and to plot them on a complex plane: $\sqrt[3]{1+i};\ \sqrt[6]{-3}.$
8. Solve the equations:

a) $2x^2 - 3x + 7 = 0$, b $\cos x = 3$, c) $\sin x = 2$.

9. Define roots of the equation $z^8 - 2\sqrt{3}z^4 + 4 = 0$ and plot them on a complex plane.
10. Represent in the indicative form the complex numbers:

$$1 + i,\ -1 + i,\ -5,\ \sqrt{3} + i.$$

11. Divide a multinomial $3x^6 + 2x^3 - 2x + 5$ by a multinomial $2x^2 + 3$ in descending powers.

12. Define the multiplicity of zero $x = 1$ for a multinomial

$$f(x)= 3x^5 - 8x^4 + 4x^3 + 6x^2 - 7x + 2$$

and the expansion of this multinomial in product of irreducible multinomials on the field R and C.

CHAPTER 4
VECTOR SPACES

On some set *K*, which has the internal law of a commutative group, can be determined also by means of some other set *L*, the external law of a composition - mapping $K \times L$ into *K*.. The most important set of such type is ***a vector space*** (or ***linear space).***

Definition. The set *K* is referred to as ***a vector (linear)*** space above the field *P* if it has the internal law **(+)** - addition and the external law **(·)** - multiplication by an element from the field *P*, having the following properties:

1. Addition on set *K* has the internal law of the commutative group. . $\forall x \in K$, $\forall y \in K$ and $\forall z \in K$ we have:

   $x + y = y + x;$

   $x + (y + z) = (x + y) + z;$

   $\exists e \in K$, so, that x +e = e + x = x (a neutral element),

   $\exists \bar{x} \in K$, so that $\bar{x} + x = e$ (a symmetric element).

2. The external law of multiplication, so that $\forall x \in K$, $\forall y \in K$ and $\forall \lambda \in P$, $\forall \mu \in P$,

   $\lambda \cdot (x + y) = \lambda \cdot x + \lambda \cdot y$

   $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$

   $\lambda \cdot (\mu \cdot x) = (\lambda \cdot \mu) \cdot x$

   $\varepsilon \cdot x = x$, where ε is a neutral element of multiplication in the field *P*.

Elements from vector space *K* are referred to as vectors and they are usually designated by lower case Latin letters with arrows above then ($\vec{a}, \vec{в}, \vec{x}$ etc.) or by lower case in thick print. Elements of the field *P* more often are designated by lower case Greek letters ($\alpha, \beta, \gamma, \lambda$ etc.). The neutral element of addition e in *K* is referred to as a zero vector and it is designated as $\vec{0}$. The neutral element of addition in *P* is designated by 0 (zero), and multiplication ε - by 1 (one). The element $\bar{x}$ symmetric *x* is referred to as opposite to a vector $\vec{x}$ and it is designated $-\vec{x}$, i.e. $\bar{x} = -\vec{x}$.

Corollary fact from definition. 1) In vector space it can be only one zero vector and for each vector can be only one opposite vector. Let's assume that there are two zero vectors $\vec{0}_1$, and $\vec{0}_2$, then it follows from definition, that their sum should be equal to each of them, i.e. $\vec{0}_1 + \vec{0}_2 = \vec{0}_1$, or $\vec{0}_1 + \vec{0}_2 = \vec{0}_2$ and, hence, $\vec{0}_1 = \vec{0}_2$. Similarly if any vector $\vec{x}$ has two opposite $-\vec{x}_1$ and $-\vec{x}_2$ the sum $(-\vec{x}_1) + \vec{x} + (-\vec{x}_2)$ should be equal both $-\vec{x}_1$ and $-\vec{x}_2$, hence $-\vec{x}_1$ = $-\vec{x}_2$.

2) If $\lambda \vec{x} = \vec{0}$, then either λ, or $\vec{x} = \vec{0}$.

3) Equality $\lambda \vec{x} = \mu \vec{x}$ is executed for any $-\vec{x}_1$ = $-\vec{x}_2$. If $\vec{x} \neq \vec{0}$, then, after we add both parts of equality $-\mu \vec{x}$ we shall obtain $\lambda \vec{x} - \mu \vec{x} = \mu \vec{x} - \mu \vec{x} = \vec{0}$ that is $(\lambda - \mu)\vec{x} = \vec{0}$, but $\vec{x} \neq \vec{0}$, hence λ - μ = 0 и λ = μ.

4) Equality $\lambda \vec{x} = \lambda \vec{y}$ is executed for any $\vec{x}$ and $\vec{y}$ if λ = 0.If λ ≠ 0, then $\lambda \vec{x} - \lambda \vec{y} = \vec{0}$ or $\lambda(\vec{x} - \vec{y}) = \vec{0}$. Since λ ≠ 0, then $\vec{x} - \vec{y} = \vec{0}$ whence $\vec{x} = \vec{y}$.

§ 1. VECTOR SPACE OF MULTINOMIALS ABOVE FIELD *P* FACTORS

As know (Book 2, гл.3, §1) addition on the multinomial set above the field *P* has the internal law of commutative group. Now we shall define on the set *P* {*x*} of multinomials by means of the field *P* the external law of a composition.

Multiplication by an element from R. Let λ∈*P*; we shall put $\lambda f(x) = \lambda\alpha_n x^n + ... + \lambda\alpha_1 x + \lambda\alpha_0$; $\lambda f(x)$ is a multinomial, which all coefficients is essence of element product λ by coefficients of a multinomial *f (x)*.

It is obviously that $\forall f(x) \in P[x]$, $\forall g(x) \in P[x]$ и $\forall \lambda \in P, \forall \mu \in P$ we have:

$$\lambda[f(x) + g(x)] = \lambda f(x) + \lambda g(x);$$
$$(\lambda + \mu) f(x) = \lambda f(x) + \mu f(x);$$
$$\lambda[\mu f(x)] = (\lambda\mu) f(x);$$

$\varepsilon f(x) = f(x)$, where ε = 1 – is a neutral element of multiplication in *P*.

Thus, operations of addition of multinomials and its multiplication by a number from *P* transform set $P[x]$ of multinomials into vector space above the field *P* of coefficients, and the multinomial in relation to this set is a vector and it can be designated as $\vec{f}(x)$.

§ 2. VECTOR SPACES P^n ABOVE FIELD *P*

Any field *P* (field *R* of real or *C of* complex numbers) is a vector space above itself with addition as the internal law and multiplication as external law (K = *L =P).*

Product of any finite number *n* of sets *P* is also vector space above the field P. This vector space is designated as $P^n = PxPx...xP = \prod_{i=1}^{n} P_i$. Elements (vectors) of this space are the ordered sets from *n* numbers ($\alpha_1, \alpha_2, \ldots, \alpha_n$), named ***components*** or ***coordinates*** of a vector: $\vec{x} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, where $\vec{x} \in P^n$, and $\alpha_i \in P, i = 1,2,...,n.$ Internal and external laws of a composition in this space are as follows:

$$\vec{x} + \vec{y} \Rightarrow (\alpha_1, \alpha_2, ..., \alpha_n) + (\beta_1, \beta_2, ..., \beta_n) = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, ..., \alpha_n + \beta_n);$$
$$\lambda\vec{x} \Rightarrow \lambda(\alpha_1, \alpha_2, ..., \alpha_n) = (\lambda\alpha_1, \lambda\alpha_2, ..., \lambda\alpha_n), \qquad (4.1)$$

here $\vec{x} \in P^n, \vec{y} \in P^n, \lambda \in P, \alpha_i \in P, \beta_i \in P, i = 1,2,...,n.$

Theoretically the components of a vector can arrange not only in row $\vec{x} = (\alpha_1, \alpha_2, ..., \alpha_n)$, but also in column $\vec{x} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$.

Depending on an arrangement these spaces are referred to as space ***of row – vectors*** of length *n*, or ***column –vectors*** of height *n.*

Let's consider a case, when *P* = *R* and vector spaces $P^n = R^n$ are real. If *n* = 1,2,3, then, how we have already defined, between point set of arithmetic space R^n and point set of oriented geometrical space it is possible to determine the biunique mapping which has presentation: $R^1 \to$ point set of the coordinate axis; $R^2 \to$ point set of the coordinate plane; R^3- point set of oriented geometrical space. Mapping here is understood as a way of definition of point coordinates of space.

By analogy it reasonable to assume, that in geometrical space there are also evident vector spaces which can be put in biunique conformity with vector spaces R^n above the field R where n = 1, 2, 3. Lets' set up such conformity.

§ 3. VECTORS IN GEOMETRICAL SPACE

Definition. In geometrical space the directed segment $\vec{AB}$, which is set by the ordered couple of points *A* and *B*, is referred to as a vector $(\vec{a})$. The first point is referred to as origin of the directed segment $\vec{AB}$ and the second point *B* - its extremity, and: $\vec{a} = \vec{AB}$.In a designation of the directed segment $\vec{AB}$ the order of points is defined by the order of their representation: *A* - the first point, *B* - the second. If points *A* and *B* are distinct, the directed segment $\vec{AB}$ is referred to as nonzero (or ***nondegenerate***) and if points *A* and *B* coincide the directed segment $\vec{AB}$ is referred to as zero (or ***degenerate***) and it is designated as $\vec{0}$.

The length of the directed segment describing the numerical value of a vector, is referred to as ***the modulus*** or ***absolute value of a vector*** and it is designated as $\left|\vec{AB}\right|$ or $|\vec{a}|$. The direction of a segment determines a straight line on which the vector is located. If vectors are located on one straight line, or on parallel straight lines such vectors are referred to as ***collinear vectors***, i.e. there is a straight line which they are parallel to. If there is a plane relating which the vectors are parallel such vectors are referred to as ***coplanar vectors.***

The zero vector is considered to be collinear to any vector, since it has no the certain direction. The length of it is equal to zero.

Equality of vectors. Two vectors are considered to be equal if their directed segments are equal. For equality of the directed segments it is possible to give three various definitions. Depending on this vector they are subdivided into three types.

3.1. Types of vectors in geometrical space

Definition 1. Two directed segments are equal, if the following conditions are satisfied:

1. The origin of segments is in the same point;
2. Lengths of segments are equal;
3. Segments belong to one straight line;
4. The directed segments have identical directions.

If for determination of vector equality we base on the given definition then any vector represented by the directed segment $\vec{AB}$ - will be equal to the vector which is represented by the same directed segment $\vec{AB}$. Vectors, satisfying this rule, are referred to as ***the bound vectors***. Bound vectors are mapped with the unique directed segment, and there is no other directed segment equal to this vector.

Definition 2. Two directed segments are equal, if the following conditions are satisfied:

1. Lengths of segments are equal;
2. Segments belong to one straight line;
3. The directed segments have identical directions.

If for determination of vector equality we base on the given definition then a set of the directed segments located on one straight line having both identical length and direction (they can be lay off from any point of this straight line) map the equal vectors, and, hence, the same vector, such set of equal among themselves (in sense of definition 2) directed segments is referred to as ***a sliding vector.***

Definition 3. Two directed segments are equal, if the following conditions are satisfied:

1. Lengths of segments are equal;
2. The directed segments have identical directions;
3. The directed segments are collinear.

If for determination of vector equality we base on the given definition then a set of the directed segments located on one straight line or on parallel straight lines, having identical length and direction, map the equal vectors. Such set equal among themselves (in sense of definition 3) directed segments is referred to as ***a free vector***.

A free vector $\vec{a}$ is designated and represented with any of the directed segments $\overrightarrow{AB}$ of that directed segment set which is the vector $\vec{a}$. In each point of the space A' it is always possible to plot the directed segment $\overrightarrow{A'B'}$, which belongs to a set of directed segments of the given vector $\vec{a}$(i.e. $\overrightarrow{A'B'} = \overrightarrow{AB}$) and this directed segment for a specific point A' will be unique. This operation is made by means of parallel shift.

Further we shall consider only free vectors, and we shall name them, as far as possible, simply vectors. It is closely related with that fact that free vectors are imposed constraints on, and all other vectors represent a special case of free vectors which are imposed additional constraints.

3.2. Vector space of free vectors above field R

On set of free vectors in geometrical space we shall set two operations - addition of vectors and multiplication by the number from the field *R*. Let' show, that the free vector set forms a vector space above the field *R* with these operations.

Addition of free vectors. Let two free vectors $\vec{a}$ and $\vec{в}$ be given. Let's plot the directed segments $\overrightarrow{AB}$ and $\overrightarrow{BC}$ which are equal to them (it can be made for any point of *B* space). Then the directed segment $\overrightarrow{AC}$, which belong to a set of directed segments of a vector $\vec{c}$, is referred to as the sum of vectors $\vec{a}$ and $\vec{в}$ and it is designated $\vec{a} + \vec{в}$. We shall notice, that all three vectors $\vec{a}$, $\vec{в}$ and $\vec{a} + \vec{в} = \vec{c}$, belong to the same set of free vectors, i.e. addition is the internal law of a composition. We shall find out its properties.

1. Addition of vectors is commutative , i.e. $\vec{a} + \vec{в} = \vec{в} + \vec{a}$. Really, we shall lay off a vector $\vec{a}$ from an any point A : $\overrightarrow{AB} = \vec{a}$, and from a point *B* we shall lay off a vector $\vec{в}$: $\overrightarrow{BC} = \vec{в}$. Then $\vec{a} + \vec{в} = \overrightarrow{AC}$. Now, first we shall lay off from a point *A* a vector $\vec{в}$: $\overrightarrow{AD} = \vec{в}$. Then by virtue of equality $\overrightarrow{AD} = \overrightarrow{BC}$ (quadrangle *ABCD* – is a parallelogram) we have $\overrightarrow{DC} = \overrightarrow{AB} = \vec{a}$, , i.e. $\overrightarrow{DC}$

is a vector $\vec{a}$ laid off from the point *D*. Thus, $\vec{в}+\vec{a}=\vec{AD}+\vec{DC}=\vec{AC}$ and therefore $\vec{a}+\vec{в}=\vec{в}+\vec{a}$.

2. Addition of vectors is associative, i.e. for any vectors $\vec{a}$, $\vec{в}$ and $\vec{c}$ it is executed $(\vec{a}+\vec{в})+\vec{c}=\vec{a}+(\vec{в}+\vec{c})$.

The proof. Let *A* – be an any point, and *A, B, D* – are such points, that $\vec{AB}=\vec{a}$, $\vec{BC}=\vec{в}$, $\vec{CD}=\vec{c}$, then

$$\vec{a}+(\vec{в}+\vec{c})=\vec{AB}+(\vec{BC}+\vec{CD})=\vec{AB}+\vec{BD}=\vec{AD},$$

$$(\vec{a}+\vec{в})+\vec{c}=(\vec{AB}+\vec{BC})+\vec{CD}=\vec{AC}+\vec{CD}=\vec{AD}.$$

1. $\vec{a}+\vec{0}=\vec{a}$, i.e. $\vec{0}$ - is a neutral element.
2. $\vec{a}+(-\vec{a})=\vec{0}$, $-\vec{a}$ - is a symmetric element.

Last two properties are obvious. Thus, addition on a set of free vectors makes Abelian group.

Multiplication of a free vector by the number from R. Product $\lambda\vec{a}$ of the number $\lambda\in R$ on a free vector $\vec{a}$ in a case of $\vec{a}\neq\vec{0}$, $\lambda\neq 0$, is referred to as vector which is collinear to the vector $\vec{a}$, which absolute value is equal to $|\lambda||\vec{a}|$ and which is directed to the same direction as the vector $\vec{a}$, if $\lambda>0\vec{a}$, and in opposite direction, if $\lambda<0$. If $\lambda=0$ or $\vec{a}=\vec{0}$,then according to the definition $\lambda\vec{a}=\vec{0}$.

The following ***condition of vector collinearity*** follows from: if two vectors $\vec{a}$ and $\vec{в}$ are related by the ratio $\vec{в}=\lambda\vec{a}$, these of a vector are collinear. Such vectors are referred to ***proportional vectors.***

Thus, multiplication of a vector by the number $\lambda\in R$ represents the external law of composition. Let's define its properties.

1. For any numbers $\lambda\in R$ and $\mu\in R$ and any vector $\vec{a}$ $\lambda(\mu\cdot\vec{a})=(\lambda\cdot\mu)\vec{a}$.
2. $1\cdot\vec{a}=\vec{a}$, $\varepsilon=1$ – is a neutral element of multiplication in *R*.
3. For any numbers $\lambda\in R$ and $\mu\in R$ and any vector $\vec{a}$

$$(\lambda+\mu)\vec{a}=\lambda\vec{a}+\mu\vec{a}.$$

4. For any vectors $\vec{a}$ and $\vec{в}$ and any number $\lambda\in R$

$$\lambda(\vec{a}+\vec{в})=\lambda\vec{a}+\lambda\vec{в}.$$

First three properties are obvious. We shall prove property 4. Let's assume, that vectors $\vec{a}$ and $\vec{в}$ are not collinear. The case of vector collinearity $\vec{a}$ and $\vec{b}$ is reduced to properties 3 and 2. We shall lay off a vector $\vec{a}$ from the point *A*: $\vec{AB} = \vec{a},$ and the vector $\vec{b}$ from the point *B:* $\vec{BC} = \vec{b}$.

Let's plot vectors $\vec{AB'} = \lambda\vec{a}$ and $AC' = \lambda(\vec{a} + \vec{b})$(fig. 2.3). It follows from similarity of triangles *ABC* and AB'C' (both in case if λ > 0, and in case if λ < 0), that $\vec{B'C'} = \lambda\vec{b}$.

But $\vec{AB'} + \vec{B'C'} = \vec{AC'}$, hence $\lambda(\vec{a} + \vec{b}) = \lambda\vec{a} + \lambda\vec{b}$.

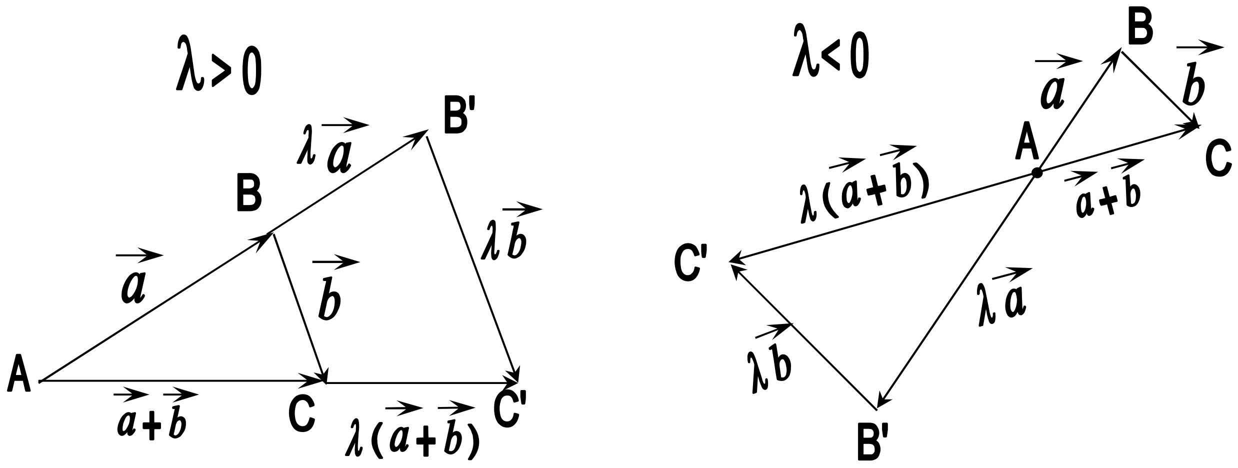


Fig. 2.3

The following ***condition vector coplanarity*** follows from the proof of property 4. For three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, to be coplanar, it is necessary and sufficient that they satisfy the ratio $\vec{c} = \lambda\vec{a} + \mu\vec{b}$, where $\lambda \in R$ and $\mu \in R$.. This ratio is read as: the vector $\vec{c}$ is ***a linear combination*** of vectors $\vec{a}$ and $\vec{b}$.

Thus, the set of free vectors on which the given operations of vector addition and vector multiplication by number from *R* are set, forms vector space above the field *R.*

Now let's consider, how it is possible to set a vector with the help the Cartesian rectangular system of coordinates, and lets' define their conformity with vectors from the space R^3.

3.3. The assignment of free vectors by means of system of coordinates and their conformity with vectors from vector space R^3

Let's choose in space Cartesian rectangular system of coordinates *x, y, z*. Let's consider an any vector $\vec{a}$ which is assigned by the directed segment $\vec{AB}$. We shall remind, that the point *A* can be any point of a space. In the chosen system of coordinates we shall define coordinates of the vector origin - points *A* and the end of this vector - point *B* (fig. 2.4).

Let coordinates of a point *A* be the triple of numbers (x_1, y_1, z_1), of the point *B* - (x_2, y_2, z_2). Then coordinates of a vector $\vec{a}$ is named the ordered triple of numbers (x, y, z), which is calculated by formulas:

$$x = x_2 - x_1; \quad y = y_2 - y_1; \quad z = z_2 - z_1, \text{ (fig. 2.4)}$$

It is written as follows $\vec{a}(x, y, z)$ or $\vec{a}(a_x, a_y, a_z)$.

If the origin of the directed segment $\vec{AB}$ coincides with the origin of coordinates $A(x_1, y_1, z_1) = O(0, 0, 0,)$, the directed segment is referred to as a radius - vector of a point *B*. In this case coordinates (x, y, z) of the vector $\vec{a}$c coincide with coordinates x_2, y_2, z_2 of the point B: $x = x_2, y = y_2, z = z_2$.

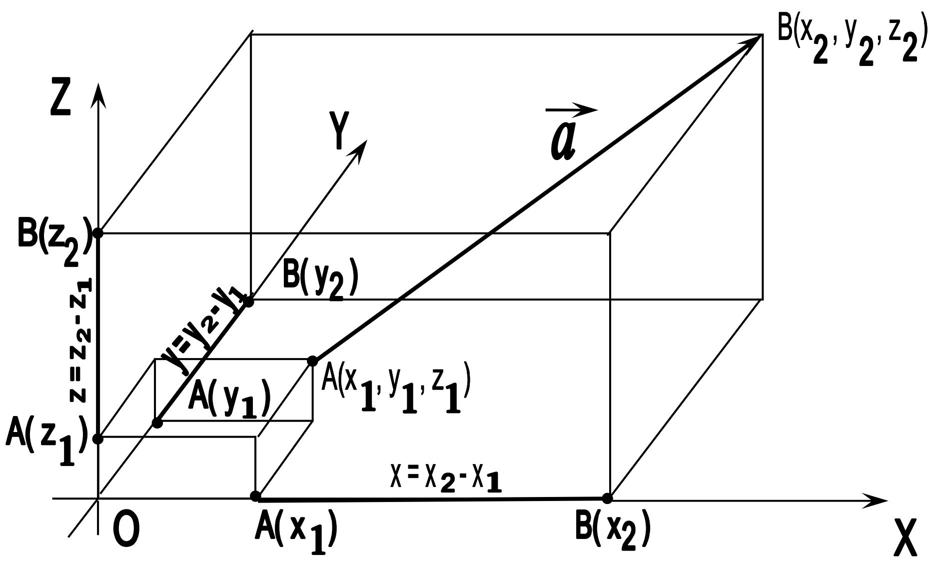


Fig. 2.4

Thus, having chosen in space the Cartesian system of coordinates, we can establish conformity with its help between any vector $\vec{a}$,set by the directed segment $\vec{AB}$ and the vector $\vec{a}$, from vector space R^3, which coordinates are determined by the ordered triple of numbers *(x, y, z)*. If the specified conformity which represents a way of a defining of coordinates by a method of mapping we designate through *f*, then

$$f : \vec{a} \rightarrow \vec{a}' = f(\vec{a}) = (x, y, z).$$

Let's show, that *f* is biunique mapping. For this purpose we shall consider the theorem of vector equality.

The theorem. Two vectors are equal only in the case when their coordinates are equal.

For the proof of this theorem, firstly we shall show, how it is possible to set a vector $\vec{a}$ by means of its length $|\vec{a}|$ and angles which it subtends with coordinate axes.

Let's consider any directed segment $\vec{AB}$ which belong to a set of the vector $\vec{a}$. We shall plot on $\vec{AB}$, as on a diagonal, a rectangular parallelepiped (fig. 2.5) with sides $AA_1 = x = x_2 - x_1 = a_x$; $AA_2 = y = y_2 - y_1 = a_y$; $AA_3 = z = z_2 - z_1 = a_z$.

It should be noticed, that all points laying on a plane, parallel to any coordinate plane, have equal coordinates of that axis to which this plane is perpendicular. If points are located on a straight line parallel to any of

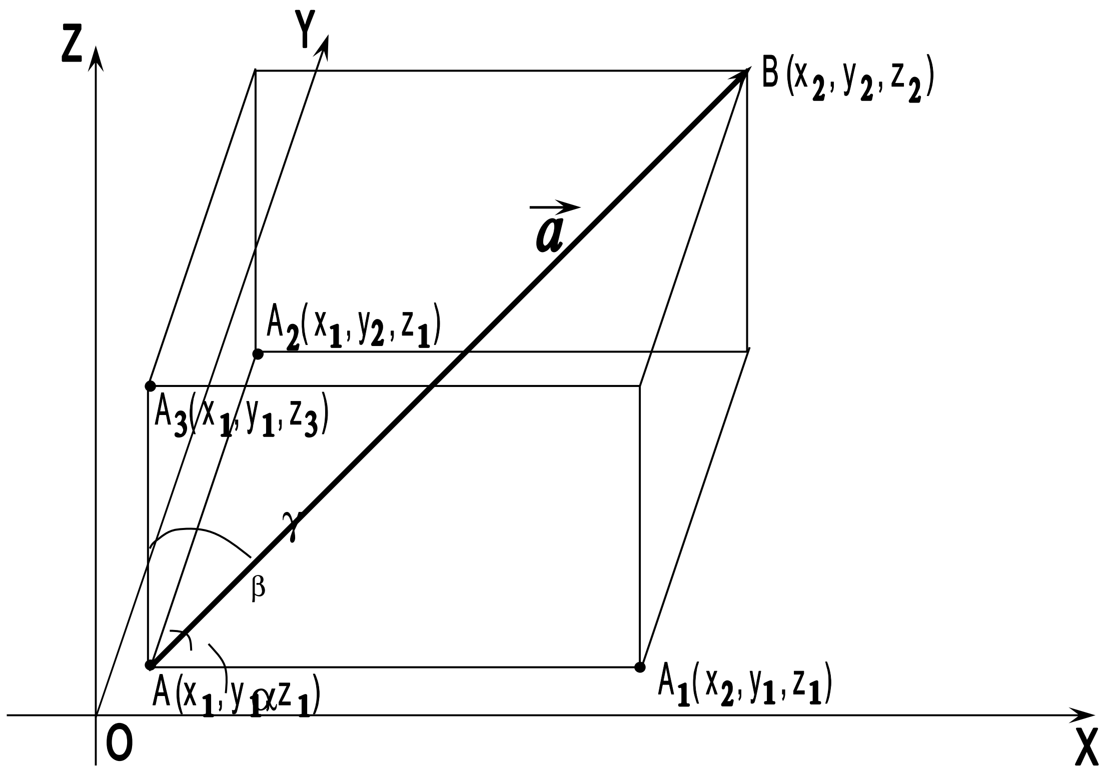


Fig. 2.5

coordinate axes, then for these points only the coordinate of that axis, which this straight line is parallel to, is changed. Two other coordinates are identical. For example, points A and A_1 (fig. 2.5) lay on a straight line parallel to an axis O_X, hence, for these points only coordinate x changes.

Now we shall designate through α, β and γ the angles which the directed segment $\vec{AB}$ subtends with axes of coordinates x, y, z accordingly or with the sides of parallelepiped AA_1, AA_2, AA_3 (fig. 2.5). From rectangular triangles AA_1B, AA_2B and AA_3B we find

$$x = x_2 - x_1 = a_x = |\vec{AB}| \cos\alpha,$$
$$y = y_2 - y_2 = a_y = |\vec{AB}| \cos\beta \qquad (4.2)$$
$$z = z_2 - z_1 = a_z = |\vec{AB}| \cos\gamma$$

where $|\vec{AB}| = |\vec{a}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} = \sqrt{x^2+y^2+z^2} = \sqrt{a_x^2+a_y^2+a_z^2}$,

$\cos\alpha$, $\cos\beta$ and $\cos\gamma$ is referred to ***direction cosine,*** and for them the ratio takes place

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (4.3)$$

Now on the basis of the obtained formulas we shall prove the theorem of vector equality. We shall consider two vectors $\vec{a}$ and $\vec{в}$ with coordinates x_1, y_1, z_1 and x_2, y_2, z_2 accordingly.

Necessity. Let's show, that if vectors are equal ($\vec{a} = \vec{в}$), also their coordinates are equal ($x_1 = x_2$; $y_1 = y_2$; $z_1 = z_2$). It follows from vector equality, that $|\vec{a}| = |\vec{в}|$, and also, that $\cos\alpha_1 = \cos\alpha_2$, $\cos\beta_1 = \cos\beta_2$, $\cos\gamma_1 = \cos\gamma_2$, since vectors are collinearand are equally directed. If vectors are collinear and are oppositely directed, $\cos\alpha_1 = -\cos\alpha_2$, $\cos\beta_1 = -\cos\beta_2$, $\cos\gamma_1 = -\cos\gamma_2$. Now it follows from formulas (4.2):

$$x_1 = |\vec{a}| \cos\alpha_1 = |\vec{в}| \cos\alpha_2 = x_2,$$
$$y_1 = |\vec{a}| \cos\beta_1 = |\vec{в}| \cos\beta_2 = y_2,$$
$$z_1 = |\vec{a}| \cos\gamma_1 = |\vec{в}| \cos\gamma_2 = z_2,$$

that was to be proved.

Sufficiency. Since coordinates of vectors $\vec{a}$ and $\vec{в}$ are equal, then

$|\vec{a}| = |\vec{в}|$ и $\cos\alpha_1 = \cos\alpha_2$, $\cos\beta_1 = \cos\beta_2$, $\cos\gamma_1 = \cos\gamma_2$.

The second condition means, that vectors $\vec{a}$ and $\vec{b}$ are collinear and directed to one direction, and taking into account $|\vec{a}| = |\vec{b}|$ such a vector are considered to be equal, i.e. $\vec{a} = \vec{b}$.

Theorem of vector equality implyies, that mapping $\vec{a} \to \vec{a}' = (x, y, z)$ is biunique. Really, each vector $\vec{a}$ from the vector space of free vectors can be put in conformity with unique vector $\vec{a}' = (x, y, z)$ from the vector space R^3 and on the contrary, each ordered triple of numbers (x, y, z), i.e. a vector from R^3, can be put in conformity with unique vector $\vec{a}$ from vector space of free vectors. For construction of this vector it is sufficient to construct a radius - vector of the point $B (x, y, z)$ in the chosen system of coordinates. Then the set of all directed segments equal to the directed segment $\vec{OB}$ is the vector $\vec{a}$ with coordinates x, y, z. It should be noticed that this conformity depends on a choice of system of coordinates.

If the vector $\vec{a}$ is located in one of coordinate planes, then one of coordinates is equal to zero, for example, if this plane is $_XO_Y$, the coordinate $z = 0$. Such vector can be represented by the directed segment laying in any of the planes, which is parallel to the plane $_XO_Y$ In this case each vector $\vec{a}$located in a coordinate plane can be put in conformity with the ordered couple of numbers (x, y), representing a vector from vector space R^2 and this conformity is biunique.

If the vector $\vec{a}$ is located on one of coordinate axes, then other two coordinates are equal to zero and thus each vector $\vec{a}$ located on a coordinate axis can be put in conformity with a vector which has coordinate x from the vector space R^1 and this conformity is biunique. Such vector can be represented by the directed segment located on any straight line, which is parallel to the corresponding coordinate axis.

Let's show now, that operations of free vectors addition and their multiplication by the number from field R are in full conformity with similar operations on the vectors from R^3, i.e. relating to the given operations these spaces are isomorphic. We list these operations without the proof since all of them are proved in the course of high school.

The sum of free vectors. Coordinates of the sum of two free vectors are equal to the sums of corresponding summand coordinates.

On a coordinate axis: $\vec{a}(x_1)$ and $\vec{b}(x_2)$;

$$\vec{a}(x_1) + \vec{в}(x_2) = \vec{c}(x_1 + x_2).$$

On a coordinate plane: $\vec{a}(x_1, y_1)$ and $\vec{в}(x_2, y_2)$;

$$\vec{a}(x_1, y_1) + \vec{в}(x_2, y_2) = \vec{c}(x_1 + x_2, y_1 + y_2).$$

In space : $\vec{a}(x_1, y_1, z_1)$ and $\vec{в}$ (x_2, y_2, z_2);

$\vec{a}(x_1, y_1, z_1) + \vec{в}$ $(x_2, y_2, z_2) = \vec{c}$ $(x_1 + x_2 , y_1 + y_2, z_1 + z_2)$ – conformity (see the formula (4.1)).

Multiplication of a free vector by a number from field R. Coordinates of the product $\lambda\vec{a}$ of the vector $\vec{a}(x, y, z)$ and the number λ are equal to the products of this number and corresponding coordinates of the vector $\vec{a}$.

$\lambda\vec{a}(\lambda x, \lambda y, \lambda z)$ - conformity (see the formula (4.1)).

Corollary fact. For two vectors $\vec{a}(x_1, y_1, z_1)$ and $\vec{в}(x_2, y_2, z_2)$ to be collinear, i.e. $\vec{в} = \lambda\vec{a}$, it is necessary and sufficient that corresponding coordinates of the vectors to be proportional: $\frac{x_2}{x_1} = \frac{y_2}{y_1} = \frac{z_2}{z_1} = \lambda$.

In addition to these two operations we shall introduce one more operation on free vectors which you met in a course of high school but which sense we shall consider later.

3.4. Scalar product of two free vectors

Definition. Scalar product $\vec{a}\cdot\vec{в}$ of two free vectors $\vec{a}$ and $\vec{в}$, if these vectors are not equal to zero, is referred to as a number which is equal to the product of their magnitudes and cosine of angle between them

$$\vec{a}\cdot\vec{в} = |\vec{a}|\cdot|\vec{в}|\cos\varphi \qquad (4.4)$$

If $\vec{a} = \vec{0}$ or $\vec{в} = \vec{0}$ (or $\vec{a} = \vec{в} = \vec{0}$), the scalar product $\vec{a}\cdot\vec{в}$, by definition, is considered to be equal to zero.

Corollary facts

1. If two vectors are perpendicular, their scalar product is equal to zero.

2. Scalar product of two vectors is expressed by the maximal number if vectors are collinear and have the identical direction (($\varphi = 0$), and it is expressed by the minimal number, if they are collinear, but oppositely directed ($\varphi = \pi$).

3. Scalar product of the vector $\vec{a}$ and $\vec{a}$ is equal to the square of the vector magnitude $\vec{a}$: $\vec{a}\cdot\vec{a} = |\vec{a}|^2$, hence $|\vec{a}| = \sqrt{\vec{a}\cdot\vec{a}}$.

Scalar product of two vectors set by their coordinates is equal to the sum of products of their corresponding coordinates:

$\vec{a}\,(x_1, y_1, z_1)$ $\vec{в}\,(x_2, y_2, z_2)$; $\vec{a}\cdot\vec{в} = x_1\cdot x_2 + y_1\cdot y_2 + z_1\cdot z_2$ (4.5)

EXERCISES

1. With the given vectors $\vec{a}$ and $\vec{в}$, construct the vectors $2\vec{a} - \vec{в}$ and $\vec{в} - \vec{a}/2$.

2. Define, at what values α and β , the vectors $\vec{a}(2, \alpha, 1)$ and $\vec{в}(3, -6, \beta)$ are collinear.

3. Ascertain that points *A(*3, -1, 2*)*, *B(*1, 2, -1*)*, *C(*-1,1,-3*)*, *D(*3,-5, 3*)* serve as vertexes of a trapeze.

4. The vector $\vec{a}$ makes with axes of coordinates the acute angles *α, β, γ*, and, $\alpha = 45^o$, $\beta = 60^o$. Determine its coordinates, if $|\vec{a}| = 3$.

5. Determine the direction cosines of the direction *L*, set by the directed segment $\vec{AB}$, where *A(*1,0,-1*)* and *B(*3,1,-3*)*.

6. Define, whether points *A,B,C,D* lie in one plane:

a) *A(*1,2,3*)*, *B(*7,3,2*)*,*C(*-3,0,6*)* and *D(*9,2,4*);*

*b) A(*1,1,3*)*, *B(*5,3,2*)*,*C(*-3,0,6*)* and *D(*9,2,4*);*

*c) A(*1,2,3*)*, *B(*-2,1,1*)*,*C(*-1,3,2*)* and *D(*3,-4,3.*)*

7. The height lowered from the vertex *A* of the triangle *ABC*, divides the opposite side in the ratio 3:1. Define the coordinates of top *A* if *B(*-1,1*)*, *C(*3, 5*)*, the length of height is equal to 2.

8. Vectors $\vec{a}(1,-1,2)$ и $\vec{в}(2,-2,1)$ are given. Define a projection of the vector $\vec{c} = 3\vec{a} - \vec{в}$ onto the direction of a vector $\vec{в}$.

§ 4. VECTOR SUBSPACE

Definition. Let there be a vector space *K* above the field *P*, and let *G* be a subset *K* which with laws induced from *K*, makes a vector space above P; then G is referred to as ***a vector subspace*** of the space K or ***a linear variety*** in K.

It follows from the definition that the sum of any vectors from G is a vector belonging to the same set G, and product of a number from the field P and a vector from G, belongs to same set G.

Thus, every vector subspace is a vector space in itself and on the contrary, any vector space can be considered as a vector subspace. For example, R^m is a vector subspace of the space R^n for any $m<n$ and, in its turn, R^n is a subspace R^{n+1}.

4.1. Subspace generated by the linear combination of vectors

Definition. Let any system m of the vectors $\vec{a}_1, \vec{a}_2, ...\vec{a}_m$ be given which belong to vector space K above the field P. Let's multiply each vector $\vec{a}_i \in K$ by the number $\lambda_i \in P$, $i = 1,2,...,m$ and add the results. The obtained expression

$$\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + ... + \lambda_m\vec{a}_m = \sum_{i=1}^{m} \lambda_i \vec{a}_i$$

is referred to as ***a linear combination*** of vectors with coefficients $\lambda_1, \lambda_2, ...\lambda_m$.

As the coefficients $\lambda_1, \lambda_2, ...\lambda_m$ are numbers from field P, which are picked out arbitrarily (there may be also zeros among λ), then the linear combinations formed by system of vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ will be an infinite set. Each linear combination of vectors determines a certain vector

$$\vec{в} = \lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + ... + \lambda_m\vec{a}_m \qquad (4.6)$$

which belongs to the same vector space K. Such vector $\vec{в}$ is referred to as ***a linear combination of the given vectors*** or also we can say, that the vector $\vec{в}$ is separated into vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$, and that infinite set G which is formed by these of vectors, will be the vector subspace of the space K. This subspace is referred to as ***a linear hull*** of the system of vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$, or subspace, ***generated by a linear combination*** of vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ from K.

Really, let

$\vec{в}_1 = \lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + ... + \lambda_m\vec{a}_m$ and $\vec{в}_2 = \mu_1\vec{a}_1 + \mu_2\vec{a}_2 + ... + \mu_m\vec{a}_m$ be two any vectors from G. We have,

$$\vec{в}_1 + \vec{в}_2 = (\lambda_1 + \mu_1)\vec{a}_1 + (\lambda_2 + \mu_2)\vec{a}_2 + ... + (\lambda_m + \mu_m)\vec{a}_m \in G,$$

a neutral element $\vec{0} = 0\vec{a}_1 + 0\vec{a}_2 + ... + 0\vec{a}_m \in G$,

a symmetric element $-\vec{в}_1 = (-\lambda_1)\vec{a}_1 + (-\lambda_2)\vec{a}_2 + ... + (-\lambda_m)\vec{a}_m \in G$.

On the other part, for any $\beta \in P$ we have,

$$\beta\,\vec{в}_1 = (\beta\,\lambda_1)\vec{a}_1 + (\beta\,\lambda_2)\vec{a}_2 + ... + (\beta\,\lambda_m)\vec{a}_m \in G,$$

hence, $G \subset K$ has properties of a vector space and consequently it is a vector subspace of the space K.

We shall consider now the basic properties of vector system and a subspace generated by them.

4.2. Linear dependence and independence of vectors

Definition 1. The system of vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ from K (where m – is finite) is referred to as ***linearly dependent***, and the vectors are referred to as ***linearly dependent*** if there will be even one set λ_1, λ_2, . . ., λ_m, of such numbers in field P, but not all these numbers are equal to zero, that

$$\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + ... + \lambda_m\vec{a}_m = \vec{0} \quad \left(\sum_{i=1}^{m} \lambda_i\vec{a}_i = 0\right) \qquad (4.7).$$

Definition 2. The system of vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m \in K$ is referred to as ***linearly independent***, and the vectors are referred to as ***linearly independent*** if the linear combination from these vectors $\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \ldots + \lambda_m\vec{a}_m$ is equal to zero vector $\vec{0}$ $\left(\sum_{i=1}^{m} \lambda_i\vec{a}_i = \vec{0}\right)$ only in that case , if $\lambda_1 = \lambda_2 = ... = \lambda_m = 0$.

The remark. One vector $\vec{a} \in K$ is linearly independent, if $\vec{a} \neq \vec{0}$, and on the contrary, the vector $\vec{0} \in K$ - is linearly dependent.

We shall give to presentation of linear dependence and independence of vectors. We shall consider system of free vectors.

The theorem 1. For two free vectors $\vec{a}_1$ and $\vec{a}_2$ to be linearly dependent, it is necessary and sufficient that they are collinear.

The proof. *Necessity*. Vectors $\vec{a}_1$ and $\vec{a}_2$ are linearly dependent. Hence $\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 = \vec{0}$, where λ_1 and λ_2 are not equal to zero at the same time. Let, for example, $\lambda_1 \neq 0$, then $\vec{a}_1 = -\dfrac{\lambda_2}{\lambda_1}\vec{a}_2$; this implyies that $\vec{a}_1$ and $\vec{a}_2$ are collinear.

Sufficiency. Vectors $\vec{a}_1$ and $\vec{a}_2$ are collinear. Hence $\vec{a}_1 = \lambda\vec{a}_2$, from here, $1 \cdot \vec{a}_1 - \lambda\vec{a}_2 = \vec{0}$ but since λ_1 = 1 ≠ 0, means vectors $\vec{a}_1$ and $\vec{a}_2$ are linearly dependent.

The remark. If two vectors are linearly independent, they are not collinear and vice versa.

The theorem 2. For three free vectors $\vec{a}_1, \vec{a}_2$ and $\vec{a}_3$ to be linearly dependent, it is necessary and sufficient that they are coplanar.

The proof of this theorem (See Book 2, Chapter 4, § 3, item 3.2.)

The remark. If three vectors are linearly independent, they are not coplanar. The converse proposition is also fair.

4.3. Theorems about linearly dependent and linearly independent vectors

The theorem 1. If the system of vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m \in K$ is linearly dependent, then after adding to it any number of new vectors from *K*, we will have again a linearly dependent system.

The proof. It follows from equality

$$\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + ... + \lambda_m\vec{a}_m + \lambda_{m+1}\vec{a}_{m+1} + ... + \lambda_{m+k}\vec{a}_{m+k} = \vec{0},$$

in which among $\lambda_1, \lambda_2, ..., \lambda_m$ are nonzero, but all $\lambda_{m+1}, \lambda_{m+2}, ..., \lambda_{m+k}$ are equal to zero.

Let the system of vectors be set, $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ from *K*. Any part of this vector system we shall name ***its subsystem***. Then the theorem 1 can be formulated as follows.

If any subsystem of the given vector system is linearly dependent, also the system is linearly dependent.

For system of linearly independent vectors the following statement is fair.

If the system consists of linearly independent vectors its any subsystem also consists of linearly independent vectors.

Corollary facts

a) If there is a vector $\vec{0}$ in the set $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$, then the set $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ is linearly dependent; it is equivalent to the statement, that if the set is linearly independent, then each vector $\neq \vec{0}$.

b) If there are two proportional vectors in some set, for example, $\vec{a}_i = \mu \vec{a}_j$ where $\mu \in P$, then the set is linearly dependent, since those is the partial set $\vec{a}_i, \vec{a}_j$; really, $\mu \vec{a}_j + (-1)\vec{a}_i = 0$, and $\lambda_i = -1 \neq 0$.

The theorem 2. The system of vectors $\vec{a}_1, \vec{a}_2, ... \vec{a}_m$ from K will be linearly dependent in only a case when one of these vectors can be presented as a linear combination of other vectors of this system.

The proof. *Necessity*. Let $\vec{a}_1, \vec{a}_2, ... \vec{a}_m$ - be a linearly dependent system of vectors. Then there will be a set of numbers $\mu_1, \mu_2, ..., \mu_m \in P$, which not all are equal to zero, and such, that $\mu_1 \vec{a}_1 + ... + \mu_m \vec{a}_m = \vec{0}$. Let's assume for definiteness, that $\mu_i \neq 0$, then

$$\vec{a}_i = \left(-\frac{\mu_1}{\mu_i}\right)\vec{a}_1 + \left(-\frac{\mu_2}{\mu_i}\right)\vec{a}_2 + ... + \left(-\frac{\mu_{i-1}}{\mu_i}\right)\vec{a}_{i-1} + \left(-\frac{\mu_{i+1}}{\mu_i}\right)\vec{a}_{i+1} + ... + \left(-\frac{\mu_m}{\mu_i}\right)\vec{a}_m$$

or

$\vec{a}_i = \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + ... + \lambda_m \vec{a}_m$, where $\lambda_j = -\dfrac{\mu_j}{\mu_i}$, j = 1,2,....,m, and j ≠ i.

Sufficiency

$\vec{a}_i = \lambda_i \vec{a}_1 + \lambda_2 \vec{a}_2 + ... + \lambda_{i-1} \vec{a}_{i-1} + \lambda_{i+1} \vec{a}_{i+1} + ... + \lambda_m \vec{a}_m$ - is a linear combination. Let's multiply this equality by (-1) and let's subtract from both parts a vector (-1)$\vec{a}_i$, we shall obtain

$$\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + ... + (-1)\vec{a}_i + ... + \lambda_m \vec{a}_m = \vec{0}.$$

For coefficients we have non-trivial combination $\lambda_i = -1 \neq 0$, hence, the system is linearly dependent.

4.4. Base and rank of vector system. Basis and dimension of

vector subspace, generated by vector system

Definition 1. In any system of vectors $\vec{a}_1, \vec{a}_2, ..., a_m$ from *К*, containing nonzero vectors, always it is possible to choose a subsystem $\vec{a}_1, \vec{a}_2, ..., a_r$, where $r \le m$, which consists of the maximal number of linearly independent vectors so, that adding of any vector from this system to the specified subsystem makes it linearly dependent; really, since there is a non-vanishing vector in system, and it is always linearly independent, then $r \ge 1$. Such subsystem of linearly independent vectors is referred to as ***base*** of initial system, and number *r* of vectors in base - ***a rank*** of this vector system.

The remark. The base of system is defined ambiguously, but number of vectors in base (rank) is always equal. For example, one of three vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$, is linearly dependent, it is possible to construct three bases of two vectors: $\vec{a}_1, \vec{a}_3;\ \vec{a}_2, \vec{a}_3;\ \vec{a}_1, \vec{a}_2$.

Properties of a base

1. All vectors of system can be presented as a linear combination of vectors of a base. (see the previous item 4.3, the theorem 2).

2. Any vector of subspace, generated by vector system, can be presented as a linear combination only the vectors forming its base and this decomposition it is unique.

The proof. Let *G* – be a subspace, generated by vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$ and let $\vec{a}_1, \vec{a}_2, ..., a_r$ $r < m$ (for $r = m$ the statement is obvious) be a base of system $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$. Then the rest of system vectors $\vec{a}_{r+1}, ..., \vec{a}_m$ can be presented as a linear combination of base vectors

$$\vec{a}_{r+2} = \beta_{21}\vec{a}_1 + \beta_{22}\vec{a}_2 + ... + \beta_{2r}\vec{a}_r, \qquad (4.8)$$

$$\vec{a}_m = \beta_{(m-r)1}\vec{a}_1 + \beta_{(m-r)2}\vec{a}_2 + ... + \beta_{(m-r)r}\vec{a}_r.$$

Now let's consider any vector $\vec{в} \in G$:

$$\vec{в} = \lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + ... + \lambda_r\vec{a}_r + \lambda_{r+1}\vec{a}_{r+1} + ... + \lambda_m\vec{a}_m.$$

Substituting this equality of a vector $\vec{a}_{r+1}, ..., \vec{a}_m$ for (4.8), we obtain

$$\vec{в} = \left[\lambda_1 + \lambda_{r+1}\left(\beta_{11} + ... + \beta_{(m-r)1}\right)\right]\vec{a}_1 + ... + \left[\lambda_r + \lambda_m\left(\beta_{1r} + ... + \beta_{(m-r)}\right)\right]\vec{a}_r$$

or $\vec{в} = \mu_1\vec{a}_1 + \mu_2\vec{a}_2 + ... + \mu_r\vec{a}_r$.

Definition 2. For vector subspace, generated by system of vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_m$, the base of this vector system is referred to as ***a basis***, and the rank of vector system is referred to as ***dimension*** of its subspace.

As a striking example we shall consider a subspace, generated by system of free vectors.

4.5. Basis and dimension of vector subspace, generated by system of free vectors

We shall consider a subspace which element is the linear combination of three free vectors $\lambda_1\vec{a}_1 + \lambda_2\vec{a}_2 + \lambda_3\vec{a}_3$. Let's assume, that this vector system is linearly dependent. The case of linearly independent vectors will be described further. We have already determined, that if the linear combination of three free vectors is linearly dependent, it means, that these of vectors are coplanar, i.e. there is a plane which they are parallel to. Obviously, that also any vector $\vec{b}$ will be coplanar, which is a linear combination of these vectors. Therefore a subspace, generated by system of such three linearly dependent vectors, represents a set of all vectors, which are coplanar to the given ones. Such system of vectors is represented by the directed segments laying in one plane, or in planes parallel to it. Further, since the system of three vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ is linearly dependent, then one of these vectors is a linear combination of two other vectors. Let this vector be $\vec{a}_3 : \vec{a}_3 = \beta_1\vec{a}_1 + \beta_2\vec{a}_2$, where $\beta_1 = -\dfrac{\lambda_1}{\lambda_3}, \beta_2 = -\dfrac{\lambda_2}{\lambda_3}$. Let's consider a condition when the rest of vectors are linearly independent, i.e. it means, that they are not collinear. Then these two ordered vectors will make basis of subspace of coplanar vectors and dimension of its subspace is equal to two. ***Hence, the basis of two-dimensional subspace of coplanar free vectors represents any two ordered noncollinear vectors.*** Usually as basic vectors of two-dimensional space we choose vectors which are represented by the directed segments which are parallel to coordinate axes O_X and O_Y on a plane and which are by absolute value to a scale segment of coordinate axes. The first vector directed parallel to an axis O_X is designated as $\vec{i}$: its coordinates are (1,0), and the second vector directed parallel to axis *OY* is designated $\vec{j}$: its coordinates are (0,1). The choice of such basis is caused by that if we represent any vector $\vec{e}$ with coordinates (x, y) of two-dimensional subspace through basis the $\vec{i}, \vec{j}$, in this case coefficients of a linear combination of basic vectors will be coordinates x and y of the vector $\vec{e}$, i.e. $\vec{e} = x\vec{i} + y\vec{j}$ $\vec{e}$, and as we already saw, this decomposition is unique.

Now we shall consider a case, when vectors $\vec{a}_1$ and $\vec{a}_2$, (one of which is not equal $\vec{0}$) are collinear, i.e. they are linearly dependent (or $\vec{a}_2 = \gamma \vec{a}_1$). Naturally any vector being a linear combination of these vectors will be collinear to them. Therefore a subspace, generated by system of vectors only one of which is linearly independent, (it is the vector which is not equal to $\vec{0}$) represents a set of collinear vectors. The basis of such subspace consists of one nonzero vector, and dimension of such subspace is equal to one. One-dimensional subspace is represented by a set of the directed segments located on one straight line or on straight lines parallel to it.

Now we shall generalize a concept of basis for a set of the vectors forming all the vector space *K*.

§ 5. BASIS AND DIMENSION OF VECTOR SPACE

Definition. Let *K* – be a vector space above the field *P;* let's assume, that there is a finite number *n* of such linearly independent vectors $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$ in this space, that every vector one $\vec{a}$ from *K* $\vec{a}$ linearly depends on $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$. Then we shall speak, that the set $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$ forms ***a basis*** of the space *K* and, that the vector space *K* has finite ***dimension*** *n*, and it is written down as *dimK = n.*

The remark. There are the vector spaces which do not have finite dimension; it is said, that they have infinite dimension; there are arbitrarily big sets of linearly independent vectors in such vector spaces. For example, vector space of multinomials. Consideration of such spaces is beyond the course of linear algebra.

There is no basis also in zero space as the system consisting of one zero vector, is linearly dependent. Dimension of zero space is not determined and it is considered to be equal to zero.

Corollary facts from definition.

1. In n – dimensional vector space K the set consisting of more than n-vectors is always linearly dependent.

2. If K has some bases, these bases contain identical number of vectors, and this number is equal to dimension K; hence, $dimK$ does not depend on a choice of basis. Really, if K has the basis which is distinct from $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$, the last will have n' vectors, and $n' \le n$. Just as in K no more than n' linearly independent vectors can exist, and so $n \le n'$, and, hence, $n = n'$.

5.1. Basis construction

Let there be n – dimensional vector space K, i.e. there is even one basis of n vectors in it. We shall choose in K an any vector $\vec{a}_1 \ne \vec{0}$. If K does not contain the vectors linearly independent on $\vec{a}_1$, then for any vector $\vec{a} \in K$ we have $\vec{a} + (-\lambda)\vec{a}_1 = \vec{0}$ or $\vec{a} = \lambda\vec{a}_1$ and $\vec{a}_1$ forms a basis of the space K which has dimension 1. Let's assume, that dimension n> 1. We shall designate through $\vec{a}_2$ vector from K which is linearly independent on $\vec{a}_1$.

Let's suppose that in such way linearly independent vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_r$ are gradually obtained. If $r < n$, then K contains the vectors linearly independent on $\vec{a}_1, \vec{a}_2, ..., \vec{a}_r$, otherwise these vectors would from the basis K, containing $r < n = dimK$ vectors what is impossible. So, there will be such vector $\vec{a}_{r+1} \in K$, that $\vec{a}_1, \vec{a}_2, ..., \vec{a}_r, \vec{a}_{r+1}$ are linearly independent. This way we can obtain n linearly independent vectors which will form a basis of the space K. That fact, that vectors for construction of basis have been chosen arbitrarily, proves that always there is an infinite set of various bases of space K (but all of them contain identical number of vectors $n = dimK$). Thus we can consider to be proved also the theorem of incomplete basis and a lemma of replacement.

The theorem of incomplete basis. Any linearly independent set of vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_r \in K$ where $r < n = dimK$ always can be added with $n - r$

other vectors from *K* so that the obtained system *n* of vectors forms a basis of the space *K*.

Lemma about replacement. Let $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$ be a basis of the space *K*. Then any vector $\vec{\ell}_i, i = 1,2,...., n$ from this basis can be replaced with other vector $\vec{a}$ from *K*, which is not a linear combination of other vectors in basis:

$$\vec{a} \neq \lambda_1\vec{\ell}_1 + ... + \lambda_{i-1}\vec{\ell}_{i-1} + \lambda_{i+1}\vec{\ell}_{i+1} + ... + \lambda_n\vec{\ell}_n.$$

Then $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_{i-1}, \vec{a}, \vec{\ell}_{i+1}, ..., \ell_n$ – is a basis *K*.

5.2. The basic properties of basis

Let $\vec{a}$ be any vector from *K* of the dimension n; since it linearly depends on the basis $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$, then there will be such numbers $\lambda, \lambda_1, . . ., \lambda_n$ which are not all equal to zero in *P*, that $\lambda\vec{a} + \lambda_1\vec{\ell}_1 + \lambda_2\vec{\ell}_2 + ... + \lambda_n\vec{\ell}_n = \vec{0}$. And $\lambda \neq 0$, as otherwise $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$ would be linearly dependent. As *P* is a field, then $\frac{1}{\lambda} \in P$ exists. After multiplication by $\frac{1}{\lambda}$ we shall obtain: $\vec{a} = \beta_1\vec{\ell}_1 + \beta_2\vec{\ell}_2 + ... + \beta_n\vec{\ell}_n$, where $\beta_i = -\frac{\lambda_i}{\lambda}$, *i=1,2,...,n.*

Thus, the vector space *K* is generated by the basis $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$, and the given expression is referred to as ***decomposition*** of the vector $\vec{a}$ in terms of the basis $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$. Numbers $\beta_1, \beta_2, ..., \beta_n$ are referred to as ***components (coordinates)*** of the vector $\vec{a}$ in basis $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$.

The theorem. (***Basic property of a basis***) Representation of any vector $\vec{a}$ from the space *K* through its basis $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$ is unique, or in other words, in the set basis the vector components are defined unequivocally.

The proof. Let's assume, that the theorem is not true and the vector $\vec{a}$ in the basis $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$ has various components $\vec{a} = \lambda_1\vec{\ell}_1 + \lambda_2\vec{\ell}_2 + ... + \lambda_n\vec{\ell}_n$ and $\vec{a} = \beta_1\vec{\ell}_1 + \beta_2\vec{\ell}_2 + ... + \beta_n\vec{\ell}_n$. Then subtracting these equalities, we shall obtain

$\vec{0} = (\lambda_1 - \beta_1)\vec{\ell}_1 + (\lambda_2 - \beta_2)\vec{\ell}_2 + ... + (\lambda_n - \beta_n)\vec{\ell}_n$. As vectors $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$ are linearly independent, then $(\lambda_1 - \beta_1) = 0, ..., (\lambda_n - \beta_n) = 0$ and hence $\lambda_1 = \beta_1, \lambda_2 = \beta_2, ..., \lambda_n = \beta_n$.

The remark. The same vector in various bases has different components.

As a striking example we shall consider the space of free vectors.

5.3. Basis and dimension of free vector space

Let's choose the system consisting of three ordered free vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. The case when this vector system is linearly dependent, is already considered by us in the previous paragraph, item 4.5. Now we shall consider a condition when the system of three vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ is linearly independent, i.e. it is the ordered triple of noncoplanar vectors.

The theorem. Adding of any free vector $\vec{a}$ to the system of three noncoplanar vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ makes it linearly dependent, or in other words: any free vector $\vec{a}$ is a linear combination of three ordered noncoplanar vectors and this representation is unique. Thus we shall establish, that set of three ordered vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ is basis of free vector space and his dimension is equal to three.

The proof. We lay off all vectors $\vec{a}, \vec{a}_1, \vec{a}_2, \vec{a}_3$ from the same point A: $\overrightarrow{AB_1} = \vec{a}_1$; $\overrightarrow{AB_2} = \vec{a}_2$; $\overrightarrow{AB_3} = \vec{a}_3$; $\overrightarrow{AB_4} = \vec{a}$. Let F –be a projection of the point B_4 onto the plane AB_1B_2 parallel to the straight line AB_3 and Q - a projection of the point F onto the straight line AB_1 parallel to the straight line AB_2. Then $\vec{a} = \overrightarrow{AB_4} = \overrightarrow{AQ} + \overrightarrow{QF} + \overrightarrow{FB_4}$. Vectors $\overrightarrow{AQ}, \overrightarrow{QF}, \overrightarrow{FB_4}$ are collinear to vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$. accordingly. If we suppose $\frac{\overrightarrow{AQ}}{\vec{a}_1} = \lambda_1, \frac{\overrightarrow{QF}}{\vec{a}_2} = \lambda_2, \frac{\overrightarrow{FB_4}}{\vec{a}_3} = \lambda_3,$ we shall obtain $\overrightarrow{AQ} = \lambda_1 \vec{a}_1, \overrightarrow{QF} = \lambda_2 \vec{a}_2, \overrightarrow{FB_4} = \lambda_3 \vec{a}_3$ and, hence $\vec{a} = \lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + \lambda_3 \vec{a}_3$, i.e. vectors $\vec{a}$, $\vec{a}_1, \vec{a}_2, \vec{a}_3$ are linearly dependent.

Thus, ***the basis of free vector space consists of three ordered noncoplanar vectors***. If as basic vectors we choose three ordered vectors which are represented by the directed segments parallel accordingly to three axes of rectangular Cartesian system of coordinates *x*, *y*, *z* and the absolute value of each vector is equal to a scale segment of these axes such basis is referred to as ***orthonormal*** basis. First two basic vectors, as well as on a plane, are designated $\vec{i}, \vec{j}$, and the third basic vector parallel to the axis O_Z, , is designated $\vec{k}$, , and these vectors are referred to as a vector ***orts***. Coordinates of these vectors will be: $\vec{i}=(1,0,0)$, $\vec{j}=(0,1,0)$, $\vec{k}=(0,0,1)$. Such choice of basic vectors is caused that in decomposition of any vector $\vec{a}(x,y,z)$on orthonormal basis $\vec{i}, \vec{j}$, $\vec{k}$ coefficients of decomposition are coordinates *x*, *y*, *z* of the vector $\vec{a}$: $\vec{a} = x\vec{i}+y\vec{j}+z\vec{k}$.

Let's consider expression of scalar product of two vectors $\vec{a}$ and $\vec{в}$, which are located on orthonormal basis, i.e. $\vec{a}=x_1\vec{i}+y_1\vec{j}+z_1\vec{k}$ и $\vec{в}=x_2\vec{i}+y_2\vec{j}+z_2\vec{k}$. Then

$$\vec{a}\cdot\vec{в}=\left(x_1\vec{i}+y_1\vec{j}+z_1\vec{k}\right)\left(x_2\vec{i}+y_2\vec{j}+z_2\vec{k}\right)=$$

$$=x_1x_2\vec{i}^{\,2}+y_1y_2\vec{j}^{\,2}+z_1z_2\vec{k}^{\,2}+(x_1y_2)\vec{i}\vec{j}+\left(y_1z_2+y_2\ \ z_1\right)\vec{j}\vec{k}+(z_1x_2+z_2x_1)\vec{k}\vec{i}\ .$$ but

since $\vec{i}, \vec{j}, \vec{k}$ are mutually perpendicular (orthogonal) vectors and the modulus of them is equal to one, then

$\vec{i}^{\,2}=\vec{j}^{\,2}=\vec{k}^{\,2}=1;\quad \vec{i}\cdot\vec{j}=\vec{i}\cdot\vec{k}=\vec{j}\cdot\vec{k}=0$, then $\vec{a}\cdot\vec{в} = x_1x_2 + y_1y_2 + z_1z_2$.

Thus, scalar product of two vectors is equal to sum of products their corresponding coordinates in the coordinates only in that case if vectors are set by their coordinates in orthonormal basis.

§ 6. ISOMORPHISM BETWEEN *n*–DIMENSIONAL VECTOR SPACES *K* AND *P*ⁿ ABOVE FIELD *P*

Let *K* - be a vector space of finite dimension *n* above the field *P*. And let $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$ - basis of this space. Let's consider vector space $P^n=\prod_{i=1}^{n}P_i$;which is product *n* of vector spaces *P* above field P. Let's put a vector $\vec{x}'=(\lambda_1,\lambda_2,...,\lambda_n)$from P^n in conformity with vector $\vec{x}=\lambda_1\vec{\ell}_1+\lambda_2\vec{\ell}_2+....+\lambda_n\vec{\ell}_n$ from *K* This mapping $f:\vec{x}\to\vec{x}'$is biunique mapping since decomposition of a vector on basis is possible only in the unique way. Let further $\vec{y} \in K$ and $\vec{y} = \beta_1 \vec{\ell}_1 + \beta_2 \vec{\ell}_2 + ... + \beta_n \vec{\ell}_n$.

Let's put in conformity the vector $\vec{y}' = (\beta_1, \beta_2, ..., \beta_n)$ from P^n with a vector $\vec{y} \in K$. Since

$$\vec{x} + \vec{y} = (\lambda_1 + \beta_1)\vec{\ell}_1 + (\lambda_2 + \beta_2)\vec{\ell}_2 + ... + (\lambda_n + \beta_n)\vec{\ell}_n,$$

that is clear, that the vector $\vec{x} + \vec{y}$ corresponds the vector $\vec{x}' + \vec{y}'$ from P^n, hence

$$f(\vec{x} + \vec{y}) = f(\vec{x}) + f(\vec{y}).$$

Further, since $\alpha \vec{x} = \alpha \lambda_1 \vec{\ell}_1 + \alpha \lambda_2 \vec{\ell}_2 + ... + \alpha \lambda_n \vec{\ell}_n$, then the vector $\alpha \vec{x}$ corresponds to the vector $\alpha \vec{x}'$ from P^n, hence, $f(\alpha \vec{x}) = \alpha f(\vec{x})$. Thus (see book 2, Chapter.1, §3), it is possible to make the following conclusion.

The vector space K of finite dimension n above the field P is isomorphic to P^n. Isomorphism between K and P^n depends on basis chosen in K, and spaces only of identical dimension can be isomorphic.

Images of vectors of basis $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$ in P^n will be $\vec{\ell}'_1 = (1,0,0,...,0)$, $\vec{\ell}'_2 = (0,1,0...,0)...$, $\vec{\ell}'_n = (0,0,...,1)$ or $\vec{\ell}'_i = (\delta_{1i}, \delta_{2i}, ..., \delta_{ni})$, where $\delta_{ij} = 0$, if $i \neq j$, and $\delta_{ii} = 1$; values δ_{ij} are referred to as ***Kronecker symbols.*** Really, since $\vec{\ell}_i = \delta_{1i}\vec{\ell}_1 + \delta_{2i}\vec{\ell}_2 + ... + \delta_{ni}\vec{\ell}_n$, then $\vec{\ell}'_i = (\delta_{1i}, \delta_{2i}, ..., \delta_{ni})$, where i = 1,2,...n.

From the mentioned above it follows, that for vectors $\vec{x}_1, \vec{x}_2, ..., \vec{x}_r$ from K to be linearly independent, it is necessary and sufficient that vectors $\vec{x}'_1, \vec{x}'_2, ..., \vec{x}'_r$ from P^n, has this property, which are correspondent to them in the case of the above-stated isomorphism. In particular we shall show, that vectors $\vec{\ell}'_1 = (1,0,0,...,0)$, $\vec{\ell}'_2 = (0,1,0...,0)...$, $\vec{\ell}'_n = (0,0,...,1)$, are a basis of the pace P^n, which is named ***canonical.***

The proof

1) Let's prove that vectors $\vec{\ell}'_1, \vec{\ell}'_2, ..., \vec{\ell}'_n$ are linearly independent. For this purpose it is necessary to prove, that the vector equation $\lambda_1 \vec{\ell}'_1 +_1 \lambda_2 \vec{\ell}'_2 + ... + \lambda_n \vec{\ell}'_n = \vec{0}$ has only the trivial solution $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$. The given equation is equivalent to the system of

scalar equations $\lambda_1 \cdot 1 = 0$, $\lambda_2 \; 1 = 0$, ..., $\lambda_n \cdot 1 = 0$, which has unique solution $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$..

2) It is obviously, that any vector $\vec{a} = (\mu_{1,} \mu_2, ..., \mu_n)$ from P^n is a linear combination of vectors $\vec{\ell}'_1, \vec{\ell}'_2, ..., \vec{\ell}'_n$ with coefficients $\mu_1, \mu_2, ..., \mu_n : \vec{a} = \mu_1 \vec{\ell}' + \mu_2 \vec{\ell}'_2 + ... + \mu_n \vec{\ell}'_{n1}$. Hence, the system $\vec{\ell}'_1, \vec{\ell}'_2, ..., \vec{\ell}'_n$ is the basis P^m.

Thus, the significance of the isomorphism theorem l consist in the following. Vector spaces can consist of everything - columns, multinomials, physical values: speed, force, intensity of an electric field etc. - the nature of their elements is of no importance, when only their properties connected to operations of addition and multiplication by number are studied. All these properties of isomorphic spaces are completely identical. From the algebraic point of view the isomorphic spaces are identical. If we shall agree to not distinguish among themselves isomorphic spaces by virtue of the isomorphism theorem, there will be only one vector space for each dimension and, P^n can serve as this space.

§ 7. VECTOR FUNCTIONS OF ONE REAL VARIABLE; MAPPINGS *R* INTO *R^n*

Vector functions of one real variable put an element of vector space in conformity with a real number. Let this space be a vector space $\boldsymbol{R}^n$ above the field R.

Definition. Let P – be some numerical set from R and let any number $t \in P$ be put in conformity with the element (vector) from R^n. In this case we can say, that a function of real variable $t \in P$ with vector values in R^n, or in short, a vector function from t is determined.

A vector function is designated through $\vec{f}$ (or by the bold lowercase Latin letter), and its value for t – through $\vec{f}(t)$; $\vec{f}(t)$ is an element of vector space R^n. Expression «the vector function from $t \in P$ with values in R^n» has the same sense, as the following expressions: a vector function determined on P, or mapping P in R^n.

We shall designate elements of canonical basis of space R^n through $\vec{\ell}'_1, \vec{\ell}'_2, ..., \vec{\ell}'_n$ If $\vec{f}$ - is the vector function determined on *P* and possessing values in R^n, then $\overrightarrow{f(t)}$ is an element from R^n and, then , it represents a set n of real numbers which value depends on t and which we shall designate through $\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)$; it will be coordinates or components of the vector $\overrightarrow{f(t)}$ on canonical. Thus, $(\varphi_1(t), ...\varphi_n(t))$.

$$\overrightarrow{f(t)} = \varphi_1(t)\vec{\ell}_1 + \varphi_2(t)\vec{\ell}_2 + ... + \varphi_n(t)\vec{\ell}_n = (\varphi_1(t), ...\varphi_n(t)).$$

Hence, for any $t \in P$ n numerical functions φ_1, φ_2, ... φ_n of one real variable are determined and, so, $\vec{f}$ is the ordered set of n numerical functions $\varphi_1, \varphi_2, ... \varphi_n$ of one real variable which are determined on the set *P*. Functions φ_i are referred to as ***coordinate functions.***

Let's suppose now, that for $\vec{f}$ - mapping of the *P* from R^n - exists inverse mapping $\overrightarrow{f^{-1}}$; it means, that for any vector $\vec{x} \in R^n$, which is value of function $\vec{f}$, the set of those numbers $t \in P$, for which $\vec{x} = \overrightarrow{f(t)}$, it is reduced to one number. Then $\overrightarrow{f^{-1}}$; will be numerical function n of the real variables (Book 1, Chapter 3, § 3).

Let's note, that complex functions of one real variable considered by us in the book 2, Chapter 2 §6, item 6.1, can be presented as vector functions of one real variable, or as mapping *R* into R^2 , since *C* as the vector space is identified with R^2.

In conclusion we shall consider a vector function of one real variable *t* which value is a radius - vector $\vec{r} = \overrightarrow{OM}$ of the point *M* in geometrical space. As it has been already mentioned (Chapter 4, §3, item 3.3) $\vec{r}$ - is a vector which origin coincides with the origin of coordinates O, and the end is some point *M* of geometrical space. Coordinates of a vector $\vec{r}$ in orthonormal basis $\vec{i}, \vec{j}, \vec{k}$ and coordinates of the point *M* coincide in the Cartesian rectangular system of coordinates, i.e., if *M (x, y, z)* then $\vec{r} = \overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$. Let coordinates of the vector $\vec{r}$, and, hence, of point *M* be the function essence of some parameter *t*, with domain of variation $P \subset R$

$$\begin{cases} x = \varphi_1(t) \\ y = \varphi_2(t) \\ z = \varphi_3(t) \end{cases}.$$

Then $\vec{r}(t) = \varphi_1(t)\vec{i} + \varphi_2(t)\vec{j} + \varphi_3(t)\vec{k}$ represents a vector function of one real variable *t* or mapping *P* into R^3. When t change, also *x, y, z* change, and the point *M* - the end of a vector $\vec{r}$ - will circumscribe some line in the space which is named ***hodograph*** of vector $\vec{r} = \vec{r}$(t), and which can be considered as the graph of the vector function $\vec{r}$(t).

Thus, the vector function of one real variable with values in R^3 is graphically represented by a line in geometrical space.

§ 8. LINEAR MAPPINGS OF VECTOR SPACES

Definition 1. Let there be two vector spaces *K* and *L* above the same field *P*. Linear mapping of the space in *K* into *L* is referred mapping $f{:}K \to L$,possessing the following properties:

$$f(\vec{x}_1 + \vec{x}_2) = f(\vec{x}_1) + f(\vec{x}_2); \forall \vec{x}_1 \in K, \forall \vec{x}_2 \in K;$$

$$f(\lambda \vec{x}) = \lambda f(\vec{x}); \quad \forall \vec{x} \in K; \forall \lambda \in P.$$

Images $f(\vec{x}),\ f(\vec{x}_1),\ f(\vec{x}_2),\ f(\vec{x}_1 + \vec{x}_2) \in L.$

It should be emphasized, that addition n in the right and left parts of first of formulas designate, generally speaking, two various operations: addition in the space *K* and in space *L*. The similar remark concerns also the second formula.

Definition 2. If *L = P,* then value of a mapping is number from *P*; in this case we can say, that *f* is ***a linear form.***

So, the orthogonal projection of a free vector onto a plane is ***a linear mapping*** of the space R^3 into R^2.

$$f(\lambda_1 \vec{x}_1 + ... + \lambda_r \vec{x}_r) = \lambda_1 f(\vec{x}_1) + \lambda_2 f(\vec{x}_2) + ... + \lambda_r f(\vec{x}_r) = \vec{0},$$

Corollary fact from definition 1. Let's consider the set *f (K),* i.e. a set of elements from *L* which serve at mapping *f* as images , at least, of one element $\vec{x} \in K$. *f (K)* is the vector space which is a vector subspace of the space L and dimension of the space $f(K)$ does not surpass the dimension K. If $\vec{x}_1, \vec{x}_2, ..., \vec{x}_r$ are linearly dependent in K, then there are such $\lambda_1, \lambda_2, ..., \lambda_r$ in P, which are not all equal to zero, that $\lambda_1 \vec{x}_1 + \lambda_2 \vec{x}_2 + ... + \lambda_r \vec{x}_r = \vec{0}$, but then

$$f(\lambda_1 \vec{x}_1 + ... + \lambda_r \vec{x}_r) = \lambda_1 f(\vec{x}_1) + \lambda_2 f(\vec{x}_2) + ... + \lambda_r f(\vec{x}_r) = \vec{0},$$

and so elements $f(\vec{x}_1), ..., f(\vec{x}_r)$ are also linearly dependent. Generally speaking, the opposite is not fair. Here we take into account, that $f(\vec{0}) = \vec{0}$. It follows from mapping linearity: $f(\vec{x} + \vec{0}) = f(\vec{x}) + f(\vec{0})$ and, then, $f(\vec{0}) = \vec{0}$. It should be noticed, however, that $\vec{0}$ in $f(\vec{0})$ and $\vec{0}$ differ in the right part of equality, since these are the neutral elements belonging to different sets.

8.1. A rank of linear mapping

Definition. A rank r of linear mapping $f: K \to L$ is referred to as dimension of vector space $f(K)$. If K has dimension n, then since dimension of the space $f(K)$ cannot surpass n, we find that $r \le n$.

If $\vec{\ell}_1, \vec{\ell}_2, ... \vec{\ell}_n$ is a basis of the space K, then $\vec{x} = \lambda_1 \ell_1 + ... + \lambda_n \vec{\ell}_n$ and $f(\vec{x}) = \lambda_1 f(\vec{\ell}_1) + ... + \lambda_n f(\ell_n)$. Thus, the vector space $f(K)$ is generated by vectors $f(\vec{\ell}_1), ..., f(\vec{\ell}_n)$, and, hence, r is a maximal number of linearly independent vectors $f(\vec{\ell}_1), ..., f(\vec{\ell}_n)$, i.e. a rank of the given system of vectors.

If all vectors $f(\vec{\ell}_1), ..., f(\vec{\ell}_n)$ are linearly independent and form the basis $f(K)$, and $f(K)$ exhausts all space L (i.e. $f(K) = L$), then mapping f will be biunique. Hence, for linear mapping f to be biunique, it is necessary and sufficient that $dim\ K = dim\ L = n$, and it is equaled to a rank of r mappings. Thus, biunique mappings are possible only between spaces of identical dimension.

We shall notice, that if linear mapping f - is biunique, it will be isomorphism.

8.2. Coordinate notation of linear mappings

We shall consider two vector spaces *K* and *L* of various dimensions above the same field *P*. Let in the space *K* of dimension *m* be chosen the basis $\vec{\ell}_1, \vec{\ell}_2, ... \vec{\ell}_m$, and in space *L* of dimension *h* - the basis $\vec{e}_1, \vec{e}_2, ..., \vec{e}_h$. Let f be a linear mapping *K* into *L*; it converts $\vec{x} \in K$ в $\vec{y} = f(\vec{x}) \in L$. After we decompose vectors $\vec{x}$ and $\vec{y}$ on bases of corresponding spaces, we shall receive

$\vec{x} = \lambda_1 \vec{\ell}_1 + \lambda_2 \vec{\ell}_2 + ... + \lambda_m \vec{\ell}_m, \quad \vec{y} = \beta_1 \vec{e}_1 + \beta_2 \vec{e}_2 + ... \beta_h e_h, \ \lambda_i \in P$ и $\beta_i \in P$

or subject to a linear mapping, we have

$$\vec{y} = f(\vec{x}) = \lambda_1 f(\vec{\ell}_1) + \lambda_2 f(\vec{\ell}_2) + ... + \lambda_m f(\vec{\ell}_m) = \sum_{j=1}^{m} \lambda_j f(\vec{\ell}_j).$$

Since elements $f(\vec{\ell}_j) \in L$, then by means of basis $\vec{e}_1, \vec{e}_2, ..., \vec{e}_h$ they can be presented as

$$f(\vec{\ell}_j) = \alpha_{1j} \vec{e}_1 + \alpha_{2j} \vec{e}_2 + ... + \alpha_{hj} \vec{e}_h, \ j = 1,2,...,m$$

or $f(\vec{\ell}_j) = \sum_{i=1}^{h} \alpha_{ij} \vec{e}_i, i = 1,2,...,h.$

Hence,

$\vec{y} = \sum_{j=1}^{m} \lambda_j f(\vec{\ell}_j) = \sum_{j=1}^{m} \lambda_j \sum_{i=1}^{h} \alpha_{ij} \vec{e}_i = \sum_{i=1}^{h} \left(\sum_{j=1}^{m} \alpha_{ij} \lambda_j \right) \vec{e}_i$ or coordinate-wise, subject to, that $\vec{y} = \sum_{i=1}^{h} \beta_i \vec{e}_i$

$$\begin{cases} \beta_1 = \alpha_{11} \lambda_1 + \alpha_{12} \lambda_2 + ... + \alpha_{1m} \lambda_m \\ \beta_2 = \alpha_{21} \lambda_1 + \alpha_{22} \lambda_2 + ... + \alpha_{2m} \lambda_m \\ .... \\ \beta_h = \alpha_{h1} \lambda_1 + \alpha_{h2} \lambda_2 + ... + \alpha_{hm} \lambda_m \end{cases} \qquad (4..9)$$

It should be noticed, that the given system contains elements λ_j, β_i and α_{ij}, which belong only to the field *P*. It allows to consider the specified system also as the characteristic of linear mapping of the space P^m into P^h. Elements of the space P^m are vectors $\vec{x}' = (\lambda_1, \lambda_2, ..., \lambda_m)$, and spaces P^h -vectors $\vec{y}' = (\beta_1, \beta_2, ... \beta_h)$. Thus, any linear mapping *f* of the vector space *K* into *L* can be compared to linear mapping of the space P^m in P^h, which will be determined by the identical expressions, describing the mapping.

The received system of expressions to the full characterizes linear mapping *f* vector space *To* in *L*. In turn this system is set, if the rectangular table of factors $_{ij}$ which are written down as follows is known;
The obtained system of expressions in full characterizes linear mapping *f* of the vector space *K* in *L*. In its turn this system is assigned, if the rectangular table of factors α_{ij} is known , which is written down as follows;

$$A = \begin{pmatrix} \alpha_{11}\alpha_{12} \,.\,.\, \alpha_{1m} \\ \alpha_{21}\alpha_{22} \,. \quad \alpha_{2m} \\ \dots\dots\dots\dots\dots\dots \\ \alpha_{h1}\alpha_{h2} \,.\,.\, \alpha_{hm} \end{pmatrix} = (\alpha_{ij}), \qquad \begin{matrix} i = 1, 2,..., h, \\ \\ j = 1, 2,..., m \end{matrix}$$

Such rectangular table of numbers is referred to as ***a matrix,*** and numbers α_{ij} are referred to as its ***members..***

A set of the members which have identical first indexes, is referred to as ***a row,*** and a set of the members which have identical second indexes, is referred to as ***a column .*** So, α_{ijj} is a member of the i - row and the j - column.

With the help of a matrix *A* system of the expressions (4.9) describing linear mapping *f* of the vector space *K* in *L* (or P^m in P^h) is written down in the following way $\vec{y}' = A(\vec{x}')$, where

$\vec{x}' = (\lambda_1, \lambda_2, ..., \lambda_m) \in P^m, \;\; \vec{y}' = (\beta_1, \beta_2, ..., \beta_h) \in P^h.$

The matrix can be also considered irrespective of spaces *K* and *L*. It can be associated with the assignment of vector system in the space of row - vectors, or in the space of column – vectors. Really, let members of the i – row of the matrix (α_{i1}, α_{i2}, . . . α_{im}) represent components of the row - vector $\vec{a}_i$ in the space P^m, then

$$A = \begin{pmatrix} \alpha_{11}\alpha_{12} \,.\,.\,.\, \alpha_{1m} \\ \alpha_{21}\alpha_{22} \,.\,.\,.\, \alpha_{2m} \\ \alpha_{h1}\alpha_{h2} \,.\,.\,.\, \alpha_{hm} \end{pmatrix} = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_h \end{pmatrix} \qquad (4.10)$$

And, hence, the assignment of the matrix *A* means the assignment of the system from *h* of row vectors $\vec{a}_1, \vec{a}_2, ..., \vec{a}_h$ in the space P^h. Similarly,

$$\vec{g}_j = \begin{pmatrix} \alpha_{ij} \\ \alpha_{2j} \\ \vdots \\ \alpha_{hj} \end{pmatrix} \in P^h, \quad \text{then } A = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1m} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2m} \\ \dots & \dots & \dots & \dots \\ \alpha_{h1} & \alpha_{h2} & \dots & \alpha_{hm} \end{pmatrix} = (\vec{g}_1, \vec{g}_2, ..., \vec{g}_m) \qquad (4.11)$$

Hence, the assignment of the matrix A means the assignment of system from m column - vectors in the space P^h.

Members of a matrix in these cases are components of vectors.

If we consider matrix A in expression (4.9) as the assigned system of the column – vectors in the space P^h, then formulas (4.9) can be written down in the following equivalent form:

$$\vec{y}' = \lambda_1 \vec{g}_1 + \lambda_2 \vec{g}_2 + ... + \lambda_m \vec{g}_m,$$

here $\vec{y}' = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_h \end{pmatrix} \in P^h$, $\vec{g}_j = \begin{pmatrix} \alpha_{1j} \\ \alpha_{2j} \\ \vdots \\ \alpha_{hj} \end{pmatrix} \in P^h$, j = 1, 2,..., m.

This expression means, that the vector $\vec{y}' \in P^h$ is a linear combination of the column –vector system $\vec{g}_1, \vec{g}_2, ...\vec{g}_m$ from P^h, assigned by the matrix A with coefficients $\lambda_1, . . . , \lambda_m$. It follows from above-stated, that the matrix can be considered separately as independent value, and on a set of matrixes, as well as on any set, introduce the internal and external laws of a composition.

EXERCISES

1. Prove: a) a linear dependence of vectors $\vec{a}_1$(2,-1, 2), $\vec{a}_2$(3, 1,-2), $\vec{a}_3$(6,-3, 6); b) a linear independence of vectors $\vec{e}_1$ (2,-1,-2), $\vec{e}_2$(3, 1, 1), $\vec{e}_3$(-4, 2, 1).

2. Prove, that vectors $\vec{a}_1$(2,-1,-1), $\vec{a}_2$(2,-3, 0), $\vec{a}_3$ (1, 1,-1) form the basis of geometrical space and define coordinates of the vector $\vec{e}$(-5,-4,-2) in this basis.

3. Prove, that vectors $\vec{a} = 2\vec{i} - \vec{j} + 2\vec{k}$, $\vec{b} = \vec{i} + 2\vec{j} - 3\vec{k}$,

$\vec{c} = 3\vec{i} - 4\vec{j} + 7\vec{k}$ are coplanar.

4. Determine the components and write down decomposition of the vector $\vec{a}$ in orthonormal basis $\vec{i}, \vec{j}, \vec{k}$, if $|\vec{a}|$=2 and this vector forms with axes the absciss and ordinates angles on-the-miter on 45°.

5. Find out, whether the given set of vectors in *n-dimensional* vector space *K* above field *P* is a vector subspace and determine its dimension: a) the set of vectors, which all coordinates are equal among themselves; b) set of vectors, which sum of coordinates it is equal to 0; c) the set of vectors, which sum of coordinates it is equal to 1.

CHAPTER 5
MATRIXES

Definition 1. The matrix A above the field P, consisting of k - rows and m - columns, is the rectangular table of elements $\alpha_{ij} \in P$, where $i = 1, 2, ..., k;\ j = 1, 2, ..., m.$

Definition 2. Product of k- row number and m- columns of the matrix $k \times m$ (k by m), which is equal to number of matrix members α_{ij}, is referred to as ***the size*** of a matrix.

It should be noticed, that matrixes with identical number of members can have different dimension. For example, dimensions of matrixes from m - rows and n - columns $(m \times n)$ and both n rows and m columns $(n \times m)$ are not identical.

§ 1. MATRIX RANK. ELEMENTARY MATRIX TRANSFORMATIONS

As it has been already mentioned, the matrix A of the size $k \times m$ can be considered as the assignment of system from m column - vectors in the space P^k or from k row -vectors in the space P^m. It is possible to show (the proof of this theorem is omitted), that ranks of systems of column-vectors and row –vectors are identical.

Definition. The general value of a rank of column – vector system (or row –vector system), assigned by the matrix A, is referred to as ***a rank*** of this matrix and it is designated as $r(A)$.

Being based on conclusions of theorems of linearly dependent and linearly independent vectors, it is possible to establish, that $r(A) \leq \min(k, m)$, and also the following elementary transformations of a matrix which do not change its{her} rank.

Elementary matrix transformations:

1. Multiplication of a row (column) of a matrix by the number which is distinct from zero;

2. Addition of one row (column) of a matrix to another row (column) of this matrix;

3. Permutation of two rows (columns) of the given matrix.

Combining elementary transformations, we can add any row (column) matrixes to a linear combination of other rows (columns) and thus a matrix rank also does not change. By means of elementary transformations any matrix

$$A = \begin{pmatrix} a_{11}a_{12}....a_{1m} \\ a_{21}a_{22}....a_{2m} \\ .... \\ a_{k1}a_{k2}....a_{km} \end{pmatrix}$$

can be transformed into the form

$$B = \begin{pmatrix} в_{11}....0......0 \\ .... \\ 0........в_{rr}...0 \\ .... \\ 0........0......0 \end{pmatrix} \quad \text{or} \quad E = \begin{pmatrix} 1....0....0 \\ .... \\ 0....1....0 \\ .... \\ 0....0....0 \end{pmatrix}$$

where $в_{ii} \neq 0$, $i = 1,2,...r$, $r \leq min(k,m)$. It is clear, that number r of nonzero members is equal to a matrix rank: $r = r(A) = r(B) = r(E)$. In such a way it is possible to define a rank of any matrix.

Now we shall consider a matrix A of the size $k{\times}m$ as the characteristic of linear mapping $\vec{x} \to A(\vec{x})$, where $\vec{x} \in P^m$, and $A(\vec{x}) \in P^k$. In this case a matrix rank is equal to a rank of this linear mapping. Really, the system from column - vectors of a matrix A consists of m vectors belonging *to* $P^к$, and the set of mappings $A(\vec{x})$ is a linear hull of column – vector system of the matrix A. Thus, dimension of subspace is mapped $A(\vec{x})$ (a rank of linear mapping) is equal to a rank of column – vector (a matrix rank), generating this subspace.

As we already have determined, mapping $A: P^m \to P^k$ will be biunique if and only if dimensions of spaces coincide k = m and are equal to a rank of r mapping, i.e. r = k = m. Hence, the matrix determining biunique mapping should have the size m×m (square), and its rank $r(A)$ be equal to m.

§ 2. ALGEBRAIC OPERATIONS ON MATRIXES. VECTOR SPACE OF MATRIXES

Since the matrix is associated with vector system s, and operations of comparison and addition are introduced only for vectors belonging to a single space, therefore we can compare and add only matrixes of the identical sizes.

Equality. Two matrixes of the identical sizes, which corresponding members are equal among themselves, are referred to as equal.

Addition. Sum of two matrixes *A* and *B* of the identical sizes is referred to as matrix *C* of the same size which members are equal to the sums of corresponding members of added matrixes.

$$A = \begin{pmatrix} a_{11}a_{12}....a_{1m} \\ a_{21}a_{22}....a_{2m} \\ .... \\ a_{k1}a_{k2}....a_{km} \end{pmatrix} = (a_{ij}), \quad B = \begin{pmatrix} в_{11}в_{12}....в_{1m} \\ в_{21}в_{22}....в_{2m} \\ .... \\ в_{к1}в_{к2}....в_{кm} \end{pmatrix} = (в_{ij})$$

$$C = A + B = \begin{pmatrix} a_{11}+в_{11} \quad a_{12}+в_{12}....a_{1m}+в_{1m} \\ a_{21}+в_{21} \quad a_{22}+в_{22}....a_{2m}+в_{2m} \\ .... \\ a_{ki}+в_{ki} \quad a_{k2}+в_{k2}....a_{km}+в_{km} \end{pmatrix} \quad (c_{ij}) = (a_{ij}+в_{ij}),$$

where $i = 1,2,....,\ к$; $j = 1,2,....m$.

Addition is associative and commutative as exists for addition a_{ij} + $в_{ij}$ *P;* there is a neutral element - a zero matrix, designated O or (0), which all members are zeros, and $O(\vec{x}) = \vec{0} \in P^{к}$**,**whatever $\vec{x} \in P^{m}$ may be. Each matrix *A* from members $\boldsymbol{a_{ij}}$ has opposite (symmetric), designated – *A* which all elements are essence - a_{ij} и *A+(-A)* = O. Thus, operation of addition on set of matrixes of the identical sizes forms Abelian group.

Multiplication of a matrix by a number from P. Product of a matrix by a number (or numbers by a matrix) is referred to as a matrix which members are products of members of the given matrix by this number:

$$\lambda A = A\lambda = \lambda(a_{ij}) = (\lambda a_{ij}) = \begin{pmatrix} \lambda a_{11}\ \lambda a_{12} \ldots \lambda a_{1m} \\ \lambda a_{21}\ \lambda a_{22} \ldots \lambda a_{2m} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda a_{k1}\ \lambda a_{k2} \ldots \lambda a_{km} \end{pmatrix},$$

where i = 1,2,...,к; j = 1,2,...,m.

We can see, that multiplication by number is commutative and the obtained matrix has the same dimension, as multiplied matrix. Besides:

$\lambda(A+B) = \lambda A+\lambda B$, since $\lambda(a_{ij}+в_{ij}) = \lambda a_{ij}+\lambda в_{ij}$;

$(\lambda+\mu) A = \lambda A+\mu A$, since $(\lambda+\mu) a_{ij} = \lambda a_{ij}+\mu a_{ij}$;

$\lambda(\mu A) = (\lambda \mu) A$, since $\lambda(\mu a_{ij}) = (\lambda \mu) a_{ij}$;

$\varepsilon A = A$, since $\varepsilon a_{ij} = a_{ij}$, where $\varepsilon = 1 \in P$ – is a neutral element of multiplication in P, whatever matrixes A and B may be, from κ rows and m columns, and whatever $\lambda \in P$ and $\mu \in P$ may be.

Thus, the set of matrixes A, consisting of k rows and m columns forms a vector space above the field P.

We shall designate through I_{ij} a matrix of k rows and m columns, which all elements are zero, except for a member of i – row and j - that column - equal to ε = 1;• = 1; i.e. we shall put

$$I_{ij} = \begin{pmatrix} 0 \ldots 0 \ldots 0 \\ \ldots\ldots\ldots \\ 0 \ldots 1 \ldots 0 \\ \ldots\ldots\ldots \\ 0 \ldots 0 \ldots 0 \end{pmatrix} \text{ i.}$$

j

Quantity of such matrixes is equal to number of members in a matrix, i.e. to product $k \cdot m$.

Then any matrix $A = (\alpha_{ij})$ consisting of k rows and m columns takes form of:

$$A = \sum_{i=1}^{k} \left(\sum_{j=1}^{m} \alpha_{ij} I_{ij} \right),$$

and this representation is unique. Hence, matrixes I_{ij} form a basis of matrix vector space of k rows and m columns that is, this vector space has the finite dimension which is equal to product $k \cdot m$, that forms the general number of elements in a matrix.

Multiplication of two matrixes. Product of two matrixes A, with the size $m\times k$ and B, with the size $k\times n$, is referred to as the matrix C, with the size $m\times n$ which element c_{ij} is equal to the sum of member products i - row of the matrix A by corresponding elements of the j - column of the matrix B.

Let be given matrixes

$$A = \begin{pmatrix} a_{11}a_{12}....a_{1k} \\ a_{21}a_{22}....a_{2k} \\ .... \\ a_{m1}a_{m2}....a_{mk} \end{pmatrix} \text{ and } B = \begin{pmatrix} в_{11}в_{12}....в_{1n} \\ в_{21}в_{22}....в_{2n} \\ .... \\ в_{к1}в_{к2}....в_{кn} \end{pmatrix},$$

then, their product

$$C = A\cdot B = \begin{pmatrix} c_{11}c_{12}....c_{1n} \\ c_{21}c_{22}....c_{2n} \\ .... \\ c_{m1}c_{m2}....c_{mn} \end{pmatrix} = (c_{ij}),$$

where $c_{ij} = a_{i1}, в_{1j} + a_{i2}в_{2j} + \ldots + a_{ik}в_{kj} = \sum_{\gamma=1}^{k} a_{i\gamma}в_{\gamma j}$,

$i = 1,2,..., m, \quad j = 1,2, \ldots, n.$

The remark. Two matrixes A and B, taken in the certain order, can be multiplied only if column number of the first matrix is equal to row number of the second matrix, i.e. they have the sizes $m\times k$ and $k\times n$. Such matrixes are referred to as ***consistent.***

For multiplication of matrixes the following properties are fair:

1. Product of any matrix by consistent with it zero matrix is equal to zero matrix.

2. Product of matrixes is not commutative, i.e. generally $AB \neq BA$.

Thus it is supposed, that $A\cdot B$ and $B\cdot A$ make sense. If $A\cdot B = B\cdot A$, then matrixes are referred to as ***commutative (permutable)***.

3. Let A, B and C be matrixes which can be added or multiplied, and λ - some number from P

$$(A \cdot B) \cdot C = A \cdot (B \cdot C)$$
$$\lambda \cdot (A \cdot B) = (\lambda \cdot A) \cdot B = A \cdot (\lambda \cdot B)$$
$$A \cdot (B+C) = A \cdot B + A \cdot C.$$

§ 3. ISOMORPHISM BETWEEN VECTOR SPACE OF MATRIXES AND VECTOR SPACE P^n ABOVE FIELD

As we already mentioned, the matrix A with the size $k \times m$ can put in conformity the ordered system of m column vectors $(\vec{a}_1, \vec{a}_2, ..., \vec{a}_m)$ in the space P^m, or of k row - vectors $(\vec{в}_1, \vec{в}_2, ..., \vec{в}_к)$ in the space P^m. Both ordered systems of vectors $(\vec{a}_1, \vec{a}_2, ..., \vec{a}_m)$ and $(\vec{в}_1, \vec{в}_2, ..., \vec{в}_к)$ - are elements of same vector space $P^{\,n}$, where $n = k \cdot m$ which is isomorphic for vector space of matrixes with the size $k \times m$. Really,

$$(\vec{a}_1, \vec{a}_2, ..., \vec{a}_m) \in \prod_{i=1}^{m} P_i^k = P^{k \cdot m} = P^n \quad \text{and} \quad (\vec{в}_1, \vec{в}_2, ..., \vec{в}_к) \in \prod_{i=1}^{k} P_i^m = P^{m \cdot k} = P^n$$

We shall consider now the system consisting of one vector $\vec{x} = (\alpha_1, \alpha_2, ..., \alpha_n) \in P^n$. It is obvious, that this vector through the components in matrix space will be associated with matrixes of the size $1 \times n$, or $n \times 1$; $\vec{x} \to X = (\alpha_1, \alpha_2, ..., \alpha_n)$ a matrix of the size $1 \times n$;

$$\vec{x} \to X = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ .... \\ \alpha_n \end{pmatrix}$$

matrix of the size $n \times 1$. It is clear, that mapping $\vec{x} \to X$ is isomorphism, since

$$\vec{x}_1 + \vec{x}_2 \to X_1 + X_2 \quad \text{and} \quad \lambda\vec{x} = \lambda X,$$
$$\forall \vec{x}_1 \in P^n, \quad \forall \vec{x}_2 \in P^n, \quad \forall \lambda \in P.$$

Using the specified isomorphism, we shall show, how mapping $\vec{y} = A(\vec{x})$ is presented in the matrix space, where $\vec{x} \in P^m, \vec{y} \in P^k$.

Let the mapping A of the space P^m into $P^к$ is determined by formulas:

$$\beta_1 = \alpha_{11}\lambda_1 + \alpha_{12}\lambda_2 + ... + \alpha_{1m}\lambda_m$$
$$\beta_2 = \alpha_{21}\lambda_1 + \alpha_{22}\lambda_2 + ... + \alpha_{2m}\lambda_m$$
. . . .

$$\beta_к = \alpha_{к1}\lambda_1 + \alpha_{к2}\lambda_2 +. . . +\alpha_{кm}\lambda_m$$

Let's put vector $\vec{y}$ with components (β_1, β_2, . . .,$\beta_к$) from $P^к$ in conformity with a matrix:

$Y = \begin{pmatrix} \beta_1 \\ \beta_2 \\ ... \\ \beta_k \end{pmatrix}$ of the size к×1, and vector $\vec{x}$ with components (λ_1 ,λ_2,. . . ,λ_m)

matrix $X = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ ... \\ \lambda_m \end{pmatrix}$ of the size m×1. Then mapping, $\vec{y} = A(\vec{x})$, determined by

matrix

$A = \begin{pmatrix} a_{11}a_{12}....a_{1m} \\ a_{21}a_{22}....a_{2m} \\ .... \\ a_{k1}a_{k2}....a_{km} \end{pmatrix}$ of the size $k\times m$ in the matrix space is

determined by the same matrix A and it is represented in the form

$$\vec{y} = A(\vec{x}) \to \begin{pmatrix} \beta_1 \\ \beta_2 \\ ... \\ \beta_k \end{pmatrix} = \begin{pmatrix} a_{11}a_{12}....a_{1m} \\ a_{21}a_{22}....a_{2m} \\ .... \\ a_{k1}a_{k2}....a_{km} \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ ... \\ \lambda_m \end{pmatrix} \to Y = AX.$$

Finally we shall consider, howthe scalar product of two vectors from space R^n is mapped in the matrix space.

§ 4. SCALAR PRODUCT OF TWO VECTORS FROM SPACE R^n

Definition. We shall consider mapping φ of the vector space $R^n \times R^n$ into R wherein the following conformity is established

$$(\vec{x},\vec{y}) \to \varphi(\vec{x},\vec{y}) = \sum_{i=1}^{n} \alpha_i\beta_i = \alpha_1\beta_1 + \alpha_2\beta_2 + ... + \alpha_n\beta_n ,$$

here $\bar{x} \in R^n$, $\bar{y} \in R^n$; the ordered couple $(\bar{x}, \bar{y})$ is an element of the vector space $R^n \times R^n$; $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $(\beta_1, \beta_2, \ldots, \beta_n)$ are components of vectors $\vec{x}$ и $\vec{y}$; $\sum_{i=1}^{n} \alpha_i \beta_i$ is number from R.. Such mapping φ is referred to as ***scalar product*** of two vectors $\vec{x}$ and $\vec{y}$ from the space R^n and it is designated $\vec{x} \cdot \vec{y}$.

Mapping φ is not a linear mapping. Really, since $R^n \times R^n$ is a vector space, $(\vec{x}_1, \vec{y}_1) + (\vec{x}_2, \vec{y}_2) = (\vec{x}_1 + \vec{x}_2, \vec{y}_1 + y_2)$, where $x_1 = (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n})$;
$\vec{x}_2 = (\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n}); \vec{y}_1 = (\beta_{11}, \beta_{12}, \ldots, \beta_{1n}); \vec{y}_2 = (\beta_{21}, \beta_{22}, \ldots, \beta_{2n})$.
It is easy to show, that
$\varphi[(\vec{x}_1, \vec{y}_1) + (\vec{x}_2, \vec{y}_2)] = \varphi[(\vec{x}_1 + \vec{x}_2, \vec{y}_1 + \vec{y}_{22})] \neq \varphi(\vec{x}_1, \vec{y}_1) + \varphi(\vec{x}_2, \vec{y}_2)$ and, hence, mapping φ is not linear mapping.

We define now how scalar product is represented in matrix space. Let two vectors be given $\vec{x} = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in R^n$ and $\vec{y} = (\beta_1, \beta_2, \ldots, \beta_n) \in R^n$. Now let's put the vector $\vec{x}$ in conformity with a matrix $X = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of the size $1 \times n$, and the vector $\vec{y}$ - with a matrix $Y = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \ldots \\ \beta_k \end{pmatrix}$ of the size $n \times 1$. Then product $\vec{x} \cdot \vec{y}$ in the matrix space is equivalent to the product

$$X \cdot Y = (\alpha_1, \alpha_2, \ldots, \alpha_n) \begin{pmatrix} \beta_1 \\ \beta_2 \\ \ldots \\ \beta_n \end{pmatrix} = \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_n\beta_n.$$

We can see, that in vector space of matrixes the mapping φ is not linear mapping

$$(\alpha_{11} + \alpha_{21}\ \alpha_{12} + \alpha_{22} \ldots \alpha_{1n} + \alpha_{2n}) \cdot \begin{pmatrix} \beta_{11} + \beta_{21} \\ \beta_{12} + \beta_{22} \\ \ldots\ldots\ldots\ldots \\ \beta_{1n} + \beta_{2n} \end{pmatrix} \neq (\alpha_{11}, \alpha_{12}, \ldots, \alpha_{1n}) \cdot \begin{pmatrix} \beta_{11} \\ \beta_{12} \\ \ldots \\ \beta_{1n} \end{pmatrix} + (\alpha_{21}, \alpha_{22}, \ldots, \alpha_{2n}) \cdot \begin{pmatrix} \beta_{21} \\ \beta_{22} \\ \ldots \\ \beta_{1n} \end{pmatrix}.$$

§ 5. SQUARE MATRIXES

Definition. A matrix which row number is equal to column number is referred to as ***square***; the equal number of n rows and columns is referred to as ***the order*** of matrix.

The set of elements α_{ii} is referred to as ***main diagonal***, and a matrix which all members are located outside of the main diagonal is zero $\alpha_{ij} = 0$, if $i \neq j$, it is referred to as ***diagonal***.

$$A_{ii} = \begin{pmatrix} \alpha_{11}...0...0 \\ .... \\ 0....\alpha_{ii}...0 \\ .... \\ 0....0....\alpha_{nn} \end{pmatrix},$$

if all elements of a diagonal matrix are equal $\alpha_{ii} = \lambda$, such matrix is referred to as ***scalar***.

The diagonal matrix, which all members are equal to one, is referred to as ***identity matrix*** and it is designated E_n *(or* I_n).

$$E_n = \begin{pmatrix} 1...0...0 \\ .... \\ 0...1...0 \\ .... \\ 0...0...1 \end{pmatrix}$$ or $E_n = (\delta_{ij})$, where $i = 1,2,...,n$; $j = 1,2,...,n$; δ_{ij} – is Kronecker symbol. Identity matrix E_n represents a neutral element concerning multiplication of matrixes A of order n: $AE_n = E_nA = A$.

The sum and product of two matrixes of n - order are always determined and the result will be matrixes of n order. However product of square matrixes is not commutative: $A \cdot B \neq B \cdot A$. For example,

$$\begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 7 \\ 1 & 2 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 1 & 0 \end{pmatrix}.$$

Square matrixes of n order determine linear mappings P^n into P^n, and identity matrix E_n is associated with system of vectors of canonical basis $\vec{\ell}_1 = (1,0,...,0), \vec{\ell}_2 = (0,1,0,...0)....\vec{\ell}_n = (0,0,...,1)$ of the space P^n.

5.1. Inverse matrix

We shall consider a matrix A which sets the mapping $\vec{x} \to A(\vec{x})$. Inverse mapping exists, if this is biunique mapping P^n onto P^n. But for this purpose it is necessary and sufficient that the matrix A be square one of the order n and which rank $r(A)$ is equal to n. Therefore the inverse matrix exists only for square matrix A which rank $r(A)$ and the order n are identical.

Definition. The square matrix representing inverse mapping for matrix A, is referred to as inverse matrix for a matrix A and it is designated A^{-1}; matrix A^{-1} is a symmetric member for matrix A concerning multiplication.

Really, let biunique mapping $\vec{x} \to A(\vec{x})$ of space P^n onto P^n be given. Inverse mapping for it will be $A(\vec{x}) \to A^{-1}[A(\vec{x})] = \vec{x}$, therefore $A^{-1}A = E_n$; just as $AA^{-1} = E_n$ and, hence, $AA^{-1} = A^{-1}A = E_n$. If A^{-1} exists, we can say, that the matrix A ***is invertible.*** Inversely, if A - is an invertible matrix, the mapping $\vec{x} \to A(\vec{x})$ is biunique.

Let A and B - two invertible matrixes of the order n; by virtue of associativity

$$ABB^{-1}A^{-1} = A(BB^{-1})A^{-1} = AE_nA^{-1} = (AE_n)A^{-1} = AA^{-1} = E_n.$$

Hence, $(AB)(B^{-1}A^{-1}) = E_n$ so, product of two invertible matrixes - is invertible matrix and $(AB)^{-1} = B^{-1}A^{-1}$.

5.2. The transposed square matrix. Symmetric matrixes

Definition 1. We can say, that matrix A^{T} of elements α'_{ij} ***is transposed*** in relation to a square matrix A of members α'_{ij}, if $\alpha'_{ij} = \alpha_{ji}$, for i = 1, 2..., n; j = 1, 2..., n.

$$A = \begin{pmatrix} \alpha_{11}\ \alpha_{12} \dots \alpha_{1n} \\ \alpha_{21}\ \alpha_{22} \dots \alpha_{2n} \\ \dots\dots\dots\dots \\ \alpha_{n1}\ \alpha_{n2} \dots \alpha_{nn} \end{pmatrix}, \qquad A^T = \begin{pmatrix} \alpha_{11}\ \alpha_{21} \dots \alpha_{n1} \\ \alpha_{12}\ \alpha_{22} \dots \alpha_{n2} \\ \dots\dots\dots\dots \\ \alpha_{1n}\ \alpha_{2n} \dots \alpha_{nn} \end{pmatrix}.$$

Members of the matrix A^T are symmetric to members of a matrix A concerning the main diagonal. The operation converting a square matrix into transposed one, is referred to as ***transposition.*** For this purpose members of every row of matrix A are set down in the same order into the columns of the matrix A^T, and number of a column coincides with number of a row. It is clear, that thus i - row A^T consists of the same members, in the same order, as i - column of a matrix A.

Matrixes A and A^T have an identical rank r (A) = r (AT), and also $(\lambda A)^T = \lambda A^T$; $(A+B)^T = A^T+B^T$; $(A\cdot B)^T = B^T A^T$; if A is invertible, then
$(A-1)\ T = (AT)-1$.

Definition 2. The square matrix A of members α_{ij} is referred to as ***symmetric***, if $A = A^T$. If $\alpha_{ij} = \alpha_{ji}$ i.e. members of matrix A which are symmetric relative to its main diagonal are equal each other. All diagonal matrixes are symmetric, for example, $E = E^T$.

EXERCISES

1. Define ranks of matrixes with the help of elementary transformations:

$$A = \begin{pmatrix} 4 & 12 & 4 & 6 & 2 \\ 3 & 4 & 1 & 2 & -2 \\ 5 & -2 & 3 & -1 & 3 \\ 2 & 6 & 2 & 3 & 1 \end{pmatrix}; \qquad B = \begin{pmatrix} 2 & 0 & 1 & 1 & 4 \\ 1 & 1 & 2 & 2 & 0 \\ 4 & 4 & 3 & 1 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 3 & 1 & 1 & 1 & 3 \end{pmatrix}.$$

2. Prove, that for any matrix A, the matrix $S=A+AT$ - is symmetric. Show, that product of matrix A by transposed matrix is always a symmetric matrix.

3. Let $A = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 2 & -4 \\ 5 & -6 \end{pmatrix}$. Determine $C = A + B + A^T + B^T$.

4. Are these matrixes $A = \begin{pmatrix} 2 & 1 \\ 4 & -2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 3 \\ -2 & 0 \end{pmatrix}$ transposed?

CHAPTER 6
DETERMINANTS

§ 1. DEFINITION AND THE PROPERTIES OF THE DETERMINANT FOLLOWING FROM DEFINITION

Definition. Let's consider a vector space of square matrixes *A* of the order *n* above the field *P.* We shall set such mapping *D* of space of these matrixes in the field *P*, wherein each square matrix

$$A = \begin{pmatrix} a_{11}\ a_{12}\ldots a_{1n} \\ a_{21}\ a_{22}\ldots a_{2n} \\ \ldots\ldots\ldots\ldots \\ a_{n1}\ a_{n2}\ldots a_{nn} \end{pmatrix}$$ is put in conformity with number *D(A)* from *P* by law

$$D(A) = \begin{vmatrix} a_{11}\ a_{12}\ldots a_{1n} \\ a_{21}\ a_{22}\ldots a_{nn} \\ \ldots\ldots\ldots\ldots \\ a_{n1}\ a_{n2}\ldots a_{nn} \end{vmatrix} = \sum_{f=\begin{pmatrix}1,2,\ldots,n \\ m_1,m_2,\ldots m_n\end{pmatrix}}^{n!} (-1)^{\nu(f)} a_{1m_1} \cdot a_{2m_2} \cdot a_{3m_3} \cdot \ldots \cdot a_{nm_n} \tag{6.1}$$

This number is referred to as ***a determinant*** of matrix *A.* Designation *D (A)* or |*A* |.

It follows from the given definition, that mapping *D* represents the numerical function prescribed on a set of square matrixes and consequently the square matrix *A* acts as a variable in it .Thus, a determinant, i.e. value *D (A)* of numerical function *D* can be considered as the numerical characteristic of the square matrix *A.* Matrix order is referred also to as the order of a determinant which it corresponds to.

The sum of the right part of equality is taken on transpositions of the second indexes of matrix members a_{ij}, where *j=1, 2.... n.* It means, that each transpostion of the second indexes a_{ij}, where *j=1, 2,....n.* , or $f = \begin{pmatrix} 1, & 2, \ldots & n \\ m_1, & m_2, \ldots & m_n \end{pmatrix}$ is conformed to a summand. Every summand consists of product *n* of members taken on one and only to one member from each row and each column. Products are added with signs determined by number of inversions $\nu(f)$ of corresponding transpositions $f = \begin{pmatrix} 1, & 2, \ldots, n \\ m_1, m_2, \ldots, m_n \end{pmatrix}.$ Number of such summand is equal to number of transpositions 1, 2,....n, i. e. n!.

Examples

1. $\begin{vmatrix} a_{11} a_{12} \\ a_{21} a_{22} \end{vmatrix} = \sum_{f=\begin{pmatrix} 1, & 2 \\ m_1, m_2 \end{pmatrix}}^{2!} (-1)^{\nu(f)} a_{1m_1} a_{2m_2} = a_{11}a_{22} - a_{12}a_{21}.$

Really, there are only two transpositions m_1, m_2 from 1,2 $f_1 = \begin{pmatrix} 1.2 \\ 1.2 \end{pmatrix}, \nu(f_1) = 0$ and $f_2 = \begin{pmatrix} 1,2 \\ 2,1 \end{pmatrix}, \nu(f_2) = 1.$

2. $\begin{vmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{vmatrix} = \sum_{f=\begin{pmatrix} 1, & 2, & 3 \\ m_1 m_2 m_3 \end{pmatrix}}^{3!} (-1)^{\nu(f)} \cdot a_{1m_1} \cdot a_{2m_2} a_{3m_3} = a_{11}a_{22}a_{33} -$

$- a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32}$. There are only 3 transpositions m_1, m_2, m_3 from 1, 2, 3 ! = 6.

$$f_1 = \begin{pmatrix} 1\,2\,3 \\ 1\,2\,3 \end{pmatrix}, \ \nu(f_1) = 0, \ f_2 = \begin{pmatrix} 1\,2\,3 \\ 2\,1\,3 \end{pmatrix}, \ \nu(f_2) = 1, \ f_3 = \begin{pmatrix} 1\,2\,3 \\ 2\,3\,1 \end{pmatrix}, \ \nu(f_3) = 2,$$

$$f_4 = \begin{pmatrix} 1\,2\,3 \\ 3\,2\,1 \end{pmatrix}, \ \nu(f_4) = 3, \ f_5 = \begin{pmatrix} 1\,2\,3 \\ 3\,1\,2 \end{pmatrix}, \nu(f_5) = 2, \ f_6 = \begin{pmatrix} 1\,2\,3 \\ 1\,3\,2 \end{pmatrix}, \ \nu(f_6) = 1.$$

The properties of a determinant following from definition

1. The determinant of the transposed matrix is equal to initial $D(A^T) = D(A)$. It follows from equality of rows and columns in relation to a determinant.

2. If we transpose two columns (rows) of a determinant, the determinant will reverse a sign. Really, if columns (rows) are interchanged, it result in permutation $f = \begin{pmatrix} 1, & 2, \ldots. \ n \\ m_1, m_2, \ldots m_n \end{pmatrix}$, and transposition, as we have determined, results in change of permutation parity (Book 1, Chapter.2, § 2, item 2, 3). Hence, all summands of a determinant reverse a sign.

3. Determinant which two rows (columns) are identical, is equal to zero. Really, if we permute in a determinant two identical rows (column), then, on the one hand, we shall change nothing, and on the other hand, according to item 2, we shall reverse a sign of a determinant, i.e. $D(A) = -D(A)$, hence $D(A) = 0$.

4. If we multiply all elements of a column (row) of a determinant by the same number, then the determinant also will be multiplied by this number.

$$\begin{vmatrix} a_{11}...\lambda a_{1j}...a_{1n} \\ .... \\ a_{i1}.....\lambda a_{ij}...a_{in} \\ .... \\ a_{m}....\lambda a_{nj}....a_{mn} \end{vmatrix} = \lambda \begin{vmatrix} a_{11}...a_{1j}...a_{1n} \\ .... \\ a_{i1}.....a_{ij}....a_{in} \\ .... \\ a_{n1}....a_{nj}....a_{nn} \end{vmatrix}.$$

Thus, if all elements of some row (column) contain common multiplier it can be taken out a sign of the determinant.

3. If each element of any column (row) is the sum of two summands, then the determinant is equal to the sum of two determinants which columns (rows) are corresponding summands, and the others coincide with columns (rows) of the given determinant:

$$\begin{vmatrix} a_{11}...a_{1j}+b_{1j}...a_{1n} \\ .... \\ a_{i1}...a_{ij}+b_{ij}....a_{in} \\ .... \\ a_{n1}...a_{nj}+b_{nj}...a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11}...a_{ij}...a_{1n} \\ .... \\ a_{i1}...a_{ij}...a_{in} \\ .... \\ a_{ni}...a_{nj}...a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11}...b_{1j}...a_{1n} \\ .... \\ a_{i1}...b_{ij}...a_{in} \\ .... \\ a_{n1}...b_{nj}...a_{nn} \end{vmatrix}$$

Properties 4 and 5 result from distributivity of multiplication concerning addition. Property 5 can be considered as a rule for addition of determinants.

Corollary facts. 1. The value of a determinant will not change, if elements of any column (row) are added the corresponding elements of other column (row) multiplied by the same number.

2. If *And* – is a matrix of order n, $D(\lambda A)=\lambda^n D(A)$..

3. $D(A)\cdot D(B)=D(A\cdot B)$. Even if $A\cdot B\neq B\cdot A$, then, nevertheless $D(A\cdot B)=D(A)\cdot D(B)=D(B\cdot A)$.

§ 2. DECOMPOSITION OF A DETERMINANT ON THE LINE (COLUMN) ELEMENTS. THE THEOREM OF ANOTHER'S ADDITIONS

Definition 1. Complementary minor of some member a_{ij} of the square matrix A of order n, is referred to as determinant D_{ij} of a matrix of order n-1 which results from deletion of i - rows and j - column (intersected on this member).

Example

$$A=\begin{pmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{pmatrix};\ D_{31}=\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

– complementary minor of the member a_{31}.

Definition 2. Algebraical complement A_{ij} of the member a_{ij} is referred to as its additional minor D_{ij} multiplied by $(-1)^{i+j}$

$$A_{ij}=(-1)^{i+j}\cdot D_{ij}$$

It is valid the following statement which we shall postulate: if we multiply members of some row (column) by their algebraical complements, and we add these products, then we will have the value of a determinant.

$$D(A)=a_{i1}A_{i1}+a_{i2}A_{i2}+...+a_{in}A_{in}=\sum_{j=1}^{n}a_{ij}A_{ij}\ \text{- decomposition in i - row.}$$

$$D(A)=a_{1j}A_{1j}+a_{2j}A_{2j}+...+a_{nj}A_{nj}=\sum_{i=1}^{n}a_{ij}A_{ij}\ \text{- decomposition in j - column.}$$

The given decomposition allow us reduce the calculation a determinant of the n – order to the calculation n determinants of the order n - 1. In addition to these formulas frequently also the following theorem can be useful.

The theorem (about another's complements). If we multiply elements of some row (column) by algebraical complements of corresponding members of other row (column) and then we add these products, the sum will be equal to zero.

$$a_{i1}A_{k1}+a_{i2}A_{k2}+...+a_{in}A_{kn}=\sum_{j=1}^{n}a_{ij}A_{kj}=0.$$

a_{ij}, where j = 1, 2,..., n – members of i – row, and $A_{\kappa j}$, where j = 1, 2,..., n algebraical complements of k –row members.

The proof. We shall consider a determinant of the matrix B which results from a matrix A by substituting k – row members for i – row members. As it is a determinant with two equal rows, it is equal to zero

$$D(B)=\sum_{j=1}^{n}b_{kj}B_{kj}=0.$$

Let's notice, that $в_{\kappa j}=a_{ij}$, a $B_{\kappa j}=A_{\kappa j}$, then $\sum_{j=1}^{n}b_{kj}B_{kj}=\sum_{j=1}^{n}a_{ij}A_{kj}=0$, is to be proved.

Example: $\begin{vmatrix}3 & 2 & -1\\ 1 & 1 & 2\\ 4 & -2 & 1\end{vmatrix}=3\begin{vmatrix}1 & 2\\ -2 & 1\end{vmatrix}-2\begin{vmatrix}1 & 2\\ 4 & 1\end{vmatrix}-1\begin{vmatrix}1 & 1\\ 4 & -2\end{vmatrix}=3\cdot5+2\cdot7+6=35$

Now we shall give geometrical interpretation to a determinant.

§ 3. GEOMETRICAL REPRESENTATION OF A DETERMINANT

We shall consider the ordered triple of noncoplanar free vectors $\vec{a},\vec{в},\vec{c}$ and we shall put it in conformity with the ordered triple of the directed segments $\overrightarrow{DA},\overrightarrow{DB},\overrightarrow{DC}$ originated from one point in the oriented space. On these directed segments as on the sides, we shall construct a parallelepiped (fig. 2.6).

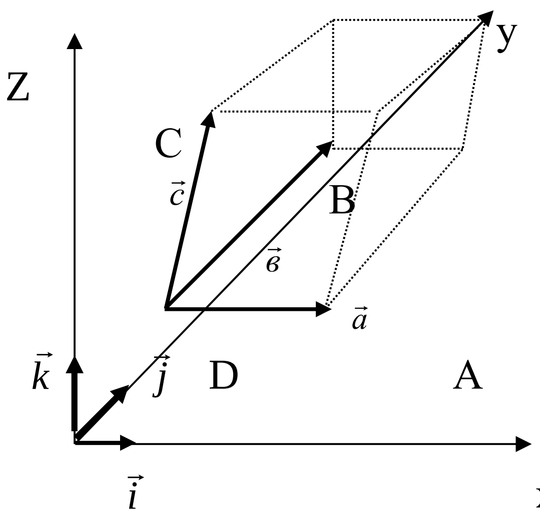


Fig.2.6

There is an infinite set of oriented parallelepipeds, each of them is put in conformity the same ordered triple three $\vec{a},\vec{b},\vec{c}$ of vectors. These parallelepipeds turn out carryovers of any of them and have on this the same volume V_{p}. If vectors are coplanar, the volume of such ***degenerate*** parallelepiped is assumed to be equal to zero.

Let's determine volume V_p of the parallelepiped constructed on vectors $\vec{a},\vec{b},\vec{c}$, in coordinates. For this purpose we shall choose in space an orthonormal basis $\vec{i},\vec{j},\vec{k}$, and connect with it system of coordinates x, y, z (fig. 2.6). And let three vectors specified by their coordinates be given concerning this basis: $\vec{a} = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$; $\vec{b} = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$; $\vec{c} = x_3\vec{i} + y_3\vec{j} + z_3\vec{k}$. We shall introduce two operations on free vectors.

3.1. Vector product of two free vectors

Definition. Vector product of two vectors $\vec{a}$ and $\vec{b}$ is referred to as vector $\vec{p}$ so, that a) $|\vec{p}| = |\vec{a}||\vec{b}|\sin\varphi$, where φ - an angle between vectors $\vec{a}$ and $\vec{b}$, b) $\vec{p}\perp\vec{a}$ and $\vec{p}\perp\vec{b}$, c) if $\vec{p}\neq\vec{0}$, then vectors $\vec{a},\vec{b},\vec{p}$ form the right triple. Vector product is designated $[\vec{a}\times\vec{b}]$.

According to condition a) $\vec{p} = \vec{0}$ only if, vectors $\vec{a}$ and $\vec{b}$ are collinear. Therefore for a set of vectors of the space R' vector product will consist only of one zero vector. If $\vec{p}\neq\vec{0}$, then $|\vec{p}|$ is numerically equal to the area of the parallelogram constructed on vectors $\vec{a}$ and $\vec{b}$, reduced to common origin (fig. 2.7). It should be noted, that as against the scalar product $(\vec{a}\cdot\vec{b})$, which is a mapping $R^3\times R^3$ into R, vector product, as well as addition, represents the internal law of a composition for space of free vectors R^3.

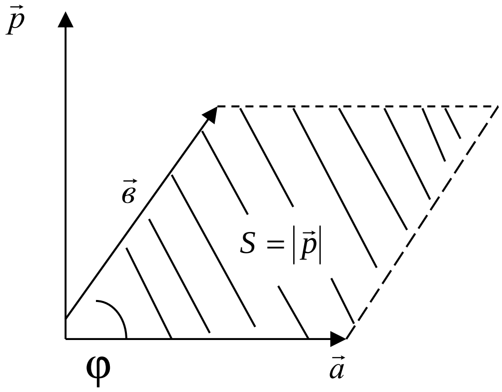


Fig. 2.7

The basic properties of vector product are reduced to the following:

1. $[\vec{a}\times\vec{в}] = -[в\times a]$ – is noncommutative;
2. $\lambda[\vec{a}\times\vec{в}] = [\lambda\vec{a}\times\vec{в}] = [\vec{a}\times\lambda\vec{в}]$;
3. $[\vec{a}\times(\vec{в}+\vec{c})] = [\vec{a}\times\vec{в}] + [\vec{a}\times\vec{c}]$ – is distributive relative to the addition.
4. Neutral element does not exist.

Let's consider how the vector product is represented in coordinate form.

$$\vec{p} = [\vec{a}\times\vec{в}] = [(x_1\vec{i} + y_1\vec{j} + z_1\vec{k})\times(x_2\vec{i} + y_2\vec{j} + z_2\vec{k})].$$

We oped the brackets taking into account that

$[\vec{i}\times\vec{i}] = [\vec{j}\times\vec{j}] = [\vec{k}\times\vec{k}] = \vec{0}$, a

$[\vec{i}\times\vec{j}] = \vec{k}, [\vec{j}\times\vec{i}] = -\vec{k}, [\vec{j}\times\vec{k}] = \vec{i}, [\vec{k}\times\vec{j}] = -\vec{i}, [\vec{k}\times\vec{i}] = \vec{j}, [\vec{i}\times\vec{k}] = -\vec{j},$ we obtain

$\vec{p} = (y_1z_2 - y_2z_1)\vec{i} + (z_1x_2 - x_1z_2)\vec{j} + (x_1y_2 - y_1x_2)\vec{k}.$

Hence $\vec{p} = \begin{vmatrix} y_1 z_1 \\ y_2 z_2 \end{vmatrix}\vec{i} - \begin{vmatrix} x_1 z_1 \\ x_2 z_2 \end{vmatrix}\vec{j} + \begin{vmatrix} x_1 y_1 \\ x_2 y_2 \end{vmatrix}\vec{k} = \begin{vmatrix} \vec{i} \ \vec{j} \ \vec{k} \\ x_1 y_1 z_1 \\ x_2 y_2 z_2 \end{vmatrix}$, (6.2)

here $\begin{vmatrix} y_1 z_1 \\ y_2 z_2 \end{vmatrix} = x_4$; $-\begin{vmatrix} x_1 z_1 \\ x_2 z_2 \end{vmatrix} = y_4$; $\begin{vmatrix} x_1 y_1 \\ x_2 y_2 \end{vmatrix} = z_4$, coordinates of vector $\vec{p} = x_4\vec{i} + y_4\vec{j} + z_4\vec{k}$.

3.2. The mixed product of three free vectors

Definition. If we multiply a vector $\vec{p} = [\vec{a}\times\vec{в}]$ scalar by vector $\vec{c}$, the obtained number is referred to as ***the mixed product*** of three vectors $\vec{a}, \vec{в}$ and $\vec{c}$. It is designated $[\vec{a}\times\vec{в}]\cdot\vec{c}$.

It is not difficult to show, that absolute value of the mixed product of three vectors is equal to volume V_p of the parallelepiped constructed on these vectors, i.e. $|[\vec{a}\times\vec{в}]\cdot\vec{c}| = V_p$ Really, $|\ \vec{p}\ | = |[\vec{a}\times\vec{в}]|$- is area S of the parallelogram constructed on vectors $\vec{a}$ and $\vec{в}$, and $\left|\ \frac{\vec{p}\cdot\vec{c}}{|\vec{p}|}\ \right| = |\vec{c}|\ |\ \cos(\vec{p}\vec{c})\ |$- is height h of a parallelepiped which basis is a parallelogram with area S

since $\vec{p}\perp\vec{a}$ и $\vec{p}\perp\vec{в}$. Hence, $\left|[\vec{a}\times\vec{в}]\cdot\vec{c}\right| = \left|\,|\vec{p}|\cdot|\vec{c}|\;\cos(\vec{p}\,\vec{c})\right| = S\cdot h = V_p$ - volume of a parallelepiped.

Let's express the mixed product $[\vec{a}\times\vec{в}]\cdot\vec{c}$ (and volume V_p of a parallelepiped) through coordinates of vectors. Taking into account (6.2), and also, that $\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1$ and $\vec{i}\cdot\vec{j} = \vec{i}\cdot\vec{k} = \vec{j}\cdot\vec{k} = 0$ we obtain

$$[\vec{a}\times\vec{в}]\cdot\vec{c} = \vec{p}\cdot\vec{c} = \left(\begin{vmatrix} y_1 z_1 \\ y_2 z_2 \end{vmatrix}\vec{i} - \begin{vmatrix} x_1 z_1 \\ x_2 z_2 \end{vmatrix}\vec{j} + \begin{vmatrix} x_1 y_1 \\ x_2 y_2 \end{vmatrix}\vec{k}\right)\cdot\left(x_3\vec{i} + y_3\vec{j} + z_3\vec{k}\right) =$$

$$= x_3\begin{vmatrix} y_1 z_1 \\ y_2 z_2 \end{vmatrix} - y_3\begin{vmatrix} x_1 z_1 \\ x_2 z_2 \end{vmatrix} + z_3\begin{vmatrix} x_1 y_1 \\ x_2 y_2 \end{vmatrix} = \begin{vmatrix} x_1 y_1 z_1 \\ x_2 y_2 z_2 \\ x_3 y_3 z_3 \end{vmatrix} \text{ and } V_p = \left|\begin{vmatrix} x_1 y_1 z_1 \\ x_2 y_2 z_2 \\ x_3 y_3 z_3 \end{vmatrix}\right|$$

Thus, absolute value of a determinant of the third order is equal to volume of the parallelepiped constructed on three vectors which coordinates in unite orthonormal basis $\vec{i},\vec{j},\vec{k}$ are row – vectors of a corresponding matrix and, accordingly, elements of rows of a determinant. Basically, vector coordinates can be placed in columns of a matrix (determinant), since value of a determinant in transposing of a matrix does not change. Hence, we can make the following ***conclusion***.

For three vectors to be coplanar, it is necessary and sufficient that the determinant of the matrix specified in coordinates of these vectors, in orthonormal basis be equal to zero.

The concept of a parallelepiped and a determinant as its volume, is distributed to the vector space R^n, which dimension $n > 3$. Similar formation from n vectors of the space R^n and a set of points of this space, enclosed in borders of these vectors which are considered as volume and limited to these vectors, is referred to as ***parallelotope.***

Let parallelotope be formed by n vectors $\vec{a}_1, \vec{a}_2 ... \vec{a}_n$, which decomposition by canonical basis $\vec{\ell}_1, \vec{\ell}_2, ... \vec{\ell}_n$ of the space R^n is of the form of $\vec{a}_j = \alpha_{1j}\vec{\ell}_1 + \alpha_{2j}\vec{\ell}_2 + ... + \alpha_{nj}\vec{\ell}_n$, $j = 1,2,...,n$, then the volume V_p of such parallelotope is equal to absolute value of determinant *D (A)*, where *A* – is a square matrix which $\vec{a}_j$ are column – vectors (row - vectors), i. e.

$$V_p = |\ D(A)\ | = \begin{Vmatrix} \alpha_{11}\alpha_{12}\dots\alpha_{1n} \\ \alpha_{21}\alpha_{22}\dots\alpha_{2n} \\ \dots\dots\dots\dots \\ \alpha_{n1}\alpha_{n2}\dots\alpha_{nn} \end{Vmatrix} \text{ and } A = \begin{pmatrix} \alpha_{11}\alpha_{12}\dots\alpha_{1n} \\ \alpha_{21}\alpha_{22}\dots\alpha_{2n} \\ \dots\dots\dots\dots \\ \alpha_{n1}\alpha_{n2}\dots\alpha_{nn} \end{pmatrix} = (\vec{a}_1, \vec{a}_2 \dots \vec{a}_n)$$

§ 4. APPLICATION OF DETERMINANTS FOR THE DETERMINING OF A MATRIX RANK

We shall consider a matrix *A* above the field *P* which has size m×n and we shall present it as system of n column –vectors in the space P^m.

$$A = (\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n) = \begin{pmatrix} a_{11}a_{12}\dots a_{1n} \\ a_{21}a_{22}\dots a_{2n} \\ \dots\dots\dots\dots \\ a_{m1}a_{m2}\dots a_{mn} \end{pmatrix},$$

Elements of a matrix a_{ij} – numbers from *P*. To determine rank *r* of the given system of vectors or matrixes *A*, specified in the coordinates of these vectors, it is necessary to define possible greatest number of linearly independent vectors which can be chosen from this system, or, in other words, number of basic vectors of this system.

Before we shall consider a special case, when m = n. Let's show, that for such square matrix of the order n the following theorem is valid.

The theorem 1. For n of vectors $\vec{a}_1, \vec{a}_2, \dots a_n$ from P^n to be linearly independent, it is necessary and sufficient, that a determinant of a square matrix *A*, formed of coordinates of these vectors $D(A) = D(\vec{a}_1, \vec{a}_2, \dots \vec{a}_n) \neq 0$.

The proof. *Necessity*. Let $\vec{a}_1, \vec{a}_2, \dots a_n$ be linearly independent, the the matrix *A* - is invertible (book 2, Chapter.5, §5, item 5.1) that is, there is a matrix invertible to it - matrix A^{-1}, such, that $A \cdot A^{-1} = E$, where *E* - an identity matrix. Then having taken an advantage of property of determinant multiplication, we shall receive: $D(AA^{-1}) = D(A) \cdot D(A^{-1}) = D(E) = 1$, and, so, $D(A) \cdot D(A^{-1}) = 1$, hence $D(A) \neq 0$.

Sufficiency. The statement, that if $D(A) \neq 0$, then system of vectors is linearly independent, is equivalent to the statement, that if $D(A) = 0$, then the system of vectors is linearly dependent. We shall prove the last.

Since *D (A) =0* , then either one of rows or one of columns of a determinant are equal to zero, or two rows (columns) of a determinant are equal or proportional, and, at last, one of rows (columns) of a determinant is a linear combination of other rows (columns) of a determinant. For system of vectors $\vec{a}_1, \vec{a}_{2,} ... \vec{a}_n$ it means, that in system there is either a zero – vector, or two equal or proportional vectors, or a vector which is a linear combination of other vectors of system. In all these three cases as it follows from theorems of linearly dependent and linearly independent vectors, the system of vectors $\vec{a}_1, \vec{a}_{2,} ... \vec{a}_n$ will be linearly dependent, as was to be proved.

Thus, it follows from the above-stated theorem, that if determinant *D (A)* of the square matrix *A* of the order *n* is not equal to zero, then rank of the matrix *A* is equal *n*: *r (A) =n*. If *D (A) = 0,* then *r(A)< n.*

Now we shall generalize the obtained result to specify the process, allowing to determine the exact value of rank *r (A)* by means of determinants for a matrix *A* of any size. This process is based on the theorem for which we shall give only the formulation, and we shall omit the proof. But before we give the formulation of the theorem, we shall introduce a concept ***of the basic minor and minors bordering it*** for a matrix *A*.

Definition 1. A minor of the order *h* of the matrix *A* is referred to as the determinant from *h* rows and *h* columns which is obtained as a result of deletion of rows and columns of this matrix so that only *h* rows and *h* columns remained, or in other words, the minor is a determinant of a square matrix formed with of elements located at intersection of *h* various rows and *h* various columns of the initial matrix.

It is obvious, that the best order of a minor of a matrix in the size $m \times n$ is equal to the minimal number from *m* or *n*, $h_{max} = min(m, n)$.

Definition 2. If $h < min(m, n)$, then matrix of the order *h* can be added some *i*- rows and *i* – columns of the initial matrix where *i = 1,2,..., min(m, n)-h* and we can obtain the minors of higher orders *h + i*. Such minors are referred to as ***bordering*** for ***the basic*** minor *h*.

Definition 3. If as the basic minor of a matrix

$$A=\begin{pmatrix} a_{11}a_{12}....a_{1n} \\ a_{21}a_{22}....a_{2n} \\ .... \\ a_{m1}a_{m2}....a_{mn} \end{pmatrix},$$

we shall choose the minor of the 1-st order located in the upper left corner a_{11}, all minors bordering minors of the higher orders obtained by addition of the next rows and columns, are referred to as ***the main*** minors of a matrix *A*.

$$D_1(A)=a_{11}, D_2(A)=\begin{vmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{vmatrix}, D_3(A)=\begin{vmatrix} a_{11}a_{12}a_{13} \\ a_{21}a_{22}a_{23} \\ a_{31}a_{32}a_{33} \end{vmatrix}, \ldots, D_S(A)=\begin{vmatrix} a_{11}a_{12}...a_{1S} \\ a_{21}a_{22}...a_{2S} \\ .... \\ a_{S1}a_{S2}...a_{SS} \end{vmatrix},$$

where S = *min(m, n).*

The following theorem is valid.

The theorem 2. If there is a minor of the *r* - order which is not equal to zero in the matrix *A*, and all minors of the *(r+1)* - order, bordering this minor, are equal to zero, then *r* is a rank of the matrix *A*: *r = r (A)*. The minor of the order *r*, distinct from zero is referred to as ***basic.***

The remark. If all minors of the *(r+1)* - order are equal to zero, also all minors of higher orders also are equal to zero.

In view of this theorem the process of definition of a matrix rank is reduced to the following. It is necessary to choose in a matrix as the basic a minor of any order which is distinct from zero. Then it is necessary to calculate minors of higher orders which are bordering it. Then the highest order of the bordering minor which is distinct from zero, also will be a rank of a considered matrix

Example. Define a rank of a matrix $A=\begin{pmatrix} 1 & -2 & 1 & 3 \\ 1 & 3 & -1 & 1 \\ 3 & 4 & -1 & 5 \end{pmatrix}$. Let's choose as the basic, a minor of the 1-st order located in the upper left corner $|1| \neq 0$. A bordering minor bordering of the second order $\begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} = 5 \neq 0,$ and bordering minors of the third order

$$\begin{vmatrix} 1 & -2 & 1 \\ 1 & 3 & -1 \\ 3 & 4 & -1 \end{vmatrix} = 1\begin{vmatrix} 3 & -1 \\ 4 & -1 \end{vmatrix} + 2\begin{vmatrix} 1 & -1 \\ 3 & -1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = 1 + 4 - 5 = 0, \quad \begin{vmatrix} 1 & -2 & 3 \\ 1 & 3 & 1 \\ 3 & 4 & 5 \end{vmatrix} = 1\begin{vmatrix} 3 & 1 \\ 4 & 5 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} = 11 + 4 - 15 = 0,$$

hence, a matrix rank A: $r(A) = 2$.

The remark. If as the basic minor we choose other minor distinct from zero, but located in the other place of a matrix, the result will be the same.

§ 5. ARRAYING OF INVERSE MATRIX

We already saw, that for the matrix A to be convertible, it is necessary and sufficient that it be square and its rank $r(A)$ should be equal to the order n of the matrix A. Now, using a determinant of a matrix, we can formulate this statement as follows. For the square matrix A have the inverse matrix A^{-1}, it is necessary and sufficient, that its determinant $D(A) \neq 0$. Members of inverse matrix A^{-1} are defined by the formula:

$$A^{-1} = \begin{pmatrix} \frac{A_{11}}{D(A)} \; \frac{A_{21}}{D(A)} \ldots\ldots \frac{A_{n1}}{D(A)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{A_{1i}}{D(A)} \; \frac{A_{2i}}{D(A)} \ldots\ldots \frac{A_{ni}}{D(A)} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \frac{A_{in}}{D(A)} \; \frac{A_{2n}}{D(A)} \ldots\ldots \frac{A_{nn}}{D(A)} \end{pmatrix}.$$

Here, $D(A)$ – is a determinant of a matrix $A = (a_{ij})$, where i = 1,2,...,n, j = 1,2,...,n. A_{ij} – are algebraical complements of the member a_{ij} of the matrix A. It should be noticed that A_{ij} are not located on a place of the member a_{ij}, but they are located on a place of the member a_{ji}. Hence, matrix A^{-1} is transposed to a matrix $\left(\frac{A_{ij}}{D(A)}\right)$,which members A_{ij} are located on a place of members a_{ij} which algebraic complements they are, then

$$A^{-1} = \frac{1}{D(A)}\begin{pmatrix} A_{11}A_{21}\ldots\ldots A_{n1} \\ A_{12}A_{22}\ldots\ldots A_{n2} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ A_{1n}A_{2n}\ldots\ldots A_{nn} \end{pmatrix} = \frac{1}{D(A)}\begin{pmatrix} A_{11}A_{12}\ldots\ldots A_{1n} \\ A_{21}A_{22}\ldots\ldots A_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ A_{n1}A_{n2}\ldots\ldots A_{nn} \end{pmatrix}^T = \frac{1}{D(A)}(A_{ij})^T.$$

We shall prove, that arrayed matrix A^{-1} is inverse to A. For it we need to show, that $AA^{-1} = E$.

$$A\cdot A^{-1}=(a_{ij})\cdot\frac{1}{D(A)}(A_{ij})^T=\frac{1}{D(A)}\left(\sum_{e=1}^{n}a_{ie}A'_{\ell j}\right)=\frac{1}{D(A)}\left(\sum_{e=1}^{n}a_{ie}A_{j\ell}\right).$$

Members of the transposed matrix $A'_{\ell j}=A_{j\ell}$. It follows from the theorem of another's complements that if $i \neq j$, then

$$\sum_{\ell=1}^{n}a_{i\ell}A_{j\ell}=0, \text{ a } \sum_{\ell=1}^{n}a_{i\ell}A_{i\ell}=D(A).$$

We obtained a diagonal matrix with equal members on the main diagonal, and it is a scalar matrix, therefore

$$AA^{-1}=\frac{1}{D(A)}\begin{pmatrix} D(A)......0......0 \\ .... \\ 0..........D(A)..0 \\ .... \\ 0............0...D(A)\end{pmatrix}=\frac{D(A)}{D(A)}\begin{pmatrix}1....0....0 \\ .... \\ 0....1....0 \\ .... \\ 0....0....1\end{pmatrix}=E.$$

Example. Define a matrix, inverse for a matrix $A=\begin{pmatrix}2 & 5 & 7\\ 6 & 3 & 4\\ 5 & -2 & -3\end{pmatrix}$. First let’s show that the given matrix has inverse matrix . $D(A)=\begin{vmatrix}2 & 5 & 7\\ 6 & 3 & 4\\ 5 & -2 & -3\end{vmatrix}=-1.$ Since $D(A) \neq 0$, then the given matrix has inverse one. Let’s calculate algebraical complements:

$$A_{11}=(-1)^{1+1}\begin{vmatrix}3 & 4\\ -2 & -3\end{vmatrix}=-1,\quad A_{12}=(-1)^{1+2}\begin{vmatrix}6 & 4\\ 5 & -3\end{vmatrix}=38,\quad A_{13}=-27,$$

$$A_{21}=1,\ A_{22}=-41,\ A_{23}=29,\ A_{31}=-1, A_{32}=34,\ A_{33}=-24.$$

Thus,

$$A^{-1}=\frac{1}{-1}\begin{pmatrix}-1 & 1 & -1\\ 38 & -41 & 34\\ -27 & 29 & -24\end{pmatrix}=\begin{pmatrix}1 & -1 & 1\\ -38 & 41 & -34\\ 27 & -29 & 24\end{pmatrix}.$$

Let’s test

$$AA^{-1}=\begin{pmatrix}2 & 5 & 7\\ 6 & 3 & 4\\ 5 & -2 & -3\end{pmatrix}\cdot\begin{pmatrix}1 & -1 & 1\\ -38 & 41 & -34\\ 27 & -29 & 24\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

EXERCISES

1. Solve the equations

$$\begin{vmatrix}x & 1 & 1\\ 1 & x & 1\\ 1 & 1 & x\end{vmatrix}=0;\qquad \begin{vmatrix}x & -2 & 2\\ 1 & x & -1\\ 1 & -x & 1\end{vmatrix}=0.$$

2. Are points A(1,1,2), B(-2,1,2), C(3,0,2), Д(2,2,1) lying in one plane?

3. Prove that addition the members of any determinant column to corresponding members of other column of the same determinant, multiplied by the same number which is not equal to zero, does not change volume of a determinant.

4. Do vectors $\vec{a}(3,2,1)$, $\vec{в}(1,-1,-2)$ and $\vec{c}(0,3,1)$ form the basis of vector space R^3? If they do, determine the coordinates of a vector $\vec{d}$ (1,2,3) in this basis.

5. Vectors are given: $\vec{a}=1\vec{i}-2\vec{j}+2\vec{k}$; $\vec{в}=3\vec{i}-4\vec{k}$. Define their vector product, angler between them and the area of the parallelogram constructed on these vectors.

6. To calculate volume V_p of the parallelepiped constructed on vectors: $\vec{a}(3,4,6)$, $\vec{в}(4,1,1)$, $\vec{c}(2,0,3)$.

7. To define a matrix rank

$$A=\begin{pmatrix}-2 & 4 & 2 & 3 & 1\\ 1 & 6 & 3 & 2 & 2\\ 2 & 12 & 6 & 4 & 4\\ 3 & -2 & -1 & 5 & 3\end{pmatrix}$$

8. Is matrix $A=\begin{pmatrix}2 & 1 & 0 & 5\\ 3 & 3 & 1 & 4\\ 1 & 1 & 0 & 0\\ 2 & 1 & 4 & 6\end{pmatrix}$ invertible? If it is, define its inverse matrix.

CHAPTER 7
LINEAR EQUATION SYSTEMS

§ 1. DEFINITIONS. CONSISTENT AND INCONSISTENT SYSTEMS

Linear system k of the equations with n unknown $x_1, x_2, \ldots, x_n$, is referred to as a set of equalities

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = в_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = в_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{к1}x_1 + a_{к2}x_2 + \ldots + a_{кn}x_n = в_к \end{cases} \qquad (7.1)$$

Coefficients a_{ij} and ***free members*** $в_i$, $i = 1,2,..., к$, $j = 1,2,..., n$ – are known and belong to the field R of real numbers or to the field C of complex numbers. Further we shall consider the field R of real numbers as this field.

Solve system (7.1) means to determine the ordered set of numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ from R (or C) so, that in substituting at replacement $x_1, x_2, \ldots, x_n$ for $\lambda_1, \lambda_2, \ldots, \lambda_n$, accordingly, each equation of the system becomes correct equality. The ordered set of numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$, is referred to as ***solution of system*** (7.1).

The system of the linear equations is referred to as ***consistent*** if it has solutions, and, ***inconsistent*** if it has no solutions.

If two consistent systems have identical solutions, such systems are referred to as ***equivalent systems*** .

The consistent system of the linear equations is referred to as ***certain*** if it has only one solution and to ***uncertain*** if there is a set of solutions.

Gaussian method gives answers to these questions.

§ 2. GAUSSIAN METHOD

With system (7.1) of linear equations it is possible to make the following operations which do not break equivalence of system of the equations:

a) to add to both parts of the equation corresponding parts of other equation multiplied by on some number;

b) to permute the equations in system;

c) to exclude from system the equations $0x_1 + 0x_2 + \ldots + 0x_n = 0$. As this equality is identity, and any values $x_1, x_2, \ldots, x_n$ satisfy it.

With the help of these operations any system of the linear equations can be reduced to triangular

$$\begin{aligned} c_{11}x_1 + c_{12}x_2 + \ldots + c_{1r}x_n + \ldots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \ldots + c_{2r}x_r + \ldots + c_{2n}x_n &= d_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{rr}x_r + \ldots + c_{rn}x_n &= d_r \\ \ldots\ldots\ldots\ldots\ldots \\ + c_{nn}x_n &= d_n \end{aligned} \qquad (7.2)$$

or trapezoid form

$$\begin{aligned} c_{11}x_1 + c_{12}x_2 + \ldots + c_{1r}x_n + \ldots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \ldots + c_{2r}x_r + \ldots + c_{2n}x_n &= d_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{rr}x_r + \ldots + c_{rn}x_n &= d_r. \end{aligned} \qquad (7.3)$$

At reduction of system to triangular or trapezoid form there can be equations $0x_i + 0x_{i+1} + \ldots + 0x_n = d_i$, $i = 1,2, \ldots, n$ If $d_i = 0$, these equations are identities and they are excluded from system, but if $d_i \neq 0$, then this equation is not satisfied with any values x_j. In this case the system has no solutions, it is inconsistent.

The consistent system of the equations reduced to a triangular kind (7.2) has the unique solution and, hence, it is certain. If the consistent system is reduced to trapezoid kind (7.3), and $r < n$, then giving to $x_{r+1}, x_{r+2}, \ldots, x_n$ any values, from system (7.3) we can define $x_1, x_2, \ldots, x_r$ and construct the solution of system. However, taking into account, that $x_{r+1}, x_{r+2}, \ldots, x_n$ can take any values from *R*, we obtain uncertain system, and number of its solutions is an infinite set. Unknown which take any values, are referred to as ***free, auxiliary, independent*** and their quantity is equal to

n - r.

Examples

1. Solve the system by Gaussian method

$$\begin{cases} 4x_1 + 2x_2 + x_3 = 4 \end{cases}$$

$x_1 + 3x_2 + 2x_3 = 2$
$2x_1 - x_2 + x_3 = 5.$

Let's exclude from the 2 - nd and 3 – rd equations of the given system the unknown x_1. For this purpose we multiply the second equation by **-4**, and the third equation by **-2** and add to the first one:

$$4x_1 + 2x_2 + x_3 = 4$$
$$-10x_2 - 7x_3 = -4$$
$$4x_2 - x_3 = -6.$$

Now we shall multiply the third equation of the obtained system by 5/2 and we shall add the second equation to it:

$$4x_1 + 2x_2 + x_3 = 4$$
$$-10x_2 - 7x_3 = -4$$
$$-\frac{19}{2}x_3 = -19$$

The system is reduced to a triangular kind. From last equation of system we define $x_3 = 2$, from the second $x_2 = -1$, from the first $x_1 = 1$. The system has the unique solution (1,-1, 2).

2. The system is given

$$\begin{cases} 2x_1 - x_2 + x_4 = 4 \\ 4x_1 - 2x_2 + x_3 + x_4 = 7 \\ 6x_1 - 3x_2 + 2x_3 - x_4 = 8 \end{cases}$$
$$8x_1 - 4x_2 + 3x_3 - x_4 = 11$$

The remark. In solving the system by Gaussian method the unknowns in the equations of system can be excluded not only from the beginning, but also from the end.

Thus we do in solving of the given system. For this purpose we shall multiply the last equation by 1, 1,-1 consistently, and we shall add it with three first ones; we shall obtain an equivalent system

$$\begin{cases} 8x_1 - 4x_2 + 3x_3 - x_4 = 11 \\ 10x_1 - 5x_2 + 3x_3 = 15 \\ 12x_1 - 6x_2 + 4x_3 = 18 \end{cases}$$
$$-2x_1 + x_2 - x_3 = -3$$

Now we shall multiply the last equation by 3 and by 4 consistently, and we shall add it to two previous ones; we shall obtain an equivalent system:

$$\begin{cases} 8x_1 - 4x_2 + 3x_3 - x_4 = 11 \\ -2x_1 + x_2 - x_3 = -3 \\ 4x_1 - 2x_2 = 6 \\ 4x_1 - 2x_2 = 6 \end{cases}$$

Then, we shall multiply the penultimate equation by-1 and add it to the last equation, we have:

$$\begin{cases} 8x_1 - 4x_2 + 3x_3 - x_4 = 11 \\ -2x_1 + x_2 - x_3 \quad -3 \\ 4x_1 - 2x_2 = 6 \\ 0x_1 - 0x_2 = 0 \end{cases}$$

Last equation is identity and it can be excluded from system. Finally

$$\begin{cases} 8x_1 - 4x_2 + 3x_3 - x_4 = 11 \\ -2x_1 + x_2 - x_3 = -3 \\ 2x_1 - x_2 = 3 \end{cases}$$

Thus, the system is reduced to resulted to a trapezoid kind. If we suppose x_1 the auxiliary unknown and give to it any values, for example, β, we find the solution of system (β, 2β-3, 0, 1). Since β can take any values from *R*, the system is not certain and it has infinitely many solutions.

§ 3. MATRIX AND VECTOR FORMS OF NOTATION OF LINEAR EQUATION SYSTEMS. KRONECKER-CAPELLI THEOREM

It is possible to connect the following matrixes with system (7.1) of linear equations:

1. Matrix *A* of coefficients a_{ijj} if unknowns of the system are x_1, x_2, . . . , x_n.

$$A = \begin{pmatrix} a_{11}a_{12}...a_{1n} \\ a_{21}a_{22}...a_{2n} \\ .... \\ a_{k1}a_{k2}...a_{kn} \end{pmatrix} = (a_{ij}), \quad i = 1,2,...k, \quad \text{j} = 1,2,...\text{n}.$$

This matrix is named ***basic matrix.***

2. If we add a column of free members $в_1, в_2, \dots в_к$ of the system to the basic matrix A we shall obtain the so-called ***expanded*** matrix A^* of the given system

$$A^* = \begin{pmatrix} a_{11}a_{12}\dots a_{1n}в_1 \\ a_{21}a_{22}\dots a_{2n}в_2 \\ \dots\dots\dots\dots\dots \\ a_{k1}a_{k2}\dots a_{kn}в_к \end{pmatrix}.$$

3. Matrix – column of free members $B = \begin{pmatrix} в_1 \\ в_2 \\ \dots \\ в_к \end{pmatrix}$, matrix format $к\times 1$.

4. Matrix – column of unknowns

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}, \text{ matrix format } n\times 1.$$

Using definition of matrix product, system (7.1) can be written down as

$$AX = B \qquad (7.4)$$

This form of notation of the linear equations system is referred to as ***matrix***. If thus we consider the matrix A as some mapping of the space R^n into R^k, and if we associate matrixes X and B with column - vectors $\vec{x} \in R^n$ and $\vec{в} \in R^k$.accordingly. Then the solution of system (7.1) can be reduced to a problem of determining of vectors $\vec{x}_j \in R^n$,which are prototypes of a vector $\vec{в} \in R^k$ if mapping R^n into R^k, set by a matrix A, i.e. $A(\vec{x}_j) = \vec{в}$.

Besides of matrix, the system of the linear equations can be written down also in the vector form. For this purpose a matrix A is connect with system from n column - vectors $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ in the space $R^к$.

$$A = \begin{pmatrix} a_{11}a_{12}...a_{1n} \\ a_{21}a_{22}...a_{2n} \\ .... \\ a_{k1}a_{k2}...a_{kn} \end{pmatrix} = (\vec{a}_1, \vec{a}_2, ..., \vec{a}_n), \quad \vec{a}_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ ... \\ a_{kj} \end{pmatrix}, \quad j = 1,2,...n.$$

Then the system (7.1) will become $\vec{a}_1 x_1 + \vec{a}_2 x_2 + ... + \vec{a}_n x_n = \vec{в},$ (7.5)

here $\vec{в} = \begin{pmatrix} в_1 \\ в_2 \\ ... \\ в_к \end{pmatrix} \in R^k$.

In terms of the equation (7.5) the problem of solution of system (7.1) can be reduced to a problem of determining of linear dependence of vector system $\vec{a}_1, \vec{\vec{a}}_2, ..., \vec{a}_n, \vec{в}$. So the system (7.1) has the solution if the vectors $\vec{a}_1, \vec{\vec{a}}_2, ..., \vec{a}_n, \vec{в}$ are linearly dependent. Really, it follows from (7.5), that the vector $\vec{в}$ is a linear combination of vectors $\vec{a}_1, \vec{\vec{a}}_2, ..., \vec{a}_n$ and, hence, it belongs to the subspace, generated by vectors $\vec{a}_1, \vec{\vec{a}}_2, ..., \vec{a}_n$. If the vector $\vec{в}$ does not belong to the subspace, generated by vectors $\vec{a}_1, \vec{\vec{a}}_2, ..., \vec{a}_n$, i. e. vectors $\vec{a}_1, \vec{\vec{a}}_2, ..., \vec{a}_n, \vec{в}$ are linearly independent, the system (7.1) has no solutions. In other words the system (7.1) has the solution if the rank $r^*(A^*)$ of vector system of vectors $\vec{a}_1, \vec{\vec{a}}_2, ..., \vec{a}_n, \vec{в}$ does not exceed the rank $r(A)$ of vector system $\vec{a}_1, \vec{\vec{a}}_2, ..., \vec{a}_n$, and it means, that they should be equal. Now if we connect system of vectors $\vec{a}_1, \vec{\vec{a}}_2, ..., \vec{a}_n, \vec{в}$ with expanded matrix A^*, then the aforesaid can be considered as the proof of the following theorem.

Kronecker-Capelli Theorem* (*a consistency condition of the linear equation system): the linear equation system is solvable (consistent), only if the rank $r\ (A)$ of the basic matrix A is equal to the rank $r^*(A^*)$ of the expanded matrix A^* : $r(A) = r^*(A^*)$.

§ 4. KRAMER'S SYSTEM

We suppose, that the number of the equations in system (7.1) is equal to number of unknowns ($k = n$) and that column – vectors $\vec{a}_1, \vec{a}_2 ... \vec{a}_n$ from R^n are linearly independent; in this case (7.1) is referred to as ***Kramer's system.***

Since column – vectors $\vec{a}_1, \vec{a}_2 ... \vec{a}_n$ are linearly independent, they form basis of the space R^n, hence, any column - vector $\vec{в} \in R^n$ is represented by unique way, in the form (7.5). Thus, Kramer's system always has the solution, and moreover it is unique.

For defining of this solution we shall write down Kramer's system in the matrix form (7.4): $AX = B$. Basic matrix A of the Kramer's systems – is square, of the order n, and its determinant is distinct from zero: $D(A) \neq 0$, since column – vectors of a matrix are linearly independent. Therefore the matrix A has inverse matrix A^{-1}. We shall multiply both parts of the equation (7.4) by A^{-1} from the left:

$A^{-1}AX = A^{-1}B.$

Since $A^{-1}A = E$ and $EX = X$, then $X = A^{-1}B$ or

$$\begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} = \frac{1}{D(A)} \begin{pmatrix} A_{11} A_{21} ... A_{n1} \\ A_{12} A_{22} ... A_{n2} \\ .... \\ A_{1n} A_{2n} ... A_{nn} \end{pmatrix} \cdot \begin{pmatrix} в_1 \\ в_2 \\ . \\ в_n \end{pmatrix}.$$

Multiplying A^{-1} by B, we obtain

$$\begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix} = \frac{1}{D(A)} \begin{pmatrix} A_{11}в_1 + A_{21}в_2 + ... + A_{n1}в_n \\ A_{12}в_1 + A_{22}в_2 + ... + A_{n2}в_n \\ .... \\ A_{1n}в_1 + A_{2n}в_2 + ... + A_{nn}в_n \end{pmatrix} \qquad (7.6)$$

Hence $x_j = \dfrac{1}{D(A)} \left(A_{ij}в_1 + A_{2j}в_2 + ... + A_{nj}в_n \right),$

where $j=1, 2,...n,$ and $A_{1j}в_1 + A_{2j}в_2 + ... + A_{nj}в_n$ – a matrix determinant which is obtained from the basic A by substituting of members j- column,

i. e. coefficients at the determined unknown x_j for column of free members $в_1$, $в_2$,...$в_n$ of system. Thus,

$$x_j = \frac{\begin{vmatrix} a_{11}\dots a_{1(j-1)}в_1 a_{1(j+1)}\dots a_{1n} \\ a_{21}\dots a_{2(j-1)}в_2 a_{2(j+1)}\dots a_{2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\dots a_{n(j-1)}в_n a_{n(j+1)}\dots a_{nn} \end{vmatrix}}{\begin{vmatrix} a_{11}\dots a_{1(j-1)}a_{1j} a_{1(j+1)}\dots a_{1n} \\ a_{21}\dots a_{2(j-1)}a_{2j} a_{2(j+2)}\dots a_{2n} \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}\dots a_{n(j-1)}a_{nj} a_{n(j+1)}\dots a_{nn} \end{vmatrix}} = \frac{|A_j|}{|A|} \ .$$

Now all aforesaid we shall formulate as the following rule.

Kramer's rule. If determinant *D (A)* the basic matrix *A* of the system of *n* linear equations with *n* unknowns is distinct from zero *(D(A)≠0),* then system has the unique solution and this solution is defined by the formula:

$$x_j = \frac{D(A_j)}{D(A)}, \ j=1,2, \dots, n, \qquad (7.7)$$

where $D(A_j)$ – is a determinant obtained from *D(A)* by substituting *j*- column for column of free members of system.

An example. Solve system of the equations.

$$\begin{cases} 3x\text{-}3y+2z=2, \\ 4x\text{-}5y+2z=1, \\ 5x\text{-}6y+4z=3. \end{cases}$$

Let's calculate a determinant of the basic matrix *A*:

$$D(A) = \begin{vmatrix} 3 & -3 & 2 \\ 4 & -5 & 2 \\ 5 & -6 & 4 \end{vmatrix} = 3\begin{vmatrix} -5 & 2 \\ -6 & 4 \end{vmatrix} + 3\begin{vmatrix} 4 & 2 \\ 5 & 4 \end{vmatrix} + 2\begin{vmatrix} 4 & -5 \\ 5 & -6 \end{vmatrix} = -24 + 18 + 2 = -4 .$$

Since *D(A) ≠ 0*, then this is Kramer's system and, hence, it has one solution which we determine be the formula:

$$x_j = \frac{D(A_j)}{D(A)}, \ j = 1, 2, 3.$$

$$x_1=\frac{\begin{vmatrix}2&-3&2\\1&-5&2\\3&-6&4\end{vmatrix}}{-4}=\frac{2\begin{vmatrix}-5&2\\-6&4\end{vmatrix}+3\begin{vmatrix}1&2\\3&4\end{vmatrix}+2\begin{vmatrix}1&-5\\3&-6\end{vmatrix}}{-4}=\frac{-16-6+18}{-4}=\frac{-4}{-4}=1;$$

$$x_2=\frac{\begin{vmatrix}3&2&2\\4&1&2\\5&3&4\end{vmatrix}}{-4}=\frac{3\begin{vmatrix}1&2\\3&4\end{vmatrix}-2\begin{vmatrix}4&2\\5&4\end{vmatrix}+2\begin{vmatrix}4&1\\5&3\end{vmatrix}}{-4}=\frac{-4}{-4}=1;$$

$$x_3=\frac{\begin{vmatrix}3&-3&2\\4&-5&1\\5&-6&3\end{vmatrix}}{-4}=\frac{3\begin{vmatrix}-5&1\\-6&3\end{vmatrix}+3\begin{vmatrix}4&1\\5&3\end{vmatrix}+2\begin{vmatrix}4&-5\\5&-6\end{vmatrix}}{-4}=\frac{-4}{-4}=1.$$

Answer: $x_1 = x = 1$; $x_2 = y = 1$; $x_3 = z = 1$.

§ 5. HOMOGENEOUS SYSTEM OF THE LINEAR EQUATIONS

The system of the linear equations is referred to as homogeneous if the right parts of these equations are equal to zero:

$$\begin{cases} a_1x_1 + a_{12}x_2 + \dots + a_{1n}x_n = 0, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n = 0. \end{cases} \qquad (7.8)$$

The homogeneous system is always consistent, since the expanded matrix differs from the basic one with a column representing a zero - vector. As the system containing a zero - vector, is always linearly dependent, the rank of the expanded matrix coincides with a rank of the basic matrix. Consistency of homogeneous system is obvious, as it always has the trivial solution $x_1 = x_2 = \dots = x_n = 0$. This solution will be the unique if the homogeneous system is Kramer's system i.e. when $k = n$ and determinant $D(A)$ of basic matrix A is distinct from zero. In other words, when the rank $r(A)$ of the basic matrix is equal to number n of the unknowns of system: $r(A) = n$. If $r(A) < n$, the homogeneous system of the linear equations has uncountable set of solutions and a set of solutions of system forms a vector subspace. We shall show it. For this purpose we shall write down system (7.8) in the vector form in space R^n of row - vectors. In this case each equation of system represents scalar product of two vectors

from R^n: $\vec{a}_i = (a_{i1}, a_{i2},..., a_{in}), i = 1,2,..k$ и $\vec{x} = (x_1, x_2,...x_n)$:

$$\begin{cases} \vec{a}_1 \cdot \vec{x} = 0 \\ \vec{a}_2 \cdot \vec{x} = 0 \\ \dots\dots\dots\dots \\ \vec{a}_k \cdot \vec{x} = 0 \end{cases} \qquad (7.9)$$

Let's prove, that if vectors $\vec{x}^0 = \left(x_1^0, x_2^0,....x_n^0\right)$ and $\vec{y}^0 = \left(y_1^0, y_2^0,...y_n^0\right)$ are solutions of system (7.9) then $\vec{x}^0 + \vec{y}^0$ and $\lambda\vec{x}^0$ will be solutions of this system. Really, since scalar product is distributive relative to addition of vectors also is associative relative to multiplication by number, we have:

$$\vec{a}_i\left(\vec{x}^0 + \vec{y}^0\right) = \vec{a}_i\vec{x}^0 + \vec{a}_i\vec{y}^0 = 0,$$
$$\vec{a}_i\left(\lambda\vec{x}^0\right) = \lambda\left(a_i\vec{x}^0\right) = 0, \quad i = 1, 2,..., k.$$

This implyies that $\vec{x}^0 + \vec{y}^0$ and $\lambda\vec{x}^0$ are also solutions of homogeneous system. Besides neutral *(*0,0...., 0*)* and symmetric $\left(-x_1^0, -x_2^0,....-x_n^0\right)$ elements also belong to the space of solutions. Thus, a set of solutions of homogeneous system forms a vector subspace. Now we shall define subspace dimension of system solutions we shall construct its basis. As we have already mentioned, a subspace of solutions contains nonzero vectors, if $r(A) < n$. The condition $r(A) < n$ is always satisfied, if the number *k* of the system equations is less than number *n* of unknowns. That fact, that a rank of basic matrix *A* is equal to $r(A)$, means, that matrix *A* contains a minor of the order *r*, distinct from zero; nevertheless minors of higher orders are equal to zero, including (if it exists) minor of the order *n*. Without limiting a generality, we can consider that this minor is the main minor of matrix *A* of the order *r*.

$$D_r(A) = \begin{vmatrix} a_{11}a_{12}...a_{1r} \\ a_{21}a_{22}...a_{2r} \\ .... \\ a_{r1}a_{r2}...a_{rr} \end{vmatrix} \neq 0 .$$

We can always obtain this, by permutation of the equations in system. Then the others *κ-r* equations of system are linear combinations of the first *r* equations of system and consequently, without breaking equivalence of the system, these equations can be excluded from the system. The rest *r* equations of the system we shall write down in the following form

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1r}x_r = -a_{1(r+1)}x_{r+1} - ... - a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2r}x_r = -a_{2(r+1)}x_{r+1} - ... - a_{2n}x_n \\ .... \\ a_{r1}x_1 + a_{r2}x_2 + ... + a_{rr}x_r = -a_{r(r+1)}x_{r+1} - ... - a_{rn}x_n \end{cases} . \qquad (7.10)$$

Let's notice, that if we give some numerical values to the unknowns $x_{r+1}, . . ., x_n$ in the system (7.10) we shall receive Kramer's system since $D_r(A) \neq 0$ and, hence, other unknowns $x_1, x_2, . . ., x_r$ can be determined unequivocally by Kramer's rule (7.7). Let's define unknowns $x_1, x_2, . . ., x_r$ giving consistently following values (1,0..., 0), (0,1,0..., 0)..., (0,0..., 1) for unknowns $x_{r+1}, x_{r+2}, . . ., x_n$. Such choice is caused by that each set from *n-r* numbers is a vector of canonical basis of the space R^{n-r}. Let's suppose, that for each specified set of values $x_{r+1}, x_{r+2}, . . ., x_n$ for $x_1, x_2, . . ., x_r$ the following *n-r* sets from *r* numbers $(\alpha_{11}, \alpha_{12}, ..., \alpha_{1r}), (\alpha_{21}, \alpha_{22}, ..., \alpha_{2r}),$ are obtained accordingly.... . . $(\alpha_{(n-r)1}, \alpha_{(n-r)2}, ..., \alpha_{(n-r)r})$.

It is obvious that vectors

$$\begin{aligned} \vec{y}_1 &= (\alpha_{11}, \alpha_{12}, ..., \alpha_{1r}, 1, 0, ..., 0), \\ \vec{y}_2 &= (\alpha_{21}, \alpha_{22}, ... \alpha_{2r}, 0, 1, 0, ..., 0), \\ &.................................... \\ \vec{y}_{n-r} &= (\alpha_{(n-r)1}, \alpha_{(n-r)2}, ..., \alpha_{(n-r)r}, 0, 0, ..., 1) \end{aligned} \qquad (7.11)$$

are solutions of the system (7.10). Number of coordinates of vectors $\vec{y}_1, \vec{y}_2, ... \vec{y}_{n-r}$ is equal to *n* and they belong to the space R^n.

We shall prove, that vectors $\vec{y}_1,...,\vec{y}_{n-r}$ are linearly independent. Really, if we write down the equality $\lambda_1\vec{y}_1 + ...\lambda_{n-r}\vec{y}_{n-r} = \vec{0}$ in the scalar form

$$\sum_{i=1}^{n-r} \lambda_i y_{ij} = 0, \ \mathrm{j} = 1,2,...,\mathrm{n},$$

using components (7.11) it is satisfied only under condition of $\lambda_1 = \lambda_2 = ... = \lambda_{n-r} = 0$. It is immediate from the equations, for which $j \geq r+1$. It is not difficult to show that any solution $\vec{y} = (\beta_1, \beta_2, \ldots, \beta_n)$ of homogeneous system (7.10) is a linear combination of vectors $\vec{y}_1, \vec{y}_2, ...\vec{y}_{n-r}$ with coefficients

$\gamma_1 = \beta_{r+1}, \gamma_2 = \beta_{r+2}, ..., \gamma_{n-r} = \beta_n$, i.e.

$$\vec{y} = \beta_{r+1}\vec{y}_1 + \beta_{r+2}\vec{y}_2 + ... + \beta_n\vec{y}_{n-r} = \gamma_1\vec{y}_1 + \gamma_2\vec{y}_2 + ... + \gamma_{n-r}\vec{y}_{n-r}, \quad (7.12)$$

where $\gamma_1 = \beta_{r+1}, ..., \gamma_{n-r} = \beta_n$ can take any values from *R*. For the proof of it if we solve the system (7.10) for unknowns $x_{r+1},...,x_n$ we suppose values $(\beta_{r+1}, 0...., 0), (0, \beta_{r+2}, 0,..0),.....,(0,0,...,\beta_n)$.

Thus, vectors $\vec{y}_1, \vec{y}_2, ...\vec{y}_{n-r}$ with components (7.11) form a subspace basis of homogeneous system (7.8) solutions of dimensions *n-r*. The expression (7.12) determining all set of subspace solutions, is referred to as ***general solutions*** of homogeneous system. Set of linearly independent solutions $\vec{y}_1,.....\vec{y}_{n-r}$ of the system is referred to as ***fundamental*** system of decisions. Variables $x_{r+1},...,x_n$ are referred to as ***free***, $x_1,...,x_r$ – ***basic***.

The remark. The definition of the fundamental solutions indicated above, is not obligatory and in solving of specific problems a choice of values $x_{r+1},...x_n$ can be another.

Example

Let the homogeneous system of the equations be given

$$\begin{cases} x_1 + 2x_2 - 5x_3 + 3x_4 = 0, \\ 2x_1 + 5x_2 - 6x_3 - x_4 = 0, \\ 5x_1 + 12x_2 - 17x_3 + x_4 = 0, \end{cases}$$

in which number of unknown is n = 4, and number of the equations is k = 3. Since $\kappa < n$ then $r(A) < n$ and, hence, the system has infinite number of solutions. For definition of fundamental and the general solutions of the system we shall define a rank *r (A)* of the basic matrix

$$A = \begin{pmatrix} 1 & 2 & -5 & 3 \\ 2 & 5 & -6 & -1 \\ 5 & 12 & -17 & 1 \end{pmatrix}.$$

Let's consider the principal minors: $D_2(A) = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1 \neq 0;$

$D_3(A) = \begin{vmatrix} 1 & 2 & -5 \\ 2 & 5 & -6 \\ 5 & 12 & -17 \end{vmatrix} = 0.$ For matrix *A* there is one more minor of the third order $D_3'(A) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 5 & -1 \\ 5 & 12 & 1 \end{vmatrix}$, it is also equal to zero. Thus, all minors of the third order of the matrix *A* are equal to zero, and among minors of the second order there is a minor distinct from zero. Hence, the rank *r (A)* of the matrix *A* is equal to 2. It means also, that the third equation of system is a linear combination of first two ones and it can be excluded from the system. Really, we can obtain the third equation, if we multiply the second equation by 2 and add it with the first one. After deletion of the third equation from the system of the third equation, we shall rewrite the rest two equations in the following form

$$\begin{cases} x_1 + 2x_2 = 5x_3 - 3x_4 \\ 2x_1 + 5x_2 = 6x_3 + x_4 \end{cases}$$

Supposing $x_3 = 1$, a $x_4 = 0$, we shall obtain the fundamental solution $\vec{y}_1$ of the system

$$\begin{cases} x_1 + 2x_2 = 5 \\ 2x_1 + 5x_2 = 6 \end{cases} \Rightarrow x_1 = 13, \quad x_2 = -4, \quad \vec{y}_1 = (13, -4, 1, 0).$$

Supposing $x_3 = 0$, and $x_4 = 1$, we shall define $\vec{y}_2$

$$\begin{cases} x_1 + 2x_2 = -3 \\ 2x_1 + 5x_2 = 1 \end{cases} \Rightarrow x_1 = -17, \; x_2 = 7, \; \vec{y}_2 = (-17, 7, 0, 1).$$

The general solution of the system

$$\vec{y} = \gamma_1\vec{y}_1 + \gamma_2\vec{y}_2 = \gamma_1(13,-4,1,0) + \gamma_2(-17,7,0,1) =$$
$$= (13\gamma_1 - 17\gamma_2, -4\gamma_1 + 7\gamma_2, \gamma_1, \gamma_2),$$

where γ_1 and γ_2 are any numbers from *R*.

Where

γ_1 и γ_2

any numbers from *R*.

So, system solutions make a vector subspace of the dimensions $n - r = 4 - 2 = 2$.

§ 6. HETEROGENEOUS SYSTEM OF THE LINEAR EQUATIONS

If in system of the linear equations (7.1) only one of free members $в_i$ is distinct from zero such system is referred to as ***heterogeneous.***

Let be given the heterogeneous system of the linear equations which in the vector form can be presented as

$$\vec{a}_i\vec{x} = в_i,\ i = 1,2,...,к, \qquad (7.13)$$

$\vec{a}_i = (a_{i1}, a_{i2},..., a_{in},) \in R^n$, $\vec{x} = (x_1, x_2,..., x_n) \in R^n$.

Let's consider corresponding homogeneous system

$$\vec{a}_i\vec{x} = 0,\ \text{i} = 1,2,...\text{k}. \qquad (7.14).$$

Let the vector $\vec{x}_1 = (\alpha_1, \alpha_2,..., \alpha_n)$ be the solution of heterogeneous system (7.13), and the vector $\vec{y} = (\beta_1, \beta_2,..., \beta_n)$ be the solution of homogeneous system (7.14). Then, it is easy to see, that the vector $\vec{z} = \vec{x}_1 + \vec{y}$ also is the solution of heterogeneous system (7.13). Really

$$\begin{cases} \vec{a}_1\vec{z} = в_1 \\ \vec{a}_2\vec{z} = в_2 \\ \dots\dots\dots \\ \vec{a}_к\vec{z} = в_к \end{cases} \Rightarrow \begin{cases} \vec{a}_1(\vec{x}_1 + \vec{y}) = в_1 \\ \vec{a}_2(\vec{x}_1 + \vec{y}) = в_2 \\ \dots\dots\dots\dots\dots \\ \vec{a}_к(\vec{x}_1 + \vec{y}) = в_к \end{cases} \Rightarrow \begin{cases} \vec{a}_1\vec{x}_1 + \vec{a}_1\vec{y} = в_1 \\ \vec{a}_2\vec{x}_1 + \vec{a}_2\vec{y} = в_2 \\ \dots\dots\dots\dots\dots \\ \vec{a}_к\vec{x}_1 + \vec{a}_к\vec{y} = в_к \end{cases} \Rightarrow \begin{cases} \vec{a}_1\vec{x}_1 = в_1 \\ \vec{a}_2\vec{x}_1 = в_2 \\ \dots\dots\dots \\ \vec{a}_к\vec{x}_1 = в_к \end{cases}$$

Now, using the formula (7.12) of the general solution of the homogeneous equation, we have

$$\vec{y} = \gamma_1 \vec{y}_1 + \gamma_2 \vec{y}_2 + ... + \gamma_{n-r} \vec{y}_{n-r},$$

and therefore

$$\vec{z} = \vec{x}_1 + \gamma_1 \vec{y}_1 + \gamma_2 \vec{y}_2 + ... + \gamma_{n-r} \vec{y}_{n-r}, \qquad (7.15)$$

where $\gamma_1, ..., \gamma_{n-r}$ are any numbers from R, and $\vec{y}_1, ..., \vec{y}_{n-r}$ – are fundamental solutions of homogeneous system.

Thus, the solution of heterogeneous system is s set of its partial solution and the general solution of corresponding homogeneous system.

The solution (7.15) is referred to as ***the general solution of heterogeneous system of the linear equations.*** It follows from (7.15), that the consistent heterogeneous system of the linear equations has the unique solution if the rank *r (A)* of the basic matrix *A* coincides with number *n* of unknowns of the system (Kramer's system) if *r* $r(A) < n$, the system has infinite number of solutions and this number of solutions is equivalent to solution subspace of corresponding homogeneous equation system of dimension *n-r*.

Examples

1. Let be given the heterogeneous system of the equations, in which the number of the equations is $k = 3$, and the number of unknows is $n = 4$.

$$\begin{cases} x_1 - x_2 + x_3 - 2x_4 = 1 \\ x_1 - x_2 + 2x_3 - x_4 = 2 \\ 5x_1 - 5x_2 + 8x_3 - 7x_4 = 3 \end{cases}$$

We shall determine ranks of the basic matrix *A* and expanded matrix *A** of the given system. As *A* and *A** are not zero matrixes and $k = 3 < n$, therefore $1 \le r(A), r^*(A^*) \le 3$. Let's consider minors of the second order of matrixes *A* and *A**:

$$D_2(A) = D_2(A^*) = \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix} = 0; \quad D_2'(A) = D_2'(A^*) = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \ne 0.$$

Thus, among minors of the second order of matrixes *A* and *A** there is a minor distinct from zero, therefore $2 \le r(A), r^*(A^*) \le 3$. Now we shall consider minors of the third order

$D_3(A) = D_3(A^*) = \begin{vmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ 5 & -5 & 8 \end{vmatrix} = 0$, since the first and the second column are proportional. Similarly to the minor $D_3'(A) = D_3'(A^*) = \begin{vmatrix} 1 & -1 & -2 \\ 1 & -1 & -1 \\ 5 & -5 & -7 \end{vmatrix} = 0$.

$$D_3''(A) = D_3''(A^*) = \begin{vmatrix} -1 & 1 & -2 \\ -1 & 2 & -1 \\ -5 & 8 & -7 \end{vmatrix} = -1\begin{vmatrix} 2 & -1 \\ 8 & -7 \end{vmatrix} - 1\begin{vmatrix} -1 & -1 \\ -5 & -7 \end{vmatrix} - 2\begin{vmatrix} -1 & 2 \\ -5 & 8 \end{vmatrix} = 6 - 2 - 4 = 0.$$

And so all minors of the third order of the basic matrix *A* are equal to zero, hence, *r (A)* = 2. For expanded matrix *A** still there are minors of the third order

$$D_3'''(A^*) = \begin{vmatrix} 1 & -1 & 1 \\ 1 & -1 & 2 \\ 5 & -5 & 3 \end{vmatrix} = 0; \quad D_3''''(A^*) = \begin{vmatrix} -1 & 1 & 1 \\ -1 & 2 & 2 \\ -5 & 8 & 3 \end{vmatrix} = 5 \neq 0.$$

Hence, among minors of the third order of expanded matrix *A** there is a minor distinct from zero, therefore $r^*(A^*) = 3$. It means, that $r(A) \neq r^*(A^*)$ and then, on the basis of Kronecker-Capelli theorem, we can conclude, that the given system is inconsistent.

2. Solve system of the equations

$$3x_1 + 2x_2 + x_3 + x_4 = 1$$
$$3x_1 + 2x_2 - x_3 - 2x_4 = 2$$

For the given system $k = 2 < n = 4$ and consequently $1 \leq r(A),\ r^*(A^*) \leq 2$. Let's consider for matrixes *A* and *A** the minors of the second order $D_2(A) = D_2(A^*) = \begin{vmatrix} 3 & 2 \\ 3 & 2 \end{vmatrix} = 0;$ $D_2'(A) = D_2'(A^*) = \begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = -6 \neq 0.$ Thus, $r(A) = r^*(A^*) = 2$, and, hence, the system is consistent. As basic variables we shall choose any two variables for which the minor of the second order formed of coefficients of these variables is not equal to zero. Such variables can be, for example

x_3 and x_4, since. $D_2'(A) = \begin{vmatrix} 1 & 1 \\ -1 & -2 \end{vmatrix} = -1 \neq 0.$ Then we have

$$\begin{cases} x_3 + x_4 = 1 - 3x_1 - 2x_2 \\ -x_3 - 2x_4 = 2 - 3x_1 - 2x_2. \end{cases}$$

Let's define the partial solution $\vec{x}_1$ of the heterogeneous system. For this purpose we shall put $x_1 = x_2 = 0$.

$$\begin{cases} x_3 + x_4 = 1 \\ -x_3 - 2x_4 = 2 \end{cases}$$

The solution of this system: $x_3 = 4$, $x_4 = -3$, hence, $\vec{x}_1 = (0,0,4,-3)$.

Now we shall define the general solution of the corresponding homogeneous equation

$$\begin{cases} x_3 + x_4 = -3x_1 - 2x_2 \\ -x_3 - 2x_4 = -3x_1 - 2x_2 \end{cases}$$

We put: $x_1 = 1$, $x_2 = 0$

$$\begin{cases} x_3 + x_4 = -3 \\ -x_3 - 2x_4 = -3 \end{cases}$$

Solution of this system $x_3 = -9$, $x_4 = 6$.

Thus $\vec{y}_1 = (1,0,-9,6)$.

Now we shall put $x_1 = 0$, $x_2 = 1$

$$\begin{cases} x_3 + x_4 = -2 \\ -x_3 - 2x_4 = -2 \end{cases}$$

Solution: $x_3 = -6$, $x_4 = 4$, and then $\vec{y}_2 = (0,1,-6,4)$.

After we determined the partial solution $\vec{x}_1$, of the heterogeneous equation and fundamental solutions $\vec{y}_1$ and $\vec{y}_2$ of the corresponding homogeneous equation, we write down the general solution of the heterogeneous equation.

$$\vec{z} = \vec{x}_1 + \gamma_1\vec{y}_1 + \gamma_2\vec{y}_2 = (0,0,4,-3) + \gamma_1(1,0,-9,6) + \gamma_2(0,1,-6,4) =$$

$= (\gamma_1, \gamma_2, 4 - 9\gamma_1 - 6\gamma_2, -3 + 6\gamma_1 + 4\gamma_2)$, where γ_1 and γ_2 are any numbers from R.

EXERCISES

1. Solve system of the equations by Gaussian method and with the help of determinants

$$\begin{cases} 2x_1 + x_2 + 3x_3 + 4x_4 = 11; \\ 7x_1 + 3x_2 + 6x_3 + 8x_4 = 24; \\ 3x_1 + 2x_2 + 4x_3 + 5x_4 = 14; \\ x_1 + x_2 + 3x_3 + 4x_4 = 10; \end{cases}$$

2. Define basis and subspace dimension, formed by set of solutions of homogeneous equation system:

a) $\begin{cases} 3x_1 + 5x_2 - x_3 + 2x_4 = 0; \\ 2x_1 + 4x_2 - x_3 + 3x_4 = 0; \\ x_1 + 3x_2 - x_3 + 4x_4 = 0; \end{cases}$ b) $\begin{cases} x_1 + 4x_2 - 3x_3 + 6x_4 = 0; \\ 2x_1 + 5x_2 + x_3 + 2x_4 = 0; \\ x_1 + 7x_2 - 10x_3 + 20x_4 = 0; \end{cases}$

3. Is the system of the equations consistent t? If it is consistent, solve it:

a) $\begin{cases} x_1 + x_2 + x_3 = 3 \\ x_1 + x_2 - 3x_3 = -1 \\ 2x_1 + x_2 - 2x_3 = 1 \\ x_1 + 2x_2 - 3x_3 = 1 \end{cases}$ b) $\begin{cases} x_1 - 2x_2 - 3x_3 = -3 \\ x_1 + 3x_2 - 5x_3 = 0; \\ -x_1 + 4x_2 + x_3 = 3 \\ 3x_1 + x_2 - 13x_3 = -6 \end{cases}$

c) $\begin{cases} 2x_1 + x_2 - x_3 - x_4 + x_5 = 1 \\ x_1 - x_2 + x_3 + x_4 - 2x_5 = 0 \\ 3x_1 + 3x_2 - 3x_3 - 3x_4 + 4x_5 = 2 \\ 4x_1 + 5x_2 - 5x_3 - 5x_4 + 7x_5 = 3 \end{cases}$ d) $\begin{cases} 2x_1 - x_2 + x_3 - 5x_4 = 4 \\ 2x_1 + 3x_2 - 3x_3 + x_4 = 2 \\ 8x_1 - x_2 + x_3 - x_4 = 1 \\ 4x_1 - 3x_2 + 3x_3 + 3x_4 = 2 \end{cases}$

e) $\begin{cases} x_1 + 2x_2 + x_3 - x_4 + x_5 = -1 \\ 2x_1 + 5x_2 + 6x_3 - 5x_4 + x_5 = 0 \\ x_1 - 2x_2 + x_3 - x_4 - x_5 = 3 \\ x_1 + 3x_2 + 2x_3 - 2x_4 + x_5 = -1 \\ x_1 - 4x_2 + x_3 + x_4 - x_5 = 3 \end{cases}$

4. Define the solution of the system with the help of inverse matrix

$$\begin{cases} x_1 + 4x_2 - 7x_3 + 6x_4 = 0 \\ x_1 - 3x_2 - 6x_4 = 9 \\ 2x_1 + x_2 - 5x_3 + x_4 = 8 \\ 2x_2 - x_3 + 2x_4 = -5 \end{cases}$$

CHAPTER 8
MATRIX REDUCTION

Let K be a vector space of finite dimension n above the field P. And let f be a linear mapping of the space K into K. With the help of usual isomorphism of spaces K and P^n we come to linear mapping P^n into P^n. This mapping determines a square matrix from n rows and n columns, dependent on chosen basis in K. We shall try to find in K such concrete basis relative to which bounded with f the matrix would have the most simple form.

§ 1. A MATRIX OF TRANSITION FROM ONE BASIS TO ANOTHER

Let $\vec{\ell}_1, \vec{\ell}_2, ... \vec{\ell}_n$ - initial basis of the space K, and $\vec{\ell}'_1, \vec{\ell}'_2, ... \vec{\ell}'_n$ - its new basis. We shall express vectors $\vec{\ell}'_1, \vec{\ell}'_2, ... \vec{\ell}'_n$ through the vectors $\vec{\ell}_1, \vec{\ell}_2, ... \vec{\ell}_n$, forming the first basis. We have $\vec{\ell}'_j = \tau_{1j}\vec{\ell}_1 + \tau_{2j}\vec{\ell}_2 + ... + \tau_{nj}\vec{\ell}_n$, $j = 1, 2..., n$. Coordinates τ_{ij} of vectors $\vec{\ell}'_j$ in the basis $\vec{\ell}_1, \vec{\ell}_2, ... \vec{\ell}_n$ can be written down as a matrix:

$$T = \begin{pmatrix} \tau_{11}\tau_{12}...\tau_{1n} \\ \tau_{21}\tau_{22}...\tau_{2n} \\ .... \\ \tau_{n1}\tau_{n2}...\tau_{nn} \end{pmatrix}, -$$

here matrix columns – are coordinates of vectors $\vec{\ell}'_1, \vec{\ell}'_2, ... \vec{\ell}'_n$ on basis $\vec{\ell}_1, \vec{\ell}_2, ... \vec{\ell}_n$.

Definition. Matrix T, which column - vectors are formed of the vector coordinates of new basis expressed through initial basis, is referred to as ***a matrix of transition*** from one basis to another.

The matrix of transition T possesses the following properties:

1. As $\vec{\ell}'_1, \vec{\ell}'_2, ... \vec{\ell}'_n$ and $\vec{\ell}_1, \vec{\ell}_2, ... \vec{\ell}_n$ - are bases of the same space K, the number of them is identical, and decomposition in terms of the basis is unique. Therefore matrix T is always square also is defined unequivocally.

2. Column – vectors of the matrix *T* are linearly independent (these are vectors of the basis). Thus, the rank *r (T)* of the transition matrix *T* is equal *n*; it means, that determinant *D(T)* $\neq 0$ and matrix T is always has inverse T^{-1}, which will be a matrix of transition from $\vec{\ell}'_1, \vec{\ell}'_2, ...\vec{\ell}'_n$ to $\vec{\ell}_1, \vec{\ell}_2, ...\vec{\ell}_n$.

The matrix of transition *T* represents a biunique mapping $\vec{x} = T(\vec{x}')$ of the space paces P^n onto itself. Really, let $\vec{a}$ – be any element from *K*. We have

$$\vec{a} = \lambda_1 \vec{\ell}_1 + \lambda_2 \vec{\ell}_2 + \ldots + \lambda_n \vec{\ell}_n = \lambda'_1 \vec{\ell}' + \lambda_2{}' \vec{\ell}'_2 + \ldots + \lambda'_n \vec{\ell}'_n;$$

if we express $\vec{\ell}'_j$ through $\vec{\ell}_i$ and *T*, we shall obtain

$$\lambda_j = \tau_{j1} \lambda'_1 + \tau_{j2} \lambda'_2 + \ldots + \tau_{jn} \lambda'_n, \quad j = 1,2, \ldots n.$$

Vectors $\vec{x} = (\lambda_1, \lambda_2, \ldots \lambda_n)$ and $\vec{x}' = (\lambda'_1, \lambda'_2 \ , \ldots \lambda'_n)$ belong to the space P^n, and thus $\vec{x} = T(\vec{x}')$. Decomposition in terms of bases is unique and invertible (there is inverse matrix T^{-1}), hence $\vec{x} = T(\vec{x}')$ - is a biunique mapping.

As an evident illustration of a transition matrix we shall consider it for geometrical space in which the matrix of transition is connected to transformation of coordinate system and it defines linear mapping R^3 onto R^3.

1.1. The matrix of transition connected to the system of coordinate's transformation in geometrical space

We shall write down a transition matrix in geometrical space for orthonormal bases. Let's choose as the first basis $\vec{i}, \vec{j}, \vec{k}$ and we shall connect it with it system of coordinates x, y, z, and as the second $\vec{i}\,', \vec{j}\,', \vec{k}\,'$ and connected to it the system of coordinates x ', y ', z ' (Fig. 2.8). Then

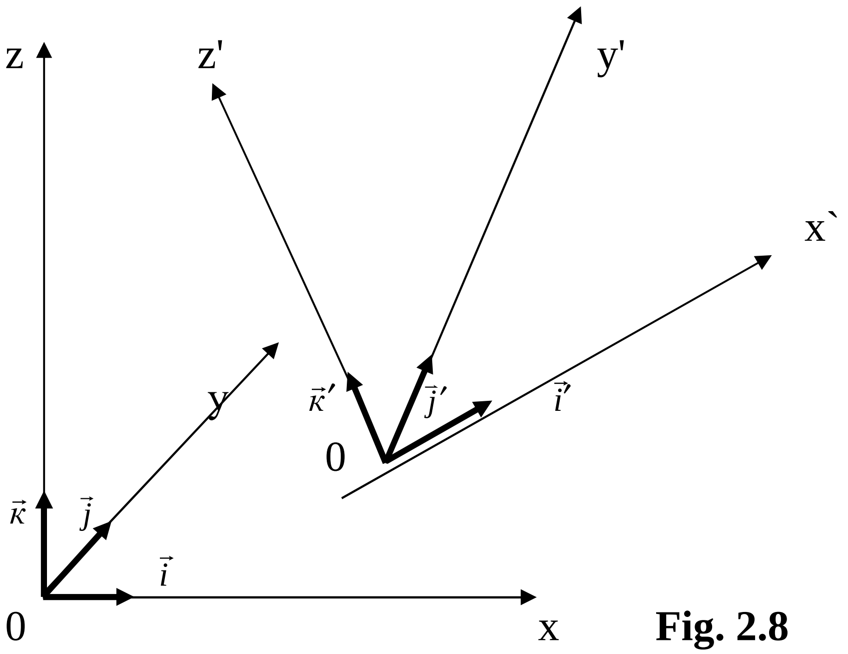


Fig. 2.8

$$\vec{i}' = \tau_{11}\vec{i} + \tau_{21}\vec{j} + \tau_{31}\vec{k}$$
$$\vec{j}' = \tau_{12}\vec{i} + \tau_{22}\vec{j} + \tau_{32}\vec{k} \qquad (8.1)$$
$$\vec{k}' = \tau_{13}\vec{i} + \tau_{23}\vec{j} + \tau_{33}\vec{k}\ .$$

If we multiply the first row by $\vec{i}, \vec{j}, \vec{k}$ in sequence, taking into account, that

$$\vec{i}\vec{j} = \vec{i}\vec{k} = \vec{j}\vec{k} = 0, \text{and } \vec{i}\vec{i} = \vec{j}\vec{j} = \vec{k}\vec{k} = 1,$$

then we shall obtain $\tau_{11} = \vec{i}'\vec{i} = \cos(\vec{i}'\vec{i});\ \tau_{21} = \vec{i}'\vec{j} = \cos(\vec{i}'\vec{j});\ \tau_{31} = \cos(\vec{i}'\vec{k}).$

If we do the same with the second and third rows of equality, we can define: $\tau_{12} = \cos(\vec{j}'\vec{i});\ \ \tau_{22} = \cos(\vec{j}'\vec{j});\ \ \tau_{32} = \cos(\vec{j}'\vec{k});\ \tau_{13} = \cos(\vec{k}'\vec{i});$ $\tau_{23} = \cos(\vec{k}'\vec{j});\ \ \tau_{33} = \cos(\vec{k}'\vec{k}).$

Thus, the transition matrix T of one orthonormal basis to another orthonormal basis, connected with transformation of coordinate system in geometrical space, has the form

$$T = \begin{pmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{pmatrix} = \begin{pmatrix} \cos(\vec{i}'\vec{i}) & \cos(\vec{j}'\vec{i}) & \cos(\vec{k}'\vec{i}) \\ \cos(\vec{i}'\vec{j}) & \cos(\vec{j}'\vec{j}) & \cos(\vec{k}'\vec{j}) \\ \cos(\vec{i}'\vec{k}) & \cos(\vec{j}'\vec{k}) & \cos(\vec{k}'\vec{k}) \end{pmatrix} \qquad (8.2)$$

and its members are determined by cosines of angles which are formed in turning of new system of coordinates relative to the previous one. If turn of coordinate system at their transformation does not occur, and it is observed at parallel shift of coordinate system then $\cos(\vec{i}'\vec{i}) = \cos(\vec{j}'\vec{j}) = \cos(\vec{k}'\vec{k}) = 1$, and other cosines are equal to zero. Therefore the transition matrix for parallel shift of coordinate system is identity matrix

$$T = E = \begin{pmatrix} 1\ldots\ 0\ldots 0 \\ \ldots\ldots\ldots \\ 0\ldots 1\ldots 0 \\ \ldots\ldots\ldots \\ 0\ldots\ 0\ldots 1 \end{pmatrix}.$$

Now we shall consider transition matrix T as a matrix of linear mapping $\vec{v} = T(\vec{v}')$ of the space R^3 onto itself. Let $\vec{r} = x'\vec{i}' + y'\vec{j}' + z'\vec{k}'$ - a radius - vector of some point M in the system of coordinates x', y', z'. In the system of coordinates x, y, z the same vector has decomposition:

$$\vec{r} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k},$$

where x_0, y_0, z_0 are coordinates of the origin of coordinates x', y', z' in the system of coordinates x, y, z. Then the vector $\vec{v} = (x - x_0, y - y_0, z - z_0)$, and $\vec{v}' = (x', y', z')$ and they belong to the space R^3. Therefore mapping $\vec{v} = T(\vec{v}')$ in the coordinate form is represented as:

$$\begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} = T \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \text{ or } X - X_0 = T \cdot X' \qquad (8.3)$$

From here we obtain the formula for coordinate change of the point M in transformation of coordinate system generally, when we have both parallel shift, and turn of coordinate system.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \text{ or } X = TX' + X_0 \qquad (8.4)$$

In formulas (8.3) and (8.4) we assume as known $X' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$ and $X_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$, i. e. coordinates of the point M in new coordinate system are known and coordinates of a point in old system should be defined.

The inverse problem is more natural when X_0 и X are known, and it is required to define X'. For this case we assume $\vec{v} = (x, y, z)$, a $\vec{v}' = (x' + x_0, y' + y_0, z' + z_0)$ and then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = T \begin{pmatrix} x' + x_0 \\ y' + y_0 \\ z' + x_0 \end{pmatrix} \text{ or } \mathrm{X} = \mathrm{T}(\mathrm{X}' + \mathrm{X}_0)$$
,

whence

$$\begin{pmatrix} x' + x_0 \\ y' + y_0 \\ z' + x_0 \end{pmatrix} = T^{-1} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ or } \mathrm{X}' = \mathrm{T}^{-1}\mathrm{X} - \mathrm{X}_0. \qquad (8.5)$$

1.2. Orthogonal matrixes of transition

If we raise to the second power all rows of the equations (8.1) or multiply by each other we shall obtain following equality system:

$$\tau_{1\alpha} \tau_{1\gamma} + \tau_{2\alpha} \cdot \tau_{2\gamma} + \tau_{3\alpha} \tau_{3\gamma} = \delta_{\alpha\gamma}, \qquad (8.6)$$

where $\delta_{\alpha\gamma}$, – is Kronecker symbol, $\alpha = 1, 2, 3$, $\gamma = 1, 2, 3$.

Hence, in matrix T (8.2) sum of squares of the elements located in each column, is equal 1, and the sum of products of corresponding elements of two any various columns ($\alpha \neq \gamma$) is equal to zero. Matrixes of such type are referred to as ***orthogonal.***

The equality system (8.6) which exists for elements of orthogonal matrix T can be rewritten also as following condition $T^T \cdot T = E$ or $T^T = T^{-1}$, where T^T – is transposed matrix, and T^{-1} – is inverse matrix to T.

Then, if $\vec{\tau}_j$ is j- column vector in T with components $(\tau_{1j}, \tau_{2j}, \tau_{3j})$, the ratio (8.6) means, that scalar product $\vec{\tau}_j \cdot \vec{\tau}_i = \delta_{ji}$, j = 1, 2, 3, i = 1, 2, 3 and, so, column –vectors $\vec{\tau}_j$, j = 1, 2, 3 of the orthogonal matrix T form the orthonormal basis.

The given definition of orthogonal matrixes is applied not only for transition matrixes of the third order $n = 3$, but also for matrixes of the order $n > 3$.

Definition. Square matrix $S = (\sigma_{ij})$, where $i = 1, 2, ..., n$, $j = 1, 2, ..., n$, for which $S^T \cdot S = E$ (or $\sigma_{1i}\sigma_{1j} + \sigma_{2i}\sigma_{2j} + ... + \sigma_{ni}\sigma_{nj} = \delta_{ij}$, where δ_{ij} – Kronecker symbol), is referred to as ***orthogonal***.

It also follows from this definition, that for the matrix to be orthogonal, it is necessary and sufficient, that either its column-vectors (or row - vectors) form orthonormal basis in R^n.

Determinant $D(S)$ of the orthogonal matrix S is equal to +1 or-1. Really, since the determinant of the matrix product is equal to product of multiplier determinants, then $D(S \cdot S^T) = D(S) \cdot D(S^T) = [D(S)]^2 = D(E) = 1$ and, hence, $D(S) = \pm 1$. Values +1 and-1 correspond to various orientation of column - vectors, forming basis. So, if as column – vectors in S we choose canonical orthonormal basis $\vec{\ell}_1 = (1,0,...,0)$, $\vec{\ell}_2 = (0,1,0,...,0)$,........, $\vec{\ell}_n = (0,0,.....,1)$, we shall obtain $S=E$ and $D(S)=+1$. If we take orthonormal basis $\vec{\ell}_1 = (1,0,...,0)$, $\vec{\ell}_2 = (0,1,0,...,0)$,....., $\vec{\ell}_{n-1} = (0,0,...,1,0), -\vec{\ell}_n = (0,0,...,-1)$, then orthogonal matrix S' adequate to it will have determinant $D(S') = -1$.

§ 2. CHANGE OF LINEAR MAPPING MATRIX AT CHANGE OF BASES

Let's consider linear mapping f of n-dimensional space K above the field P in m-dimensional space F above the field P and let if in the space K the basis $\vec{v}_1, \vec{v}_2, ..., \vec{v}_m$ is set, and in the space F the basis $\vec{v}_1, \vec{v}_2, ..., \vec{v}_m$, then mapping f is associated with the matrix A, representing linear mapping $\vec{y} = A(\vec{x})$ of the space P^n into P^m, $\vec{x} \in P^n, \vec{y} \in P^m$. Let's pass in these spaces on to other bases, accordingly $\vec{\ell}'_1, \vec{\ell}'_2, ...\vec{\ell}'_n$ and $\vec{v}'_1, \vec{v}'_2, ..., \vec{v}'_m$, which are connected to initial bases with matrixes of transition $S : \vec{\ell}' \to \vec{\ell}$ и $T : \vec{v}' \to \vec{v}$. Our task is to determine, what kind the matrix A will take in bases $\vec{\ell}'$ and $\vec{v}'$. Let's designate this transformed matrix B.

We shall consider any vector $\vec{x}$ from the space P^n, and its image $\vec{y} = A(\vec{x})$ from the space P^m in bases $\vec{\ell}$ and $\vec{v}$. In changing of space bases P^n and P^m are mapped into itself by means of transition matrixes S and T. Thus vectors $\vec{x}'$and $\vec{y}'$ will be preimages of the vectors accordingly $\vec{x} = S(\vec{x}')$ and $\vec{y} = T(\vec{y}')$. Then the matrix B is set by means of the ratio and, then, $B=T^{-1}AS$. It is also the required formula for determination of interrelation between matrixes A and B, representing the same linear mapping f of the space K into the space F, in changing of the bases in them, determined by matrixes of transition S and T.

If $F=K$, and initial, and also new bases in spaces K and F coincide, then A and $S = T$ will be square matrixes of the same order. Then we shall obtain $B=T^{-1}AT$; B is referred to as matrix transformed from A by means of T; matrixes B and A are referred to as ***similar matrixes.*** If A is invertible, then $T^{-1}(A^{-1})T=(T^{-1}AT)^{-1}=B^{-1}$.

Now we shall try to define in K such concrete basis relative to which the square matrix connected with f which is determining mapping P^n into P^n would have the most simple form.

2.1. Eigenvalues, eigenvectors of the square matrix

We can easy show, that equality $B = T^{-1}AT$ results in equality of determinants: $D(B) = D(A)$. Really, from the rule of determinant multiplication, we have

$$D(B) = D(T^{-1})\cdot D(A)\cdot D(T) = D(A)D(E) = D(A)\cdot 1 = D(A).$$

On the other hand, the matrix transformed from identity matrix, is identity matrix: $T^{-1}ET = E$; hence, for any $\rho \in P$, we have

$$B - \rho E = T^{-1}(A - \rho E)T,$$

and then, determinant $D(A - \rho E)$ depends only on linear mapping f and does not depend on a choice of concrete basis in K.

If $$A = \begin{pmatrix} \alpha_{11}\ \alpha_{12} \dots \alpha_{1n} \\ \alpha_{21}\ \alpha_{22} \dots \alpha_{2n} \\ \dots\dots\dots \\ \alpha_{n1}\ \alpha_{n2} \dots \alpha_{nn} \end{pmatrix},$$

$$\text{then } A-\rho E=\begin{pmatrix} \alpha_{11}-\rho & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22}-\rho & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}-\rho \end{pmatrix} \text{ and}$$

$D(A-\rho E) = (-1)^n\rho^n + q_{n-1}\rho^{n-1} + q_{n-2}\rho^{n-2} + \ldots + q_1\rho + D(A)$, is a multinomial of the ρ power, which is exactly equal to n. We have no need to write down, what coefficients q_i, are equal to.

Definition 1. Multinomial $D(A-\rho E)$ is referred as ***characteristic*** multinomial of the mapping *f*.

Its coefficients depend only on linear mapping *f* and do not depend on a choice of basis in *K*. The same will concern to zeros of this multinomial and to their multiplicity.

Definition 2. Eigenvalues or ***characteristic*** numbers of the mapping *f* are referred to as zeros of characteristic multinomial $D(A-\rho E)$ i. e. roots of the equation $D(A-\rho E) =0$ – this equation is referred to as ***characteristic.***

If *P* is the field *C* of complex numbers, the multinomial of the power *n* has precisely *n* zeros belonging to *C*; if we count each zero many times, as its multiplicity is (fundamental theorem of algebra). Therefore henceforth we shall assume, that *P* t is the field *C*.

Let ρ_1 be an eigenvalue, so such real or complex number, that $D(A-\rho_1 E) =0$. Then matrix $A-\rho_1 E$ is noninvertible, and let there be, at least, one such nonzero vector $\vec{u}_1' \in C^n$, so that $(A-\rho_1 E)(\vec{u}')=\vec{0}$, i.e. $A(\vec{u}_1')=\rho_1\vec{u}_1'$. Inversely, if there is such nonzero vector $\vec{u}_1' \in C^n$, so that $A(\vec{u}_1')=\rho_1\vec{u}_1'$, then the reasoning which is inverse to the mentioned, we ascertain, that ρ_1 is an eigenvalue.

Definition 3. The vector $\vec{u}_1'$ is referred to as ***eigenvector***r of the matrix *A*, belonging the eigenvalue ρ_1, if $A(\vec{u}_1')=\rho_1\vec{u}_1'$, with $\vec{u}_1' \neq \vec{0}$.

If $\vec{u}_1$ is a vector from *K*, adequate to the vector $\vec{u}_1' \in C^n$, then $f(\vec{u}_1)=\rho_1\vec{u}_1$, that shows, as $\vec{u}_1$ and ρ_1 depend only on *f*.

The vector $\vec{u}_1$ is referred to as ***eigenvector*** of linear mapping *f*.

Let's list some properties of eigenvectors and eigenvalues of the matrix *A* which are also the properties of eigenvectors and eigenvalues of linear mapping *f*.

1. Each eigenvector corresponds to the unique proper number.

2. If $\vec{u}'$- is eigenvector of the matrix *A* with proper number ρ, then any vector $\lambda\vec{u}'$ which is collinear to the vector $\vec{u}'$, also is eigenvector of the matrix *A* with the same number ρ.

3. If $\vec{u}_1'$ and $\vec{u}_2'$ are eigenvectors of the matrix *A* with same proper number ρ, then their sum $\vec{u}_1'+\vec{u}_2'$ also is eigenvector of the matrix *A* with same number ρ.

It follows from the properties 2 and 3, that each proper is correspondent to the infinite set of (collinear) eigenvectors. This set together with a zero vector which always is eigenvector, forms a subspace of the space C^n if it concerns $\vec{u}'$ eigenvectors of the matrix and the space *K* if it concerns $\vec{u}$ eigenvectors of linear mapping *f*.

4. If eigenvectors $\vec{u}_1', \vec{u}_2', \cdots, \vec{u}_k'$ (or $\vec{u}_1, \vec{u}_2, \cdots, \vec{u}_k$) belong to various eigenvalues they are linearly independent.

Last item allows to solve the problem of square matrix reduction to more simple form.

2.2. Reduction of a square matrix to the diagonal form

Eigenvectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ of linear mapping *f*, belonging to various eigenvalues of this mapping, and being linearly independent, can form a basis of the space *K* of the dimension *n*. It is possible, for example, if mapping *f* has *n* various eigenvalues; let's suppose, that it exists; we shall designate them through $\rho_1, \rho_2, \ldots, \rho_n$. All of them serve as simple zeros of a characteristic multinomial.

Let $\vec{u}_i$- eigenvectors belonging to eigenvalues ρ_i for $i = 1,2,\ldots,n$, form a basis of the space *K*. Theoretically it can happen, that linear mapping has less than *n* eigenvalues, but nevertheless it has basis from eigenvectors.

Let $\vec{x} = \lambda_1\vec{u}_1 + \lambda_2\vec{u}_2 + \ldots + \lambda_n\vec{u}_n$ be any vector from *K*, and $\vec{x}' = \lambda_1, \lambda_2, \ldots, \lambda_n$ - a corresponding vector in C^n. We have

$\vec{y} = f(\vec{x}) = \lambda_1 f(\vec{u}_1) + \ldots + \lambda_n f(\vec{u}_n) = \lambda_1 \rho_1 \vec{u}_1 + \lambda_2 \rho_2 \vec{u}_2 + \ldots + \lambda_n \rho_n \vec{u}_n$. This implyies, that a corresponding vector in C^n will be a vector $\vec{y}' = (\lambda_1 \rho_1, \lambda_2 \rho_2, \ldots, \lambda_n \rho_n)$; so, it turns out from $\vec{x}'$ by means of a diagonal matrix

$$U = \begin{pmatrix} \rho_1 \ldots 0 \ldots 0 \\ \ldots\ldots\ldots\ldots \\ 0 \ldots \rho_i \ldots 0 \\ \ldots\ldots\ldots\ldots \\ 0 \ldots 0 \ldots \rho_n \end{pmatrix}; \quad \vec{y}' = U(\vec{x}').$$

Thus, if we take eigenvectors $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ as basis in K, then mapping of space C^n into C^n, corresponding to the mapping *f*, is set by diagonal matrix *U*. If $\vec{\ell}_1, \vec{\ell}_2, \ldots, \ell_n$ - any basis in *K*, then

$\vec{u}_j = \alpha_{1j} \vec{\ell}_1 + \alpha_{2j} \vec{\ell}_2 + \ldots + \alpha_{nj} \vec{\ell}_n$ for $j = 1, 2, \ldots, n$ and transition matrix

$$T = \begin{pmatrix} \alpha_{11}\ \alpha_{12} \ldots \alpha_{1n} \\ \alpha_{21}\ \alpha_{22} \ldots \alpha_{2n} \\ \ldots\ldots\ldots\ldots\ldots \\ \alpha_{n1}\ \alpha_{n2} \ldots \alpha_{nn} \end{pmatrix}.$$

Let *A* – be a matrix representing the mapping *f* when $\vec{\ell}_1, \vec{\ell}_2, \ldots, \ell_n$ are taken as basis in *K*; then $U = T^{-1}AT$. Hence, there is such invertible matrix *T*, that the matrix transformed from *A* by means of *T*, will be diagonal matrix *U*. Matrix *U* is not unique since it is possible to change the order of vectors $\vec{u}_1, \vec{u}_2, \ldots \vec{u}_n$; however, if there is diagonal matrix

$$W = \begin{pmatrix} \beta_1 0 \ldots 0 \\ 0 \quad \beta_2 ..0 \\ \ldots\ldots\ldots \\ 0 \quad 0 \ldots \beta_n \end{pmatrix},$$ transformed from *A*, then $D(W - \rho E) = D(A - \rho E)$,

i. e. (Book 2, Chapter. 3, § 4),

$(-1)^n(\rho - \beta_1)(\rho - \beta_2) \cdot \ldots \cdot (\rho - \beta_n) = (-1)^n(\rho - \rho_1) \cdot (\rho - \rho_2) \cdot \ldots \cdot (\rho - \rho_n)$, so, numbers $\beta_1, \beta_2, \ldots, \beta_n$ accurate within the sequence order of are eigenvalues, and *W* there is one of matrixes of kind *U*.

We shall notice, that vector subspace vector of eigenvectors belonging to one eigenvalue, has the dimension equal to one. Really, if $\vec{u}_1$ and $\vec{v}_1$- are two eigenvectors belonging to eigenvalue ρ_1, then they both belong to the vector subspace, which is complement of the *n-1*-dimensional the vector space generated by vectors $\vec{u}_2,...,\vec{u}_n$, so, to the vector subspace of the dimension, is one. Hence $\vec{v}_1 = \lambda\vec{u}_1$, $\lambda \in C$ (if $\lambda \neq 0$).

If all eigenvalues are not distinct, the it is not always possible to define the diagonal matrix representing a linear mapping. However and in this case it is possible to define a matrix revealing eigenvalues and having has the form, which is easy for calculations. For consideration of this case we refer the reader to the special literature.

For real space R^n complex roots of the characteristic equation cannot be eigenvalues since they do not need equality $A(\vec{x}) = \lambda\vec{x}$, as coordinates of the vector $\vec{x}$ and members of the matrix *A* belong to the field *R* of real numbers. Therefore linear mapping R^n into R^n , set by the matrix *A* above the field *R* of real numbers, for which the characteristic equation has only complex-conjugate roots (i. e. none real root), has no eigenvalues (a power of such characteristic multinomial should be even). However, if linear mapping R^n into R^n is set by a symmetric matrix *A*, then all roots of the characteristic equation of such matrix are real; all eigenvectors belonging to them can be chosen as real. In this case eigenvectors of the matrix *A* form a basis, and in this basis the matrix of linear mapping has a diagonal kind. Let's consider it by the example of reduction of symmetric real matrix *A* to a diagonal kind, which determines the square-law form on R^n.

§ 3. REAL LINEAR AND SQUARE-LAW FORMS

Let's consider the vector space R^n above the field *R* in which the basis $\vec{\ell}_1, \vec{\ell}_2, ..., \vec{\ell}_n$ is given and let $\vec{x} = \mu_1\vec{\ell}_1 + \mu_2\vec{\ell}_2 + ...\mu_n\vec{\ell}_n$ - be any vector of this space, $\mu_i \in R$.

Definition 1. Real linear form φ is referred to as linear mapping of space R^n into *R*, which every $\vec{x} \in R^n$ puts in conformity with number

$\varphi(\vec{x}) = \sum_{i=1}^{n} \lambda_i \mu_i$ from *R*, where λ_i and μ_i - are numbers from *R* . The linear form also is named the homogeneous simple form, and it is mostly written down in the following form:

$\varphi(\vec{x}) = a_1 x_1 + a_2 x_2 + ... + a_n x_n$, where $\vec{x} = (x_1, x_2, ..., x_n) \in R^n$.

Definition 2. ***Real square-law form*** ω is referred to as linear mapping R^n into *R*, which each $\vec{x} \in R^n$ puts in conformity number $\varpi(\vec{x}) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \sigma_{ij}\mu_i\mu_j\right)$ from *R* where μ_K - are coordinates of the vector $\vec{x}$, σ_{ij} - are numbers from *R* for which the equality $\sigma_{ij} = \sigma_{ji}$ is satisfied.

From definition follows, that $\omega(\lambda\vec{x}) = \lambda^2\omega(\vec{x})$. Therefore the square-law form is the homogeneous form of the second power.

An example

$\vec{x} = (x_1, x_2, x_3)$; $\omega(\vec{x}) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} \sigma_{ij} x_i x_j\right) =$

$$= \sigma_{11}x_1^2 + \sigma_{22}x_2^2 + \sigma_{33}x_3^2 + 2\sigma_{12}x_1x_2 + 2\sigma_{13}x_1x_3 + 2\sigma_{23}x_2x_3 .$$

3.1. Reduction of the square-law form to the canonical type

The square-law form can be written down also by means of a matrix. For this purpose let's put the vector $\vec{x} = (\mu_1, ..., \mu_n)$ from R^n in conformity with two matrixes: a column - matrix $X = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{pmatrix}$ and row - matrix $X^T = (\mu_1, \mu_2, . . ., \mu_n)$. It is obvious, that X^T is the transposed matrix to *X*. For coefficients σ_{ij} of the square-law form we shall introduce the real matrix

$$A = \begin{pmatrix} \sigma_{11}\ \sigma_{12} \dots \sigma_{1n} \\ \sigma_{21}\ \sigma_{22} \dots \sigma_{2n} \\ \dots\dots\dots\dots\dots \\ \sigma_{n1}\ \sigma_{n2} \dots \sigma_{nn} \end{pmatrix}. \text{ Then}$$

$$\omega(\vec{x}) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\sigma_{ij}\mu_i\mu_j\right) = \sum_{i=1}^{n}\mu_i\left(\sum_{j=1}^{n}\sigma_{ij}\mu_j\right) = \sum_{i=1}^{n}\mu_i A\cdot X = X^T AX .$$

The matrix A is referred to as ***a matrix of the square-law form*** and since for factors of square-law form $\sigma_{ij} = \sigma_{ji}$, then the matrix A is symmetric.

We shall consider, how the matrix A changes when transition into R^n from one orthonormal basis to another. Let's designate a transition matrix through T, and coordinates of the vector $\vec{x} = (\mu_1,..., \mu_n)$ in new basis through $\vec{y} = (\beta_1, \beta_2, \ldots, \beta_n)$. Then $\vec{x} = T(\vec{y})$, or in matrix form $X = TY$, where T is an orthogonal matrix. Therefore for the square-law form we have

$$\omega(\vec{x}) = X^T AX = (TY)^T ATY = Y^T T^T ATY = Y^T BY, \text{ where } B = T^T AT.$$

But since T is orthogonal, then $T^T = T^{-1}$; and $B = T^{-1}AT$, i. e. B is transformed from A by means of matrix T. Besides the transformed matrix B – is also symmetric, since

$$B^T = (T^{-1}AT)^T = (T^TAT)^T = T^TA^T(T^T)^T = T^TAT = B.$$

As $A^T = A$.

As the matrix A is symmetric, then R^n possesses at least one orthonormal basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$, made of eigenvectors of the matrix A; then if we choose basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$, as new basis, then the transformed matrix in this basis $B = U$ and has a diagonal kind

$$U = \begin{pmatrix} \rho_1\, 0 \ldots 0 \\ 0\, \rho_2 \ldots 0 \\ \ldots\ldots\ldots\ldots \\ 0\ 0\ \ldots \rho_n \end{pmatrix},$$

Here eigenvalues ρ_i of the matrix A can be both distinct and the same, but all they are real. If a matrix of the square-law form is diagonal, then the square-law form becomes:

$\omega(\vec{x}) = Z^T UZ = \rho_1 z^2{}_1 + \rho_2 z^2{}_2 + ... + \rho_n z^2{}_n$, where z_1, z_2, ... z_n – are coordinates of the vector $\vec{x}$, decomposed on the basis $\vec{u}_1, \vec{u}_2, ...\vec{u}_n$.

Thus, concerning the basis $\vec{u}_1, \vec{u}_2, ...\vec{u}_n$, made of eigenvectors of a matrix of the square-law form, the square-law form has only members with squares; we can say, that it is reduced to the ***canonical*** kind.

An example. Reduce the square-law form to canonical type

$$\omega(\vec{x}) = 3x^2_1 + 4x_1x_2 + x^2_3, \text{ where } \vec{x} = (x_1, x_2, x_3).$$

1. We make up a matrix of the square-law form:

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (see the example in the beginning of the paragraph)

2. We write down the characteristic equation

$$\begin{vmatrix} 3-\rho & 2 & 0 \\ 2 & 0-\rho & 0 \\ 0 & 0 & 1-\rho \end{vmatrix} = 0, \text{ where } (1-\rho)(\rho^2 - 3\rho - 4) = 0 .$$

Solving the last equation, we define proper numbers: $\rho_1 = 1;\ \rho_2 = 4;\ \rho_3 = -1$. We shall designate coordinates of the vector $\vec{x}$ in system of eigenvectors of a matrix through z_1, z_2, z_3. Then the square-law form becomes

$$\omega(\vec{x}) = z_1^2 + 4z_2^2 - z_3^2.$$

3. We define orthonormal eigenvectors of a matrix: $\vec{u}_1 = (k_1, \ell_1, m_1);\ \vec{u}_2 = (k_2, \ell_2, m_2);\ \vec{u}_3 = (k_3, \ell_3, m_3)$. For this purpose the equation A($\vec{u}$) = $\rho\vec{u}$ is written down in the coordinate form:

$$\begin{cases} 3k + 2\ell + 0m = \rho k, \\ 2k + 0\ell + 0m = \rho\ell, \\ 0k + 0\ell + 0m = \rho m \end{cases} \text{ or } \begin{cases} (3-\rho)k + 2\ell + 0m = 0, \\ 2k + (0-\rho)\ell + 0m = 0, \\ 0k + 0\ell + (1-\rho)m = 0. \end{cases}$$

Let's suppose $\rho = \rho_1 = 1$. Then the system becomes:

$$\begin{cases} 2k_1' + 2\ell_1' = 0 \\ 2k_1' - \ell_1' = 0 \end{cases}$$

This system has the unique solution $k_1' = 0,\ \ell_1' = 0$. Value of the component m_1 is any. For the vector $\vec{u}_1$ to be normalized i.e. that $|\vec{u}_1| = 1$, we shall assume $m_1 = 1$. We have $\vec{u}_1 = (0,0,1)$.

Since $\rho = \rho_2 = 4$, the system becomes:

$$\begin{cases} -k_2' + 2\ell_2' = 0, \\ 2k_2' - 4\ell_2' = 0, \\ -3m_2' = 0. \end{cases}$$

Hence $k_2' = 2t$, $\ell_2' = t$, $m_2' = 0$, where $t-$ is any real number. But normalizing, we obtain $|\vec{u}_2| = \sqrt{k_2^2 + \ell_2^2 + m_2^2} = 1 \Rightarrow k_2 = \frac{2}{\sqrt{5}}$; $\ell_2 = \frac{1}{\sqrt{5}}$; $m_2 = 0$. So, $\vec{u}_2 = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0)$.

For the third proper number $\rho = \rho_3 = -1$ we have system:

$$\begin{cases} 4k_3' + 2\ell_3' = 0, \\ 2k_3' + \ell_3' = 0, \\ 2m_3' = 0. \end{cases}$$

From here $k_3' = -t$, $\ell_3' = 2t$, $m_3' = 0$, where t- is any real number. Normalizing $\vec{u}_3' = (-t, 2t, 0)$, we define $k_3 = -\frac{1}{\sqrt{5}}$, $\ell_3 = \frac{2}{\sqrt{5}}$, $m_3 = 0$, i.e. the vector is $\vec{u}_3 = (-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0)$. Thus, eigenectors of the square-law form are: $\vec{u}_1 = (0,0,1)$, $\vec{u}_2 = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0)$, $\vec{u}_3 = (-\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0)$, and a canonical form of the square-law form is: $\omega(\vec{x}) = z_1^2 + 4z_2^2 - z_3^2$.

3.2. Definite square-law form. Sylvester criterion

Definition. The material square-law form $\omega(\vec{x})$ is referred to as ***positively***

Definite form, if for any $\vec{x} \neq 0$ from R^n $\omega(\vec{x}) > 0$, and ***negatively definite*** form, if for any $\vec{x} \neq 0$ from R^n $\omega(\vec{x}) < 0$.

If for all vectors $\vec{x}$ from R^n the inequalities are not strict, i. e. $\omega(\vec{x}) \geq 0$ or $\omega(\vec{x}) \leq 0$, the square-law form is referred to as accordingly ***nonpositively*** or ***nonnegatively definite form*** or ***semidefinite form***. Definite and semidefinite square-law forms are referred to as refer to ***sign-definite forms.***

Square-law forms for which any of these conditions is not satisfied, are referred to ***as indeterminate*** square-law forms. In other words, the square-law form $\omega(\vec{x})$ is referred to as nondefinite if $\vec{x} \in R^n$ are distinct from zero and the square-law form takes both positive, and negative values.

Examples. The square-law form $\omega(\vec{x}) = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}$ is positively definite, since for anyone $\vec{x} \neq 0$ $\omega(\vec{x})$ > 0; the square-law form $\omega(\vec{x}) = \frac{x_1^2}{a^2} - \frac{x_2^2}{b^2}$ is indeterminate since the sign on the right part for $\vec{x} \neq 0$ can be both positive, and negative.

As each square-law form can be written down in canonical form, the square-law form will be positively definite; if all proper numbers of the matrix specifying the square-law form, will be positive, and negatively definite if all proper numbers are negative. ***Sylvester Criterion*** also gives the answer to a question about definiteness of the square-law form**.** For the square-law form with a symmetric matrix to be positively definite, it is necessary and sufficient that the principal minors of matrix to be positive, i. e.

$$\sigma_{11} > 0, \quad \begin{vmatrix} \sigma_{11}\sigma_{12}\sigma_{13} \\ \sigma_{21}\sigma_{22}\sigma_{23} \\ \sigma_{31}\sigma_{32}\sigma_{33} \end{vmatrix} > 0 \text{ ,..., } \begin{vmatrix} \sigma_{11}\sigma_{12}...\sigma_{1n} \\ \sigma_{21}\sigma_{22}...\sigma_{1n} \\ .... \\ \sigma_{n1}\sigma_{n2}...\sigma_{nn} \end{vmatrix} > 0.$$

Criterion of negatively definite form follows from Sylvester principle.

If $\omega(\vec{x}) > 0$, то $-\omega(\vec{x}) < 0$ and inversely. Then, according to Sylvester criterion, for $-\omega(\vec{x})$ we have

$$-\sigma_{11} > 0, \begin{vmatrix} -\sigma_{11} & -\sigma_{12} \\ -\sigma_{21} & -\sigma_{22} \end{vmatrix} > 0, \begin{vmatrix} -\sigma_{11} & -\sigma_{12} & -\sigma_{13} \\ -\sigma_{21} & -\sigma_{22} & -\sigma_{23} \\ -\sigma_{31} & -\sigma_{31} & -\sigma_{33} \end{vmatrix} > 0, \cdots, \begin{vmatrix} -\sigma_{11} & -\sigma_{12} & \dots & -\sigma_{1n} \\ -\sigma_{21} & -\sigma_{22} & \dots & -\sigma_{2n} \\ \dots & & & \\ -\sigma_{n1} & -\sigma_{n2} & \dots & -\sigma_{nn} \end{vmatrix} > 0$$

or
$$\sigma_{11} < 0, \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix} > 0, \begin{vmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{31} & \sigma_{33} \end{vmatrix} < 0, \cdots, (-1)^n \begin{vmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \dots & & & \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{vmatrix}.$$

Thus, if signs of the principal minors of the square-law form alternate, the square-law form is negatively definite.

EXERCISES

1. Define proper numbers and eigenvectors of linear transformation set by the matrix $A=\begin{pmatrix}5 & 2 & -3\\ 4 & 5 & -4\\ 6 & 4 & -4\end{pmatrix}.$

2. Show , by the example of the matrix $A=\begin{pmatrix}3 & 4\\ 5 & 2\end{pmatrix}$, that characteristic numbers of inverse matrix A^{-1} are inverse values of characteristic numbers of the matrix A.

3. Define proper numbers and eigenvectors of a symmetric matrix

$$S=\begin{pmatrix}7 & -2 & 0\\ -2 & 6 & -2\\ 0 & -2 & 5\end{pmatrix}.$$

Show, that eigenvectors are orthogonal

4. Matrixes are given

$$A=\begin{pmatrix}2 & -2 & 3\\ 10 & -4 & 5\\ 5 & -4 & 6\end{pmatrix} \text{ and } T=\begin{pmatrix}2 & 0 & 1\\ 1 & 3 & -1\\ -2 & 1 & 0\end{pmatrix}$$

Show by the example of matrixes A and $B = T^{-1}AT$, that similar matrixes have identical characteristic numbers.

5. Form the orthonormal basis $\vec{i}, \vec{j}, \vec{k}$ from eigenvectors of the matrix:

a) $A=\begin{pmatrix}7 & -2 & 0\\ -2 & 6 & -2\\ 0 & -2 & 5\end{pmatrix}$; b) $A=\begin{pmatrix}1 & 2 & -4\\ 2 & -2 & -2\\ -4 & -2 & 1\end{pmatrix}$.

6. Reduce square-law forms to the canonical kind and define their eigenvectors, if

a) $\omega(\vec{z}) = 3x^2 - 48xy + 27y^2, \quad \vec{z} = (x, y);$

b) $\omega(\vec{x}) = 99x_2^2 - 12x_1x_2 + 48x_1x_3 + 130x_2^2 - 60x_2x_3 + 71x_3^2, \quad \vec{x} = (x_1, x_2, x_3);$

c) $\omega(\vec{x}) = x_1^2 + x_1x_2 + x_2^2, \ \vec{x} = (x_1, x_2,).$

BOOK 3
ANALYTIC GEOMETRY

Analytic geometry is a branch of mathematics where geometric objects correspond with certain equations in such a way that the attributes of the objects are denoted in the attributes of the equations. If to consider a geometric object as some point set, equation of a object can be called as a rule of selection of space points for the set which forms the given geometric object.

CHAPTER 1
LINES, SURFACES AND THEIR EQUATIONS

§ 1. A LINE ON A CARTESIAN PLANE

Let us suppose that some line is set on coordinate plane *xOy*. Then points of a space can be divided into two categories in which some points are inhered in the given curve and others are not inhered.

Definition 1. The equation $F(x, y) = 0$, in which coordinates x, y for all points of a line are satisfying, and coordinates of points, not inhering in this line are not satisfying, is called ***equation of a line on a plane*** in Cartesian coordinate system.

In particular, the equation of line can be presented as an explicit function $y = f(x)$.

Examples

1. *A circle of radius R with centre at the point M* (x_0, y_0). Its equation is $(x - x_0)^2 + (y - y_0)^2 = R^2$
2. *A catenary*. Its equation is

$$y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{-\frac{x}{a}}\right) = a \cdot ch\,\frac{x}{a}$$

Along this line, a flexible inextensible heavy thread (chain, wire etc.), hung with the both ends is established in a balanced state.

Definition 2. ***Parametric equations of a line*** on a plane in Cartesian coordinate system is called as the equations having the form $x = \varphi(t), \quad y = \psi(t)$, where functions $x=\varphi(t)$ and $y=\psi(t)$ have the same domain, each value t from this domain corresponds with point $M(\varphi(\mathrm{t}),\psi(\mathrm{t}))$ for a line considered and each point M of this line corresponds with some value t from function domain $x=\varphi(t)$ and $y=\psi(t)$, i.e. we find such a value t for any point M, that $\varphi(t)$ and $\psi(t)$ become coordinates of point M.

For example, parametric equations of a circle of radius R with centre at point M look like

$$x=x_0+R\cos t,\quad y=y_0+R\sin t.$$

§ 2 SURFACE IN GEOMETRICAL SPACE

Definition 1. Equation of a surface in Cartesian coordinate system is called as equation $F(x,y,z)=0$, which is satisfied with coordinates of any point, lying on a surface, and is not satisfied with coordinates of points, not lying on the surface.

In particular, equation of a surface in Cartesian coordinate system can be set as the form, which is solved for one of coordinates, for example, as

$$z=f(x,y)$$

i. e. as a numerical function of two real variables.

Definition 2. Parametric equations of surface *Φ* in Cartesian coordinate system is an equation having the form

$$x=x(u,v),\quad y=y(u,v)\ \text{ and }\ z=z(u,v),$$

where functions $x(u,v)$, $y(u,v)$ and $z(u,v)$ have the same domain D (which constitutes the multiplicity of ordered number pairs u, v), each number pair u, v from domain D corresponds with point $M(x(u,v),\ y(u,v),\ z(u,v))$ of the surface *Φ* and we find such pair (u, v) from the domain D for any point M of the surface D, that $x(u,v)$, $y(u,v)$ and $z(u,v)$ become coordinates of the point M.

For example, equation of a sphere of radius R with centre at point $M(x_0, y_0, z_0)$ looks like

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=R^2$$

Parametric equation of a sphere considered is

$$x = x_0 + R\cos v\cos u\,,\quad y = y_0 + R\cos v\sin u\,,\quad z = z_0 + R\sin v\,.$$

Domain *D* of parameters *u,v* change is $0 \le u < 2\pi,\ -\pi/2 \le v \le \pi/2$

§ 3 LINE IN GEOMETRICAL SPACE

A line in geometric space *x*, *y*, *z* can be defined as a locus of intersection points of two surfaces, therefore such a line can be given by two equations of two surfaces intersecting along this line:

$$\begin{cases} F_1(x, y, z) = 0, \\ F_2(x, y, z) = 0. \end{cases} \qquad (1.1)$$

Thus, a line in space is a point set of this surface, satisfying set of equations (1.1).

Equations having form

$$x = \varphi_1(t),\ y = \varphi_2(t),\ z = \varphi_3(t)\,, \qquad (1.2)$$

and establishing dependence of current coordinates of point *M* on some parameter *t*, also determine a line in the geometric space.

As in the case of equations of a line on a plane, such equations are called parametric: they provide the parametric representation of a line in a geometric space.

In particular, if to consider equations (1.2) in the same way as equations, establishing dependence of current coordinates of a radius - vector $\vec{r} = \overrightarrow{OM}$ of point M in orthonormal basis $\vec{i}, \vec{j}, \vec{k}$ on some parameter *t*, then equation of a line in a geometrical space can be written as a vector function: (Book 2, Ch. 4, § 7):

$$\vec{r}(t) = \varphi_1(t)\vec{i} + \varphi_2(t)\vec{j} + \varphi_3(t)\vec{k} \qquad (1.3)$$

Equation (1.3) is called ***parametric equation of a line in the vector form*** or ***vector equation of a line*** in a geometrical space.

For example, equations $x = R\cos t,\quad y = R\sin t,\quad z = \lambda t$ or equation

$$\vec{r}(t) = R\cos t \cdot \vec{i} + R\sin t \cdot \vec{j} + \lambda t\vec{k}$$

are parametric equations or vector equation of the helix located on a cylinder surface with radius *R*.

§ 4. ALGEBRAIC LINES AND SURFACES

4.1. Algebraic lines on a plane

Definition 1. A line on coordinate plane is called ***algebraic*** if equation $F(x,y)=0$ is algebraic in some Cartesian coordinate system.

Algebraic equation is an equation which we get when equating an ***entire rational function*** to zero, i. e. the function obtained on condition that only the operations of multiplication and addition are conducted on arguments and numbers. For example,

$$z=\frac{2}{3}xy^2-xy+2x^2-3$$

Remark. Subtraction is supposed to be addition when one of addends is multiplied by -1 and division by the number which is not equal to zero is considered multiplication by the number which is its reciprocal.

Definition 2. If a line is defined with algebraic equation of n-th power in Cartesian coordinate system, then it is called ***algebraic line of n-th order***.

The degree of algebraic equation

$$F(x,y,z...)=0$$

is called the degree of an entire rational function *F*, i.e. a maximum value of sum $\alpha+\beta+\gamma+...$ of arguments' indexes in the equation having form

$$a\cdot x^{\alpha}\cdot y^{\beta}\cdot z^{\gamma}\cdot....$$

and the sum of these arguments is function *F*.

There is a function of the 3rd power in the example described above.

In all Cartesian coordinate systems algebraic line is defined with algebraic equation and has the same order. Thus, algebraic nature of algebraic equation and its order is invariant (i.e. constant) in relation to transformation of Cartesian coordinate system.

In analytic geometry algebraic lines of the 1st and 2nd order, i. e. the lines, set relatively Cartesian coordinate system with equation

$$Ax+By+C=0,\quad a_{11}x^2+a_{22}y^2+2a_{12}xy+a_{10}x+a_{20}y+a_{00}=0 \qquad (1.4)$$

are principally studied on plane.

The given equations are called general equations of the 1st and 2nd order lines.

4.2. Algebraic surfaces

Definition 1. Algebraic surface is called all points $M(x, y, z)$ set for geometrical space, and their coordinates satisfy algebraic equation

$$F(x, y, z) = 0. \qquad (1.5)$$

in Cartesian coordinate system.

Definition 2. Power of entire rational function $F(x, y, z)$ is called ***order of an algebraic surface***.

Like algebraic line, nature of algebraic equation (1.5) and order of algebraic surface are invariant in relation to transformation of Cartesian coordinate system.

In analytic geometry surfaces of the 1st and 2nd order, i.e. surfaces set relatively Cartesian coordinate system with equations

$$Ax + By + Cz + D = 0, \qquad (1.6)$$

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz + a_{10}x + a_{20}y + a_{30}z + a_{00} = 0$$

are principally studied in space.

The given equations are called ***general equations of the 1st and 2nd order surfaces***.

Thus, on the basis of the aforementioned, we can say that a line and a surface can be set in geometrical and analytical way by means of equation. To make line and surface equations, not only Cartesian coordinate system, but also polar coordinate system is used.

§ 5. POLAR SYSTEM OF COORDINATES ON THE PLANE AND IN SPACE

5.1. Polar system of coordinates on the plane

We say that polar coordinate system is introduced on plane, if point *O*, called ***pole***, and half-line *Ox*, called ***polar axis*** and coming out of point *O*, are chosen, scale bar *OE*=1, and angle reading off from the axis *Ox* to any beam, coming out of the pole *O* in positive direction is pointed out.

The position of any point M on a plane, not coinciding with pole, can be determined by means of such a coordinate system with two numbers: the number ρ, which signifies distance between point M and a pole, and number φ which is angle value, made by the beam coming out of a pole and consisting bar OM and polar axis. A positive direction for reading off angle φ is supposed to be the direction counterclockwise from the axis Ox. Ordered number pair is called ***polar coordinates*** of point M. The first coordinate ρ is called a ***polar radius*** and the second coordinate φ is called a ***polar angle***.

Numbers ρ and φ which are coordinates of point M are marked in the following way: $M(\rho,\varphi)$. We consider that $\rho = 0$, and φ is any number for the pole O.

In some cases there is a sign for polar radius ρ: we consider that $\rho < 0$ if angle φ is measured from polar axis to the beam which is made by elongation bar OM beyond point M.

Polar coordinates ρ and φ determine the position of a line on a plane with one value. Inverse statement is wrong, since every point on plane corresponds with the same coordinate ρ and infinite set of polar angles which can differ from one another on $2\pi k$ where $k \in Z$. Thus, unlike Cartesian coordinate system, polar coordinate system does not provide opportunity to establish one-to-one correspondence between set of all points on coordinate plane and set of ordered real number pairs. To acquire one-to-one correspondence, polar angle φ has restrictions:

$$0 \le \varphi < 2\pi \quad (\text{or } -\pi \le \varphi < \pi).$$

These values are called principal values of polar angles.

Let us set a link between polar and Cartesian coordinates of point M.

For this purpose we superimpose right Cartesian coordinate system xOy on polar coordinate system in such a way that the origin would coincide with pole and polar axis would coincide with positive x-semiaxis. Scale bar OE of polar coordinate system is considered as scale bar of Cartesian coordinate system (fig. 3.1).

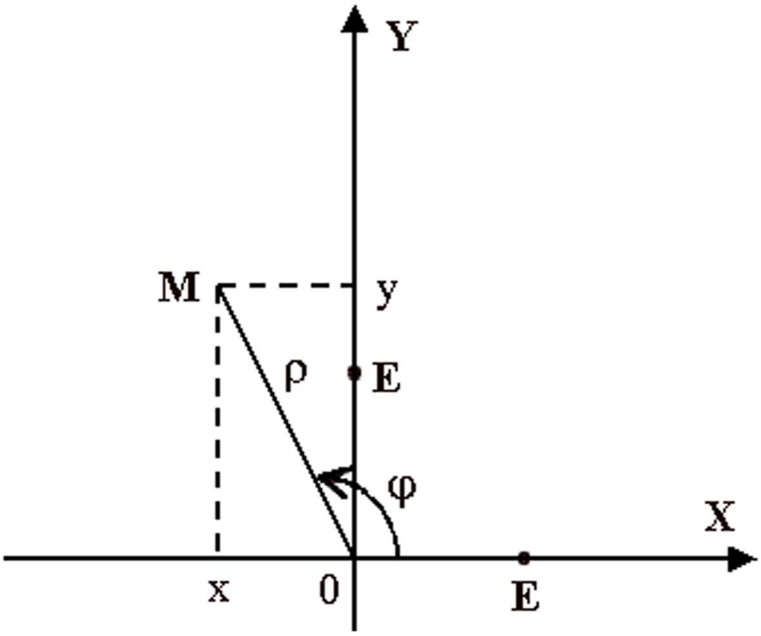


Fig. 3.1

Let us suppose that ρ and φ are polar coordinates of any plane point M, which does not coincide with the pole, and x and y are its Cartesian coordinates in the system shown above.

According to the definition of trigonometric functions, we have

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi \tag{1.7}$$

These formulas signify Cartesian coordinates of plane point, using polar coordinates. Solving the system (1.7) for ρ and φ (on condition $\rho > 0$), we acquire

$$\rho = \sqrt{x^2 + y^2} \tag{1.8}$$

$$\cos\varphi = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \sin\varphi = \frac{y}{\sqrt{x^2 + y^2}}. \tag{1.9}$$

or, if $x \neq 0$,

$$tg\varphi = \frac{y}{x}. \tag{1.10}$$

Formulas (1.8), (1.9), (1.10) allow us to calculate polar coordinates ρ and φ of point M on its Cartesian coordinates x and y. Provided that $0 \le \varphi < 2\pi$ (i.e. we consider principal values of polar angle φ), due to the formula (1.9) we have

$$\varphi = \begin{cases} arccos\dfrac{x}{\sqrt{x^2+y^2}} & \text{при } y \ge 0, \\ 2\pi - arccos\dfrac{x}{\sqrt{x^2+y^2}} & \text{при } y < 0. \end{cases}$$

In case $x \neq 0$, due to formula (1.10) we have

$$\varphi = \begin{cases} arctg\dfrac{y}{x} & \text{при } x > 0, y \ge 0, \\ \pi + \mathrm{arctg}\dfrac{y}{x} & \text{при } x \le 0, \\ 2\pi + arctg\dfrac{y}{x} & \text{при } x > 0,\ y < 0. \end{cases}$$

Equation of a line in polar coordinate system is called equation

$$\Phi(\rho, \varphi) = 0, \qquad (1.11)$$

where polar coordinates ρ and φ of any points, belonging to this line, and only coordinates of such points are satisfying.

In particular, equation of a line can take the form $\rho = \rho(\varphi)$ in polar coordinates. For example, equation $\rho = a$, where a = *const*, determines a circle with centre in pole and radius a in polar coordinates; equation $\rho = 2a\cos\varphi$ determines a circle with radius a and centre in the point $\rho = a$, $\varphi = 0$; equation $\rho = a\varphi$ determines a curve which is called the Archimedean spiral (we propose to draw this curve on your own).

5.2. Polar system of coordinates in space
Cylindrical and spherical coordinates

Let us consider coordinate plane *P*, where polar coordinate system is set, in plane. *Oz* is coordinate axis, perpendicular to plane *P*, and intersects it in the pole *O*. Coordinate axis *Oz* is directed in such a way that the direction for reading off positive values of polar angle φ from polar axis *Ox* is seen counterclockwise from the positive direction end of axis *Oz.* Complex of these elements is called ***polar coordinate system in space*** (fig. 3.2).

Coordinate plane P is called ***equatorial*** and coordinate axis *Oz* is ***zenithal***. To make it more comprehensive, we consider that scale bars is of the same length ($OE_1 = OE_2$) and starting point O for coordinate axis Oz coincides with the pole O. ***Cylindrical coordinates*** of point M which does not lie on zenithal axis are called ordered triple of numbers ρ, φ, z where ρ and φ are polar coordinates for orthogonal projection M_p of point M onto equatorial plane and z is coordinate for zenithal axis Oz *of* projection M_z of point M onto zenithal axis (fig. 3.3). We suppose that $\rho = 0$, φ can be any number, and z is determined according to the statement above for the points of zenithal axis.

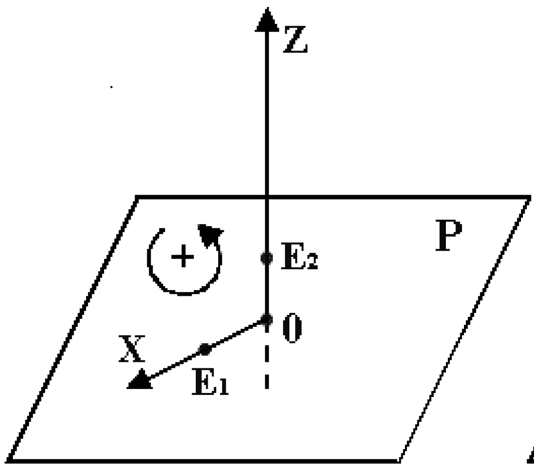


Fig. 3.2

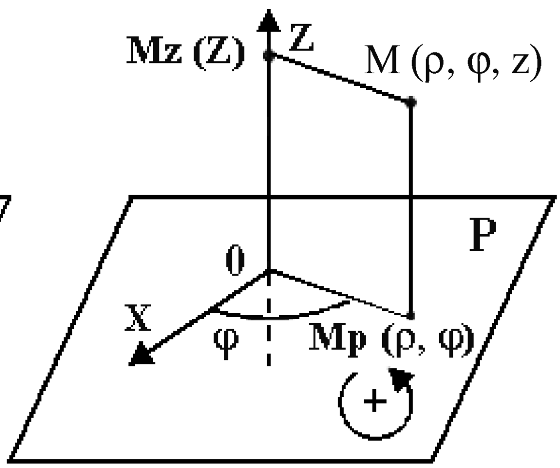


Fig. 3.3

Concerning ρ, φ and z are ***cylindrical coordinates*** of point M in space, we note them as $M\ (\rho,\ \varphi,\ z)$. We should mention that there is no one-to-one correspondence between set of all points in geometrical space and set of ordered triples of real numbers, established by means of cylindrical coordinates. Spherical coordinates of point M which does not lie on zenithal axis are called ordered triple of numbers r, φ, Θ, where r is a length of the bar OM, φ is an angle from polar axis Ox to beam OM_p (M_p is a projection of point M onto an equatorial plane) and Θ is an angle between beams OM_p and OM, which takes interval values $\left(-\pi/2, \pi/2\right)$; besides, we suppose that $\Theta = 0$ if point M lies on equatorial plane, $\Theta > 0$ if beam OM and zenithal axis make acute angle, $\Theta < 0$ if beam OM and zenithal axis make obtuse angle (fig. 3.4).

If point M lies on zenithal axis and does not coincide with pole O, then we consider that φ is any number and $\theta = +\pi/2$ or $\theta = -\pi/2$ (depending on direction coincidence or opposition between beam OM and zenithal axis). In regards of pole, we have $r = 0$, φ and Θ are any numbers. There is no one-to-one correspondence between set of all points in space and set of ordered real numbers triples, established by means of spherical coordinates.

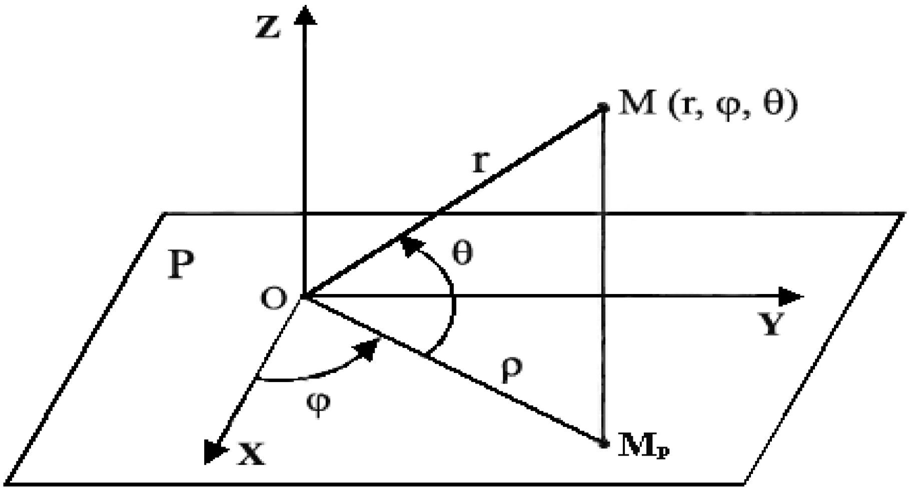


Fig. 3.4

Let us find dependency between Cartesian coordinates of point *M* (x, y, z) and its cylindrical coordinates $M(\rho, \varphi, z)$ and spherical coordinates $M(r, \varphi, \theta)$. We introduce Cartesian coordinate system and set polar axis as positive semi-axis *Ox*, axis, which is made as a result of axis *Ox* turn on angle $+\pi/2$ around pole in equatorial plane, as axis *Oy*, zenithal axis as axis *Oz* (fig. 3.4.).

Using fig. 3.4. and taking into account formula (1.7) and fig. 3.3, we find

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z = z \qquad (1.12)$$

Also $\rho = r\cos\theta$ and $z = r\sin\theta$, then we have

$$\begin{cases} x = r\cos\theta\cos\varphi, \\ y = r\cos\theta\sin\varphi, \\ z = r\sin\theta. \end{cases} \qquad (1.13)$$

Formulas (1.12) and (1.13) are right as well in cases when point M lies on zenithal axis and coincides with pole (last case requires additional conditions about values ρ and φ).

Cartesian coordinates of point *M* can be calculated on formulas (1.12) if we know its cylindrical coordinates, and on formulas (1.13) if we know its spherical coordinates.

From formulas (1.13) we have

$$x^2 + y^2 + z^2 = r^2,$$

$$x^2 + y^2 = \rho^2 = r^2 \cos^2\theta,$$

where $r\cos\theta = \sqrt{x^2 + y^2}$,

then

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2}, \\ \cos\varphi = \dfrac{x}{\sqrt{x^2 + y^2}}, \\ \sin\varphi = \dfrac{y}{\sqrt{x^2 + y^2}}, \\ \sin\theta = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}. \end{cases} \qquad (1.14)$$

Spherical coordinates *r*, *φ*, *Θ* of point M which do not lie on zenithal axis according to its Cartesian coordinates *x*, *y*, *z* (when mutual arrangement of these two coordinate system is pointed out) are calculated on these formulas.

Cylindrical coordinates *ρ*, *φ*, *z* of point M are calculated on Cartesian coordinates *x*, *y*, *z* from formulas (1.12) paying attention to formulas (1.8) и (1.9) and (1.10).

As well as Cartesian coordinates, the equation of surface in spherical coordinates is defined as $F(r,\varphi,\theta)=0$, and the equation of surface in cylindrical coordinates is $F(\rho,\varphi,z)=0$

Remark. The second spherical coordinate *φ* is often called ***longitude***, and the third spherical coordinate *Θ* is ***latitude***. Instead of latitude *Θ* we can sometimes consider angle *ψ* between positive direction of zenithal axis and beam *OM* which comes out of pole O towards given point M; value *ψ* changes from 0 to *π*. The value *ψ* is called ***zenithal distance***.

Concerning $\theta = \dfrac{\pi}{2} - \psi$, $\cos\theta$ and $\sin\theta$ should be replaced with $\sin\psi$ and $\cos\psi$ in formulas (1.13) and (1.14) (providing zenithal distance is considered as the third spherical coordinate).

CHAPTER 2
STRAIGHT LINE ON THE PLANE

§ 1. THE EQUATION OF THE STRAIGHT LINE PASSING THROUGH THE GIVEN POINT IN THE GIVEN DIRECTION

Definition. Directing vector of line is called any vector which is not equal to 0 and collinear to this line.

Theorem. In Cartesian coordinate system equation of line *p*, which passes through $M_0(x_0, y_0)$ with directing vector $\vec{a}(\ell, m)$ takes the form

$$\begin{vmatrix} x-x_0 & y-y_0 \\ \ell & m \end{vmatrix} = 0 \text{ or } \frac{x-x_0}{\ell} = \frac{y-y_0}{m} \qquad (2.1)$$

Proof. Let us consider any plane point $M(x, y)$. Point $M(x, y)$ lies on line *p* only in case vectors $\overrightarrow{M_0M}(x-x_0, y-y_0)$ and $\vec{a}(\ell, m)$ are collinear. Their collinearity is grounded on the equality (search for Book. 2, Chap. 4, § 3, p. 3.3):

$$\frac{x-x_0}{\ell} = \frac{y-y_0}{m}$$

or using theorem about disposition of a determinant on row elements (search for Book. 2, Chap. 6, § 2) this equality is written as

$$\begin{vmatrix} x-x_0 & y-y_0 \\ \ell & m \end{vmatrix} = 0$$

Equation (2.1) is called the ***canonical equation of a straight line***.

Remark. If one of denominators ℓ or m is equal to zero, equation (2.1) implies that its corresponding numerator is equal to zero as well.

§ 2. THE GENERAL EQUATION OF A LINE

Theorem 1. In Cartesian coordinate system a straight line is expressed with equation of the 1st power:

$$Ax + By + C = 0 \qquad (2.2)$$

Proof. Let us rewrite canonical equation of a straight line in the form of

$$mx - \ell y + \ell y_0 - mx_0 = 0.$$

Presuming that $m=A$, $-\ell = B$, $\ell y_0 - mx_0 = c$, we put it down as $Ax + By + C = 0$.

This is an equation of the 1st power because vector $\vec{a}(\ell, m)$ is not equal to zero, that is why A and B can not vanish at the same time ($A=m$, $B = -\ell$).

Theorem 2 (reverse variation). Any equation of the 1st power is the equation of a straight line in Cartesian coordinate system.

$$Ax + By + C = 0 \tag{2.3}$$

Proof. Let us suppose that x_0, y_0 is some solution of equation (2.3), i. e.

$$Ax_0 + By_0 + C = 0. \tag{2.4}$$

Then equation

$$A(x - x_0) + B(y - y_0) = 0 \text{ или } \begin{vmatrix} x - x_0 & y - y_0 \\ -B & A \end{vmatrix} = 0 \tag{2.5}$$

is the same as equation (2.3).

According to the theorem proved in the previous paragraph, this equation and, therefore, equation (2.3) is the equation of a line which passes through point M (x_0, y_0) and where leading vector is $\vec{a}(-B, A)$.

Theorem 3. Necessary and sufficient condition for considering vector $\vec{a}(\ell, m)$ collinear in relation to line is

$$A\ell + Bm = 0 \tag{2.6}$$

Proof. Let us mark off vector $\vec{a}(\ell, m)$ from any point M_0(x_0, y_0) belonging to the given line. End M of the vector marked off has coordinates $x_0 + \ell,\ y_0 + m$. Vector $\vec{a}(-B, A)$ is collinear to the given line, only in case point M lies on the given line, i.e. when equality is done:

$$A(x_0 + \ell) + B(y_0 + m) + C = 0$$

or

$Ax_0 + Bx_0 + C = 0$ (since point M_0 lies on the given line)

If a line is set relatively Cartesian coordinate system with equation (2.2), vector $\vec{n}(A, B)$ is perpendicular to this line.

Actually, if

$$\vec{a}\cdot\vec{n} = -B\cdot A + A\cdot B = 0$$

then vector $\vec{n}$ is perpendicular to directing vector $\vec{a}(-B, A)$ for this line, thus, vector $\vec{n}$ is perpendicular to the line itself. Vector $\vec{n}$ is called ***normal vector*** of this line.

Equation $Ax + By + C = 0$ is called the ***general equation of a straight line***.

Let us consider particular cases of disposition a straight line comparatively Cartesian coordinate system:

1. A straight line is collinear to axis *Ox* only in case *A*=0, since directing vector $\vec{a}(-B, A)$ of a straight line is collinear to axis *Ox* only in case the second coordinate of this vector is equal to zero.

Thus, the equation of a line $Ax + By + C = 0$ (provided a straight line is collinear to axis *Ox*) has the form of *By*+*C*=0 or $y = b$ (где $b = -\frac{c}{B}$).

2. In the same way we can prove that a straight line is collinear to axis *Oy* only in case *B* = 0, i.e. the general equation of a line $Ax + By + C = 0$ has the form of *Ax*+*C*=0, or $x = a \left(a = -\frac{C}{A}\right)$.

3. Necessary and sufficient condition for supposing that a straight line passes through the origin is equality *C* = 0, since only in case of *C* = 0 equation $Ax + By + C = 0$ is satisfied with origin coordinates.

Thus, the general equation of a line, crossing the origin, takes the form of *Ax*+*By*=0 and vice versa (i.e. any homogeneous equation of the 1st power *Ax*+*By*=0 determines a straight line crossing the origin).

§ 3. THE PARAMETRICAL EQUATIONS OF A STRAIGHT LINE

Theorem. Parametric equation of a line which passes through point $M_0(x_0, y_0)$ and has directing vector $\vec{a}(\ell, m)$, takes the form

$$x = x_0 + \ell t, \quad y = y_0 + mt$$

in Cartesian coordinate system.

Proof. Let us suppose that *M* (*x*, *y*) is any point of plane. The point will lie on the given plane only in case vectors $\overrightarrow{M_0M}(x - x_0, y - y_0)$ and

$\vec{a}(\ell, m)$ are collinear i. e. when they differ with the numerical factor (they are proportional)

$$\overrightarrow{M_0M} = t\vec{a} \qquad (2.7)$$

or with coordinates

$$x - x_0 = t\ell, \quad y - y_0 = tm \quad \text{which imply that}$$

$$x = x_0 + \ell t, \quad y = y_0 + mt.$$

If it takes all real values, then point M and these coordinates characterize the given whole plane.

When introducing radius-vectors $\overrightarrow{OM_0} = \vec{r}_0$ and $\overrightarrow{OM} = \vec{r}$ for points M_0 and M, we re-write ratio (2.7) in the following way:

$$\vec{r} - \vec{r}_0 = t\vec{a}$$

where we have

$$\vec{r} = \vec{r}_0 + t\vec{a} \qquad (2.8)$$

This is parametric equation of a straight line in vector form which passes through point $M_0(\vec{r}_0)$ and has directing vector $\vec{a}$.

§ 4. THE EQUATION OF A STRAIGHT LINE PASSING THROUGH TWO POINTS

Theorem. Equation of a line passing through two points $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ which are set relatively Cartesian coordinate system can be written in one of the following forms:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} \qquad (2.9)$$

or

$$\begin{vmatrix} x - x_1 & y - y_1 \\ x_2 - x_1 & y_2 - y_1 \end{vmatrix} = 0 \qquad (2.10)$$

or

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \qquad (2.11)$$

or in parametric form: $x = x_1 + (x_2 - x_1)t, \quad y = y_1 + (y_2 - y_1)t$.

Proof. Directing vector can be determined as vector

$$\vec{a} = \overrightarrow{M_1 M_2}(x_2 - x_1, y_2 - y_1)$$

then we will use the result got in § 1 and § 3.

§ 5. THE EQUATION OF A STRAIGHT LINE IN SEGMENTS

Let us suppose that a straight line does not pass through the origin and intersects the both coordinate axes in point (*a*, 0) for axis *Ox* and in point (0, *b*) for axis *Oy*.

Abscissa *a* and ordinate *b* for points of intersection line and *Ox* and *Oy* are often called line segments cut off on coordinate axes.

Equation of the given straight line looks like

$$\begin{vmatrix} x & y & 1 \\ a & 0 & 1 \\ 0 & b & 1 \end{vmatrix} = 0 \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} = 1 \qquad (2.12)$$

This equation is called intercept equation of a straight line.

§ 6. ANGULAR FACTOR OF A STRAIGHT LINE

Definition. Slope *k* of a line, set relatively Cartesian coordinate system, is called ratio of the second coordinate of directing vector $\vec{a}(\ell, m)$ to the first coordinate for this line:

$$k = \frac{m}{\ell}$$

Lines which are parallel to axis *Oy* and axis *Oy* do not have slope since the first coordinate of any directing vector is equal to zero for these lines.

Slope of every straight line which intersects axis *Oy* has some certain value without dependence on choice of directing vector.

Indeed, if (ℓ, m) and (ℓ_1, m_1) are directing vectors for the same line which intersects axis *Oy*, then they are collinear, and, therefore,

$k = \frac{m}{\ell} = \frac{m_1}{\ell_1}$, (where $\ell \neq 0$ and $\ell_1 \neq 0$).

In Cartesian coordinate system slope k of a line which intersects axis Oy is equal to tangent of angle α between axis Ox and directing vector for this line: $k = tg\alpha$

Indeed, if angle between axis Ox and vector $\vec{a}$ is equal to α, then taking account to formula (4.2) from book 2 and $\cos\gamma = 0$, then coordinates of vector $\vec{a}$ are equal to $\ell = |\vec{a}|\cos\alpha$ and $m = |\vec{a}|\cos\beta = |\vec{a}|\cos\left(\alpha - \pi/2\right) = |\vec{a}|\sin\alpha$.

Therefore, concerning these ratios, we have $k = \frac{m}{\ell} = tg\alpha$.

§ 7. THE EQUATION OF A STRAIGHT LINE WITH ANGULAR FACTOR

Equation of a line passing through point M_0 (x_0, y_0) and having slope k has the form

$$y - y_0 = k(x - x_0) \tag{2.13}$$

in Cartesian coordinate system.

Formula (2.13) follows from canonical equation of a straight line (§1). This equation includes set of all lines which belong to other line cluster with point M_0 as center, except one line, perpendicular to abscissa (parallel to axis Oy).

Definition. Set of all lines passing through one point M_0 and lying on one plane is called ***line cluster with point M_0 as center***.

The equation of a straight line which has slope k and intersects axis Oy in point (0, b) takes the form

$$y = kx + b \tag{2.14}$$

in Cartesian coordinate system.

Equation (2.14) is determined with equation (2.13), if we establish $y_0 = b,\ x_0 = 0$.

Number b is sometimes called ***initial ordinate of this line*** and equation (2.14) can be defined as ***equation of a line with given initial ordinate and given slope***.

If a line is set with general equation $Ax + By + C = 0$ in Cartesian coordinate system and if this line is not parallel to axis Oy, i.e. $B \neq 0$, and this equation is divided by B, we have

$$\frac{A}{B}x + y + \frac{C}{B} = 0, \quad \text{therefore,} \quad y = -\frac{A}{B}x - \frac{C}{B}$$

or

$$y = kx + b, \text{ где } k = -\frac{A}{B}, \ b = -\frac{C}{B}.$$

§ 8. POSITIONAL RELATIONSHIP OF TWO STRAIGHT LINES

Theorem. Let us suppose that equations of two lines are set relatively Cartesian coordinate system

$$A_1x + B_1y + C_1, \quad A_2x + B_2y + C_2 = 0. \qquad (2.15)$$

Then a necessary and sufficient condition of intersection these lines has the form

$$\frac{A_1}{A_2} \neq \frac{B_1}{B_2} \qquad (2.16)$$

Therefore, a necessary and sufficient condition of considering these lines parallel to each other has the form

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2} \qquad (2.17)$$

Therefore, a necessary and sufficient condition of coincidence these lines has the form

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \qquad (2.18)$$

A necessary and sufficient condition of coincidence two lines can be defined in another way: if there is such a number $\lambda \neq 0$ which leads to

$$A_1 = \lambda A_2, \quad B_1 = \lambda B_2, \quad C_1 = \lambda C_2$$

This theorem follows from considering conditions (2.16), (2.17) and (2.18) necessary and sufficient features that system (2.15) has only one solution, does not have solutions or is indefinite (i.e. has a lot of solutions).

Remark. Condition $\frac{A_1}{A_2} \neq \frac{B_1}{B_2}$ can be written as $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \neq 0$ and condition $\frac{A_1}{A_2} = \frac{B_1}{B_2}$ can be written as $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0$.

Condition $\frac{A_1}{A_2} \neq \frac{B_1}{B_2}$ or $\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \neq 0$ is necessary and sufficient condition to confirm that lines, set with equations (2.15) relatively Cartesian coordinate system, are parallel or coincide (otherwise they would be consider collinear).

§ 9. NORMAL EQUATION OF A LINE

Equation of a line set relatively Cartesian coordinate system $Ax + By + C = 0$ is called ***normal*** if normal vector $\vec{n}(A, B)$ for this line is ***unit***, i. e. $A^2 + B^2 = 1$.

To transform general equation of a line set regarding Cartesian coordinate system $Ax + By + C = 0$ into normal form, the left side of the given equation is multiplied by point *M*:

$$AMx + BMy + MC = 0$$

and point *M* should be chosen in such a way that vector (*AM*, *BM*) is unit:

$$(AM)^2 + (BM)^2 = 1$$

therefore,

$$M = \pm \frac{1}{\sqrt{A^2 + B^2}}$$

Thus, we always have two normal equations for each line:

$$\pm \frac{Ax + By + C}{\sqrt{A^2 + B^2}} = 0$$

It is also clear because there are two different unit vectors which are perpendicular to the given line.

Factors *M* are called normalizing factors. Radical $\sqrt{A^2 + B^2}$ is modulus of normal vector $\vec{n}(A, B)$ for this line, thus, we have $M = \pm \frac{1}{|\vec{n}|}$.

Coefficient of normal equation of a line, set relatively Cartesian coordinate system with equation

$$Ax + By + C = 0 \quad (A^2 + B^2) = 1$$

makes simple geometrical sense:

$$A = \vec{n} \cdot \vec{i} = cos\,\alpha,\ B = \vec{n} \cdot \vec{j} = cos\,\beta,\ |C| = p$$

where α and β are angles between unit vectors $\vec{i}$ and $\vec{j}$ for coordinate axes Ox, Oy and normal vector $\vec{n}(A,B)$ for straight line $Ax + By + C = 0$, p is distance between the origin of coordinates and this straight line.

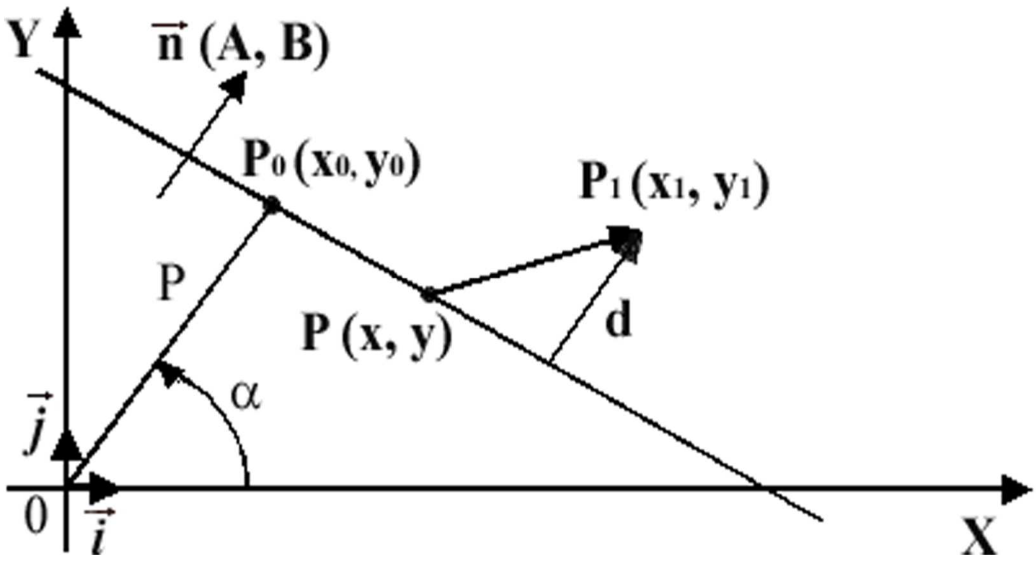


Fig. 3.5

If $C < 0$, then A and B are cosine and sine of angle α between positive direction of axis Ox and vector $\overrightarrow{OP_0}$ where P_0 is foot of perpendicular from origin of coordinate to the given straight line and $|C| = p$ is length of this perpendicular (fig. 3.5).

Indeed, if $C < 0$, then vectors $\overrightarrow{OP}_0(x_0, y_0)$ and $\vec{n}(A, B)$ are not only collinear but have the same direction, since

$$\vec{n} \cdot \overrightarrow{OP} = Ax_0 + By_0 = -C > 0$$

Angle α between positive direction of axis Ox and vector $\overrightarrow{OP_0}$ is equal to angle α between vector $\vec{i}(1,0)$ and normal vector $\vec{n}(A, B)$ for the given straight line and, therefore, $\vec{n}(\cos\alpha, \sin\alpha)$.

Thus, normal equation of a line, which does not pass through origin of coordinates, can be written as

$$x\cos\alpha + y\sin\alpha - p = 0, \qquad (2.19)$$

(this variant is frequent to use), where α and p have values pointed out above (fig. 3.5).

To transform general equation of a line $Ax + By + C = 0$ into normal form, the left side of this equation should be multiplied by normalizing factor

$$M = \pm \frac{1}{\sqrt{A^2 + B^2}},$$

beside this, it follows from condition $CM = -P < 0$ that we choose sign M in such a way that it is opposite to sign C.

§ 10. DISTANCE BETWEEN POINT AND STRAIGHT LINE

If a straight line is set relatively Cartesian coordinate system with normal equation (2.19), then distance d between point $P_1(x_1, y_1)$ and this line is equal to absolute value which is result of substituting coordinates of point P_1 into the left side of normal equation

$$d = |x_1 \cos\alpha + y_1 \sin\alpha - p|$$

Proof. Let us suppose that $P(x, y)$ is any point of the given straight line (fig. 3.5). Since vector $\vec{n}(\cos\alpha, \sin\alpha)$ is normal for this straight line according to Cartesian coordinate system, then it means that

$$d = \frac{|\vec{n} \cdot \overrightarrow{PP_1}|}{|\vec{n}|} = \frac{|\cos\alpha(x_1 - x) + \sin\alpha(y_1 - y)|}{\sqrt{\cos^2\alpha + \sin^2\alpha}} =$$

$$= |x_1 \cos\alpha + y_1 \sin\alpha - x\cos\alpha - y\sin\alpha| =$$

$$= |x_1 \cos\alpha + y_1 \sin\alpha - p|, \text{ (fig. 3.5)}$$

because $x\cos\alpha + y\sin\alpha - p = 0$ and $-x\cos\alpha - y\sin\alpha = -p$.

Remark. Sometimes sign is attributed to distance between point and straight line; such distance is called ***deviation*** and we suppose that

$$\delta = x_1 \cos\alpha + y_1 \sin\alpha - p \qquad (2.20)$$

A straight line divides a plane into two semi-planes. The semi-plane is called ***negative*** if $\delta < 0$ for points locating on this semi-plane with origin O (0, 0). If the semi-plane does not include origin of coordinates, then $\delta > 0$ and it is called ***positive*** semi-plane.

§ 11. ANGLE BETWEEN TWO STRAIGHT LINES; CONDITIONS FOR COLLINEARITY AND PERPENDICULARITY OF TWO STRAIGHT LINES

Let us suppose that two straight lines are set relatively Cartesian coordinate system with general equations

$$A_1x + B_1y + C_1 = 0, \quad A_2x + B_2y + C_2 = 0 \qquad (2.21)$$

Then angle between vectors $\vec{n}_1(A_1, B_1)$ and $\vec{n}_2(A_2, B_2)$ is equal to one of angles which are formed with these straight lines, and, therefore, cosines and sines of these angles are calculated by formulas

$$\cos\varphi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1A_2 + B_1B_2}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2^2}}; \qquad (2.22)$$

$$\sin\varphi = \frac{|[\vec{n}_1 \times \vec{n}_2]|}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1B_2 - A_2B_1}{\sqrt{A_1^2 + B_1^2} \cdot \sqrt{A_2^2 + B_2^2}}. \qquad (2.23)$$

Using formula (2.22), we find necessary and sufficient condition for perpendicularity of two straight lines:

$$A_1A_2 + B_1B_2 = 0, \qquad (2.24)$$

and using formula (2.23) we find condition for collinearity of two straight lines:

$$A_1B_2 - A_2B_1 = 0 \quad \left(\text{или} \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = 0 \right) \text{ (look at § 8)}. \qquad (2.25)$$

If the given straight lines are mutually perpendicular, then we have

$$tg\varphi = \frac{A_1B_2 - A_2B_1}{A_1A_2 + B_1B_2} \qquad (2.26)$$

Let us determine which form formula (2.26) takes, if straight lines (2.21) are set with equations which have angle coefficients $k_1 = -\frac{A_1}{B_1}$ and $k_2 = -\frac{A_2}{B_2}$. For this purpose we transform formula (2.26) and substitute values k_1 and k_2 into it:

$$tg\varphi = \frac{-\frac{A_2}{B_2} + \frac{A_1}{B_1}}{1 + \left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right)} = \frac{k_2 - k_1}{1 + k_1 k_2}. \qquad (2.27)$$

Thus, we find necessary and sufficient condition for perpendicularity and collinearity of two straight lines with angle coefficients:

$1 + k_1 k_2 = 0$ and $k_1 k_2 = -1$ is condition for perpendicularity

$k_1 = k_2$ is condition for collinearity.

CHAPTER 3
PLANE IN GEOMETRICAL SPACE

§ 1. THE EQUATION OF A PLANE PASSING THROUGH GIVEN POINT COPLANARLY TWO NON-COLLINEAR VECTORS

Theorem. In Cartesian coordinate system *x, y, z* the equation of space *P* which passes through point $M_0(x_0, y_0, z_0)$ and is coplanar to two non-collinear vectors $\vec{a}(x_1, y_1, z_1)$ and $\vec{b}(x_2, y_2, z_2)$ takes the form

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0 \qquad (3.1)$$

Proof. Let us suppose that point $M(x, y, z)$ is any point of a space. Point $M(x, y, z)$ lies on plane *P* only in case vectors $\overrightarrow{M_0M}(x-x_0, y-y_0, z-z_0)$, $\vec{a}$ and $\vec{b}$ are coplanar. Necessary and sufficient condition for coplanarity of these vectors takes the form (book II, ch.6, §3, p. 3.2)

$$\begin{vmatrix} x-x_0 & y-y_0 & z-z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0$$

§ 2. GENERAL EQUATION OF PLANE

Let us indicate that algebraic surface of the 1st order is plane. For this purpose we intend to prove the following theorems.

Theorem 1. In Cartesian coordinate system plane is determined with general equation of the 1st order relatively current coordinates.

Proof. We establish any point $M_0(x_0, y_0, z_0)$ on plane *P* and choose two non-collinear vectors $\vec{a}(x_1, y_1, z_1)$ и $\vec{b}(x_2, y_2, z_2)$ which are both collinear to plane *P*. Then taking into account the previous paragraph, equation of plane *P* can be written in the form of formula (3.1) or

$$\begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}(x-x_0) - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}(y-y_0) + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}(z-z_0) \qquad (3.2)$$

Since vectors $\vec{a}$ and $\vec{b}$ are collinear, at least one of the determinants

$$A = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad B = -\begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}, \; C = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$$

is not equal to zero. Indeed, in case all determinants are equal to zero, there is ratio

$$\frac{x_1}{x_2} = \frac{y_1}{y_2} = \frac{z_1}{z_2}$$

and, therefore, vectors are collinear. Thus, equation (3.2) is equation of the 1st order relatively x, y, z. If we suppose that

$$-x_0 \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} + y_0 \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} - z_0 \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = D$$

then equation (3.2) takes the form

$$Ax + By + Cz + D = 0 \qquad (3.3)$$

Equation (3.3) is called ***general equation of plane***.

Theorem 2 (reverse variation). General equation of the 1st order

$$Ax + By + Cz + D = 0 \qquad (3.4)$$

is equation of plane in Cartesian coordinate system x, y, z.

Proof. Let us suppose that x_0, y_0, z_0 is some solution to the given equation, i.e.

$$Ax_0 + By_0 + Cz_0 + D = 0 \qquad (3.5)$$

Equation (3.4) is equivalent to equation which is result of our subtraction equality (3.5) from equation (3.4):

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0 \qquad (3.6)$$

One of numbers A, B, C is equal to zero; if we suppose that $A \neq 0$, then equation (3.6) is equivalent to equation

$$\begin{vmatrix} x - x_0 & y - y_0 & z - z_0 \\ -B & A & 0 \\ -C & 0 & A \end{vmatrix} = 0 \qquad (3.7)$$

Indeed, last equation after revealing the determinant takes the form

$$A^2(x - x_0) + AB(y - y_0) + AC(z - z_0) = 0$$

or (because $A \neq 0$)

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Also vectors $\vec{a}(-B, A, 0)$ and $\vec{b}(-C, 0, A)$ are non-collinear since one of the determinants

$$\begin{vmatrix} A & 0 \\ 0 & A \end{vmatrix} = A^2, \quad \begin{vmatrix} 0 & -B \\ A & -C \end{vmatrix} = AB, \quad \begin{vmatrix} -B & A \\ -C & 0 \end{vmatrix} = AC$$

is not equal to zero (the first determinant is not equal to zero since $A \neq 0$). Thus, equation (3.7) and the given equation (3.4) determine plane (according to previous theorem) which passes through point (x_0, y_0, z_0) coplanarly two non-collinear vectors (provided $A \neq 0$):

$$\vec{a}(-B, A, 0) \text{ and } \vec{b}(-C, 0, A).$$

In the same way we prove that the given plane (provided $B \neq 0$) is coplanar to vectors $\vec{p}(0, -C, B)$ and $\vec{a}(-B, A, 0)$ which are not collinear to each other but provided $C \neq 0$, the given plane is coplanar to $\vec{p}(0, -C, B)$ and $\vec{b}(-C, 0, A)$ which are not collinear to each other as well.

Thus, ***every plane is surface of the 1st order and, vice versa, every surface of the 1st order is plane***.

§ 3. CONDITIONS FOR PERPENDICULARITY AND COPLANARITY OF VECTOR AND PLANE, SET WITH GENERAL EQUATION

Theorem 1. In Cartesian coordinate system x, y, z vector $\vec{n}(A, B, C)$ is perpendicular to plane, set with $Ax + By + Cz + D = 0$.

Proof. Let us choose two any different points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ from plane, set with general equation $Ax + By + Cz + D = 0$ relatively Cartesian coordinate system.

Then we have that

$$Ax_1 + By_1 + Cz_1 + D = 0, \qquad Ax_2 + By_2 + Cz_2 + D = 0,$$

therefore,

$$A(x_2 - x_1) + B(y_2 - y_1) + C(z_2 - z_1) = 0$$

or

$$\vec{n} \cdot \overrightarrow{M_1M_2} = 0 \tag{3.8}$$

Thus, if vector $\vec{n}$ is perpendicular to any straight line M_1M_2 which lies on the given plane, then it is also perpendicular to plane.

Provided $M_1(x_1, y_1, z_1) = M_0(x_0, y_0, z_0)$, $M_2(x_2, y_2, z_2) = M(x, y, z)$, equation (3.8) is called ***vector equation of a plane*** which passes through point $M_0(x_0, y_0, z_0)$ and is perpendicular to vector $\vec{n}(A,B,C)$, called ***basic*** or ***normal vector of plane***.

Theorem 2. Let us suppose that vector $\vec{a}(\ell, m, g)$ and plane are set with general equation

$$Ax + By + Cz + D = 0 \qquad (3.9)$$

relatively Cartesian coordinate system in space.

Then necessary and sufficient condition for coplanarity of vector $\vec{a}$ and given plane takes the form

$$A\ell + Bm + Cg = 0$$

Proof. Necessity. If vector $\vec{a}$ is coplanar to plane, set with general equation (3.9), then it is perpendicular to basic vector of plane $\vec{n}(A,B,C)$ and, therefore,

$$\vec{n} \cdot \vec{a} = A\ell + Bm + Cg = 0$$

Sufficiency. If there is $A\ell + Bm + Cg = 0$, then vector $\vec{a}(\ell, m, g)$ is perpendicular to vector $\vec{n}(A,B,C)$ and, thus, it is coplanar to plane which has vector $\vec{n}(A,B,C)$ as basic vector, i.e. that one, set with equation (3.9).

As a result of theorems 1 and 2, providing $A = 0$, we have equation (3.9) which takes the form

$$By + Cz + D = 0 \qquad (3.10)$$

and determines plane where normal vector $\vec{n}(0, B, C)$ is perpendicular to axis *Ox*. Thus, equation (3.10) determines plane which is parallel or passes through axis *Ox*.

In analogical way we define B=0 and $C = 0$ as necessary and sufficient conditions for considering that plane is parallel or passes through axes *Oy* and *Oz* both.

There it follows that plane is parallel or coincides with one of coordinate planes only in case two of coefficients A, B, C become zero in its general equation.

Thus, equations $Ax + D = 0,\ By + D,\ Cz + D = 0$ or $x = a,\ y = b,\ z = c$ and only similar equations of 1st power are equations of planes, parallel to coordinates planes, and in case of $a = 0,\ b = 0,\ z = 0$ they are also equations of coordinate axes *yOz, xOz, xOy*.

We should mention that equality $D = 0$ is necessary and sufficient condition for considering that plane, which is set with general equation (3.9), passes through origin of coordinates, since point O (0, 0, 0) satisfies the given equation in this case.

§ 4. THE EQUATION OF A PLANE, PASSING THROUGH THREE POINTS WHICH DO NOT BELONG TO ONE STRAIGHT LINE

Let us suppose that we have three points $M_1(x_1, y_1, z_1)$, $M_2(x_2, y_2, z_2)$, $M_3(x_3, y_3, z_3)$ and they do not lie on one straight line. These points determine plane, passing through them, in one way. Let us find equation of this plane.

We choose any point $M(x, y, z)$ in plane and draw vectors

$$\vec{a} = \overrightarrow{M_1M}(x - x_1, y - y_1, z - z_1),$$
$$\vec{b} = \overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1),$$
$$\vec{c} = \overrightarrow{M_1M_3}(x_3 - x_1, y_3 - y_1, z_3 - z_1).$$

Point $M(x, y, z)$ belongs to plane we look for only in case vectors $\vec{a}, \vec{b}, \vec{c}$ lie on one plane, i.e. when they are coplanar and, therefore, their triple product is equal to zero:

$$\left[\vec{a} \times \vec{b}\right] \cdot \vec{c} = 0.$$

We write this triple product, using coordinates of multiplied vectors

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \qquad (3.11)$$

Equation (3.11) is called equation of plane, passing through three given points.

§ 5. INTERCEPT FORM OF THE EQUATION OF A PLANE

If plane does not pass through origin of coordinates and therefore does not intersects all axes in points $(a,0,0),(0,b,0)$ è $(0,0,c)$ in Cartesian coordinate system, then we can write its equation, on the basis of previous paragraph in the following way:

$$\begin{vmatrix} x-a & y & z \\ -a & b & 0 \\ -a & 0 & c \end{vmatrix} = 0$$

or

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

This equation is called intercept form of the equation of a plane.

§ 6. MUTUAL ARRANGEMENT OF TWO PLANES

Let us suppose that two planes are set with their general equations relatively Cartesian coordinate system *x*, *y*, *z*:

$$A_1x + B_1y + C_1z + D_1 = 0 \qquad (3.12)$$

$$A_2x + B_2y + C_2z + D_2 = 0 \qquad (3.13)$$

We will consider various arrangement of these planes.

6.1. Condition for intersection of two planes and angle between them

Theorem. To assume that planes which are set with equations (3.12) and (3.13) in Cartesian coordinate system, intersect each other, we need necessary and sufficient condition: corresponding coefficients for x, y, z in equations (3.12) and (3.13) should not be proportional or at least one of determinants

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}, \quad \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}, \quad \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \qquad (3.14)$$

should not be equal to zero.

Proof. Necessary and sufficient condition for intersection of two planes is lack of collineariaty of their normal vectors $\vec{n}_1(A_1, B_1, C_1)$ and

$\vec{n}_2(A_2, B_2, C_2)$. Vectors $\vec{n}_1$ and $\vec{n}_2$ are not collinear only in case their corresponding coordinates are not proportional (book. 2, ch. 4, §3, p. 3.3), i.e. at least one of determinants (3.14) is not equal to zero. For instance

$$\frac{A_1}{A_2} \neq \frac{B_1}{B_2} = \frac{C_1}{C_2} \text{ means that } \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} \neq 0;$$

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} \neq \frac{C_1}{C_2} \text{ means that } \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} \neq 0.$$

Let us find angle between intersecting planes. This angle implies one of adjacent dihedral angles formed with these planes.

We suppose that angle between the given planes is φ. Then angle between normal vectors $\vec{n}_1(A_1, B_1, C_1)$ and $\vec{n}_2(A_2, B_2, C_2)$ for these planes is also φ or $\pi - \varphi$. We find angle φ, using formula of scalar product of two vectors $\vec{n}_1$ and $\vec{n}_2$:

$$\cos\varphi = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|} = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1^2 + B_1^2 + C_1^2} \cdot \sqrt{A_2^2 + B_2^2 + C_2^2}} \tag{3.15}$$

Adding $\varphi = \pi/2$ to this formula, we have ***condition for perpendicularity of planes:***

$$A_1A_2 + B_1B_2 + C_1C_2 = 0.$$

6.2. Condition for parallelism of two planes

Theorem. To assume that planes (3.12) and (3.13) are parallel, we need necessary and sufficient condition: corresponding coefficients for *x*, *y*, *z* in equations (3.12) and (3.13) should be proportional, but absolute terms should not be proportional to them, i.e. there should be such a number $\lambda \neq 0$, that $A_2 = \lambda A_1,\ B_2 = \lambda B_1,\ C_2 = \lambda C_1,\ D_2 \neq \lambda D_1$ or determinants

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = 0,$$

although at least one of determinants

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}, \quad \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix}, \quad \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}$$

should not be equal to zero.

Proof. Necessary and sufficient condition for parallelism of planes (3.12) and (3.13) is colleariaty of their normal vectors $\vec{n}_1(A_1, B_1, C_1)$ and $\vec{n}_2(A_2, B_2, C_2)$. According to condition of collinearity of vectors, we have $A_2 = \lambda A_1,\ B_2 = \lambda B_1,\ C_2 = \lambda C_1$ or

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = 0 \qquad (3.16)$$

Although another necessary and sufficient condition for parallelism of planes (3.12) and (3.13) is inconsistency of system which includes equations (3.12) and (3.13), i.e. any solution to equation (3.12) is not a solution to equation (3.13). It means that none of points which lie on plane, set with equation (3.12), belongs to plane, set with equation (3.13). According to Kronecker–Capelli theorem, system is inconsistent if ranks of augmented and coefficient matrixes are not equal. In case considered all minors of the 2nd order of coefficient matrix $A = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix}$ in system are equal to zero and not all minors of the 1st order are equal to zero, since one of coefficient in equations (3.12) and (3.13) both should not be equal to zero. Hence, rank of coefficient matrix is equal to 1: $r(A)=1$. Thus, to make system of equations (3.12) and (3.13) inconsistent, rank of its augmented matrix $A^* = \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix}$ should be equal to 2 and that is why we should have minor which is not equal to zero among minors of the 2nd order. The given condition is equivalent to $D_2 \neq \lambda D_1$.

6.3. Condition for coincidence of two planes

If planes (3.12) and (3.13) coincides with each other, then ranks of augmented and coefficient matrixes in system which includes equations (3.12) and (3.13) should coincide with each other as well (system is consistent) or should be equal to 1 (normal vectors $\vec{n}_1$ and $\vec{n}_2$ are collinear), i.e.

$$r\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{pmatrix} = r^*\begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{pmatrix} = 1.$$

There we have condition for coincidence of two planes:

$$\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & C_1 \\ A_2 & C_2 \end{vmatrix} = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} = \begin{vmatrix} A_1 & D_1 \\ A_2 & D_2 \end{vmatrix} = \begin{vmatrix} B_1 & D_1 \\ B_2 & D_2 \end{vmatrix} = \begin{vmatrix} C_1 & D_1 \\ C_2 & D_2 \end{vmatrix} = 0,$$

or

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}$$

§ 7. MUTUAL ARRANGEMENT OF THREE PLANES

Let us suppose that three planes are set with their general equations relatively Cartesian coordinate system *x*, *y*, *z*:

$$\begin{aligned} A_1x + B_1y + C_1z + D_1 &= 0, \\ A_2x + B_2y + C_2z + D_2 &= 0 \\ A_2x + B_3y + C_3z + D_3 &= 0 \end{aligned} \tag{3.17}$$

On the basis of previous condition, we get following necessary and sufficient conditions for mutual arrangement of three planes.

1. If determinant $D(A)$ of coefficient matrix $A = \begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$ is not equal to zero, then three planes have only one mutual point, since in case of $D(A) \neq 0$ system has only one solution: we get this solution, i.e. coordinates of one mutual points which belongs to three planes, when solving system (3.17) (for example, with usage of Cramer's formulas)

2. If $r(A) = 2$, rank $r^*(A^*)$ of augmented matrix $A^* = \begin{pmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{pmatrix}$ is equal to 3 and there are no collinear vectors among normal vectors $\vec{n}_1(A_1, B_1, C_1)$, $\vec{n}_2(A_2, B_2, C_2)$ and $\vec{n}_3(A_3, B_3, C_3)$, then system is inconsistent ($r^*(A^*) > r(A)$); planes intersect in pairs, moreover straight intersections are distinct.

3. If $r(A) = 2$, $r^*(A^*) = 3$, then there are two collinear vectors among normal vectors $\vec{n}_1(A_1, B_1, C_1)$, $\vec{n}_2(A_2, B_2, C_2)$ and $\vec{n}_3(A_3, B_3, C_3)$ (all three normal

vectors can not be collinear, because $r(A)=2$) and system is inconsistent; moreover, two planes are parallel and third one intersects them.

4. If $r(A)=2$, $r^*\left(A^*\right)=2$ and there are two collinear vectors among normal vectors $\vec{n}_1(A_1,B_1,C_1)$, $\vec{n}_2(A_2,B_2,C_2)$ and $\vec{n}_3(A_3,B_3,C_3)$, then planes are distinct and pass through one straight line.

5. If $r(A)=2$, $r^*\left(A^*\right)=2$ and there are two collinear vectors among normal vectors $\vec{n}_1(A_1,B_1,C_1)$, $\vec{n}_2(A_2,B_2,C_2)$ and $\vec{n}_3(A_3,B_3,C_3)$, then two planes coincide with each other and third one intersects them.

6. If $r(A)=1$, but coefficients of any pair are not proportional in equation (3.17), then planes are mutually parallel.

7. If $r(A)=1$, but there are only two equations which have proportional coefficients among equations (3.17), then two planes coincide with each other and the third one is parallel to them.

8. If $r^*\left(A^*\right)=1$, then all planes coincide with one another.

§ 8. NORMAL EQUATION OF PLANE

Let us assume that there is some plane which does not pass through origin of coordinates. We draw a beam from origin of coordinates (point *O*) and it is perpendicular to plane. Then we indicate point where beam and plane are intersected with letter *P* and length of perpendicular *OP* – with letter *p* (fig. 3.6).

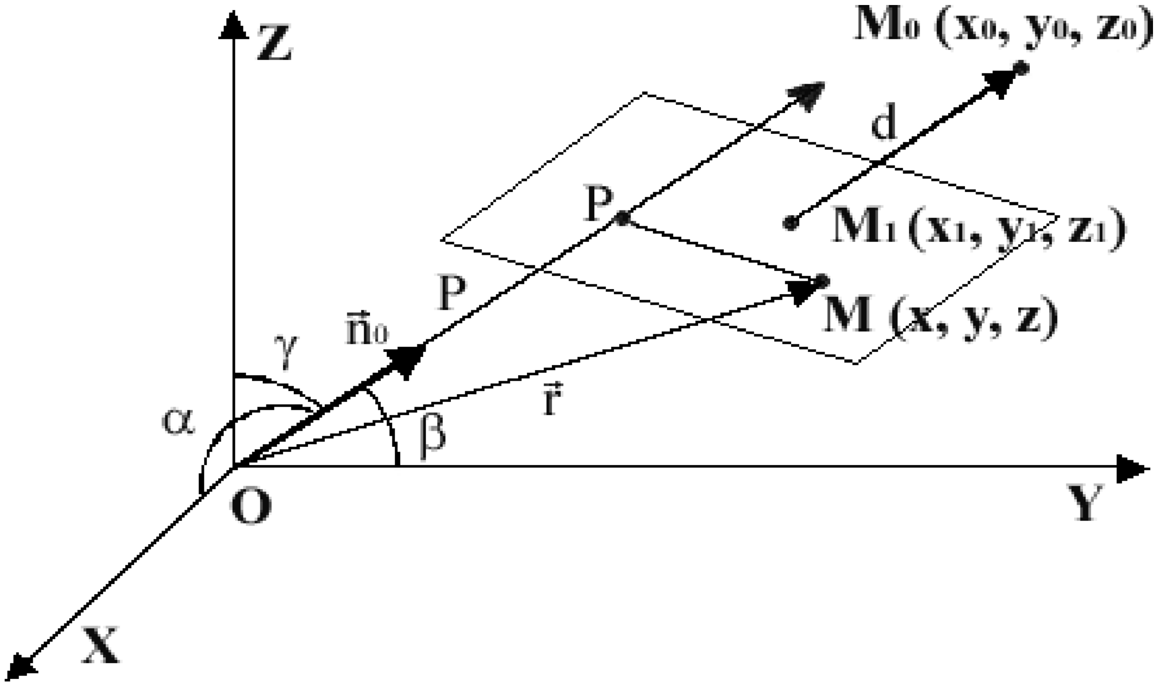


Fig. 3.6

We choose unit vector $\vec{n}_0$ with directing cosines of angles α, β, γ on beam $\overrightarrow{OP}$. Positive direction $\vec{n}_0$ is supposed to be direction from 0 to P. Let us set up equation, considering length $p = \left|\overrightarrow{OP}\right|$ and angles of slope α, β, γ between vector $\vec{n}_0$ and axes *Ox, Oy, Oz* known.

We choose any point $M(x, y, z)$. This point belongs to plane set only in case projection of its radius-vector $\vec{r}(x, y, z)$ onto beam $\overrightarrow{OP}$ is equal to *p*; this projection can be found out with usage of scalar product $\vec{r}(x, y, z)$ onto unit vector $\vec{n}_0(\cos\alpha, \cos\beta, \cos\gamma)$:

$$p = \vec{r} \cdot \vec{n}_0 = x\cos\alpha + y\cos\beta + z\cos\gamma$$

There we have equation

$$x\cos\alpha + y\cos\beta + z\cos\gamma - p = 0, \qquad (3.18)$$

which is called normal equation of plane.

We should notice that there is equality for cosines of directing angles of beam $\overrightarrow{OP}$:

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \qquad (3.19)$$

Thus, the equation of a plane set relatively Cartesian coordinate system is called normal if sum of squares of coefficients with variables is equal to 1, and absolute term is a negative number; i.e. provided

$$\begin{cases} A^2 + B^2 + C^2 = 1, \\ D < 0 \ \text{(т.к. } p = \left|\overrightarrow{OP}\right| > 0, \text{и, следовательно, } D = -p < 0) \end{cases} \quad (3.20)$$

In vector form, the normal equation of the plane has the form $\vec{r} \cdot \vec{n}_0 = p$ or $\vec{r} \cdot \vec{n}_0 - p = 0$.

If plane passes through the origin, then p = 0, and the direction of vector $\vec{n}_0$ can be chosen arbitrarily.

§ 9. TRANSFORMATION OF GENERAL EQUATION OF A PLANE TO NORMAL FORM

Let us assume that general equation of a plane is given:

$$Ax + By + Cz + D = 0 \quad (3.21)$$

We multiply both sides of this equation by number $M \neq 0$:

$$AMx + BMy + CMz + DM = 0 \quad (3.22)$$

Equation (3.22) can be transformed to normal form if conditions (3.20) are satisfied:

$$\begin{cases} (AM)^2 + (BM)^2 + (CM)^2 = 1, \\ DM < 0 \end{cases}$$

Solving this system relatively number M, we obtain

$$\begin{cases} |M| = \dfrac{1}{\sqrt{A^2 + B^2 + C^2}} \\ DM < 0 \end{cases}$$

Number M is called normalizing factor of equation (3.21).

If $D < 0$, then $M > 0$, therefore, we have

$$M = \frac{1}{\sqrt{A^2 + B^2 + C^2}}$$

If $D > 0$, then $M < 0$, therefore, we have

$$M = -\frac{1}{\sqrt{A^2 + B^2 + C^2}}$$

Thus, sign of normalizing factor is opposite to sign of absolute term for equation on plane. If $D = 0$, then number M can be chosen with any sign.

Therefore, in order to transform general equation of a plane into normal form, we need to multiply both parts of general equation by its normalizing factor.

§ 10. DISTANCE BETWEEN POINT AND PLANE

Theorem. If plane is set with normal equation

$Ax + By + Cz + D = 0$,

(where there are $A = \cos\alpha,\ B = \cos\beta,\ C = \cos\gamma,\ D = -p$) relatively Cartesian coordinate system, then distance d between point $M_0(x_0, y_0, z_0)$ and this plane is calculated by the formula

$d = |Ax_0 + By_0 + Cz_0 + D|$ (or $d = |x_0\cos\alpha + y_0\cos\beta + z_0\cos\gamma - p|$).

It means that distance between point and plane, set with normal equation relatively Cartesian coordinate system, is equal to absolute value which is a result of substituting coordinates of the given point into the left side of plane equation.

Proof. We drop a perpendicular from point M_0 to this plane and consider vector $\overrightarrow{M_1M_0}$, where point $M_1(x_1, y_1, z_1)$ is the base of perpendicular and belongs to plane. Then we have that $\left|\overrightarrow{M_1M_0}\right| = d$ (fig.3.6). Vectors $\vec{n}_0(\cos\alpha, \cos\beta, \cos\gamma)$ and $\overrightarrow{M_1M_0}(x_0 - x_1, y_0 - y_1, z_0 - z_1)$ are collinear, hence, concerning $|\vec{n}_0| = 1$, we have

$$\left|\overrightarrow{M_1M_0} \cdot \vec{n}_0\right| = \left|\overrightarrow{M_1M_0}\right| = |(x_0 - x_1)\cos\alpha + (y_0 - y_1)\cos\beta + (z_0 - z_1)\cos\gamma|.$$

There we obtain

$$d = \left|\overrightarrow{M_1M_0}\right| = |x_0\cos\alpha + y_0\cos\beta + z_0\cos\gamma - p|,$$

provided we take into account that $-x_1\cos\alpha - y_1\cos\beta - z_1\cos\gamma = -p$ (point M_1 belongs to plane).

If plane is set with general equation

$$Ax + By + Cz + D = 0,$$

Then, in order to find distance d between point $M_0(x_0, y_0, z_0)$ and plane, we should transform equation into normal form at first, and then find absolute value of its left side at point M_0:

$$d = \left| \frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2 + B^2 + C^2}} \right|.$$

Remark. Sometimes distance between point and plane obtains some sign; such distance is called deviation and we suppose that

$$\delta = \left|\overrightarrow{M_1M_0}\right| \cos\psi = x_0\cos\alpha + y_0\cos\beta + z_0\cos\gamma - p,$$

where ψ is angle between collinear vectors $\overrightarrow{M_1M_0}$ and $\vec{n}_0$.

Plane divides space into two half-spaces. There $\psi = -\pi$ and $\delta < 0$ for points in half-space which contains origin O (0, 0, 0). This half-space is called negative. Also there $\psi = 0$ and $\delta > 0$ if half-space does not contain origin (fig.3.6). This half-space is called positive.

CHAPTER 4
STRAIGHT LINE AND PLANE IN THREE-DIMENSIONAL SPACE

§ 1. EQUATIONS OF STRAIGHT LINE IN THE THREE-DIMENSIONAL SPACE

In a three-dimensional space a straight line can be set by means of different variants: by means of a point and a direction, intersection of two planes, two points, etc.

1.1. Canonical and parametric equations of the line

In Cartesian coordinate system equation of a straight line passing through point $M_0(x_0, y_0, z_0)$ and having directing vector $\vec{a}(\ell, m, k)$ are

$$\frac{x-x_0}{\ell}=\frac{y-y_0}{m}=\frac{z-z_0}{k} \tag{4.1}$$

These equations are called ***canonical equations of a straight line in a three-dimensional space*** or in parametric form:

$$x=x_0+\ell t,\ y=y_0+mt,\ z=z_0+kt. \tag{4.2}$$

Indeed, let us assume that point $M(x,y,z)$ is an arbitrary point; it lies on straight line, passing through point M_0, which is collinear to vector $\vec{a}$ only in case vectors $\overrightarrow{M_0M}(x-x_0, y-y_0, z-z_0)$ and $\vec{a}(\ell, m, k)$ are collinear, i. e. only in case coordinates of these vectors are proportional:

$$\frac{x-x_0}{\ell}=\frac{y-y_0}{m}=\frac{z-z_0}{k}.$$

Since $\vec{a}\neq\vec{0}$, then necessary and sufficient condition for collinearity of vectors $\overrightarrow{M_0M}$ and $\vec{a}$ can also be written as follows:

$\overrightarrow{M_0M}=t\vec{a}$ (the vectors are proportional),

or

$$(x-x_0, y-y_0, z-z_0)=t(\ell, m, k), \tag{4.3}$$

due to which we obtain equations (4.2).

One or two numbers among numbers ℓ, m, k can be equal to zero in equations (4.1). At the same time, all three numbers ℓ, m, k can not go to zero, since $\vec{a}\neq\vec{0}$. Let us suppose that provided one of the denominators in equation (4.1) becomes zero, then corresponding numerator becomes zero as well. For example, ratio $\frac{x-x_0}{0}$ means that $x-x_0=0$ or $x=x_0$, i.e. it determines plane which is perpendicular to axis *Ox*.

If $\ell=0$, then directing vector is perpendicular to abscissa axis. Then equations

$$\frac{x-x_0}{0}=\frac{y-y_0}{m}=\frac{z-z_0}{k}$$

determine straight line which is perpendicular to axis *Ox*.

In the same way equations with $m=0$ and $k=0$, determine straight lines which are perpendicular to axes *Oy* and *Oz* respectively. If $\ell=m=0$, $\ell=k=0$ or $m=k=0$, equations (4.1) determine straight lines which are parallel to coordinate axes *Oz, Oy, Ox* respectively.

Canonical and parametric equations of a straight line in a three-dimensional space can also be written in vector form. To do this, we introduce radius-vector $\vec{r}_0$ for point $M_0(x_0, y_0, z_0)$ and radius-vector $\vec{r}$ for point $M(x, y, z)$. Then, concerning collinearity of vectors $\vec{a}(\ell, m, k)$ and $\overline{M_0M}=\vec{r}-\vec{r}_0$, their vector product is equal to zero-vector

$$[\vec{r}-\vec{r}_0]\times\vec{a}=\vec{0} \qquad (4.4)$$

and equation (4.3) takes the form

$$(\vec{r}-\vec{r}_0)=t\vec{a} \text{ or } \vec{r}=\vec{r}_0+t\vec{a}. \qquad (4.5)$$

In coordinate expression, equation (4.4) takes the form of equations (4.1) and therefore it is called ***canonical equation of a line in vector form***, and equation (3.27) is the form of equations (4.2) and is called the ***equation of a straight line in three-dimensional space in vector-parametric form***.

1.2. Equations of a straight line passing through two points

The equations of a straight line passing through two different points $M_1(x_1, y_1, z_1)$ and $M_2(x_2, y_2, z_2)$ and set with Cartesian coordinate system can be written in the form

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \;,$$

or in parametric form

$$x = x_1 + t(x_2 - x_1),\ y = y_1 + t(y_2 - y_1),\ z = z_1 + t(z_2 - z_1).$$

Proof. Vector $\overrightarrow{M_1M_2}(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$ can be called directing vector of a straight line, and after this we can use the result of the preceding paragraph.

1.3. Straight line as line of intersection of two planes The general equation of the line

In the general case, a straight line can be set with equations of two planes

$$\begin{cases} A_1x + B_1y + C_1z + D_1 = 0 \\ A_2x + B_2y + C_2z + D_2 = 0 \end{cases}, \tag{4.6}$$

which intersect each other along this straight line. System (4.6) is called ***general equation of a straight line in three-dimensional space***.

To transform straight line, set with two intersecting planes (4.6), to canonical form, we should find some solution x_0, y_0, z_0 for system (4.6). Point $M_0(x_0, y_0, z_0)$ lies on straight line where planes (4.6) intersect each other. Then vector $\vec{a}$ with coordinates

$$\ell = \begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix},\quad m = \begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix},\quad k = \begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}$$

is directing vector of this straight line, since it is not equal to zero and coplanar to each of these planes. Indeed, using necessary and sufficient condition for coplanarity of vector and plane, we obtain

$$A_1\ell + B_1m + C_1k = A_1\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix} + B_1\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix} + C_1\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \end{vmatrix} = 0$$

and, hence, $A_2\ell + B_2m + C_2k = 0$, since vector $\vec{a}(\ell, m, k)$ is collinear to straight line where which planes (4.6) intersect each other.

Then canonical equations of a straight line (4.6) can be written in the form

$$\frac{x - x_0}{\begin{vmatrix} B_1 & C_1 \\ B_2 & C_2 \end{vmatrix}} = \frac{y - y_0}{\begin{vmatrix} C_1 & A_1 \\ C_2 & A_2 \end{vmatrix}} = \frac{z - z_0}{\begin{vmatrix} A_1 & B_1 \\ A_2 & B_2 \end{vmatrix}}$$

§ 2. ANGLE BETWEEN TWO STRAIGHT LINES IN THREE-DIMENSIONAL SPACE

Angles between two straight lines in space is any angle between two lines which are parallel to it and pass through some point in space. Thus, two straight lines in space (if they are not perpendicular) form two different angles: acute and obtuse ones. Sum of these angles is π.

Let us suppose that vectors $\vec{a}(\ell_1, m_1, k_1)$ and $\vec{b}(\ell_2, m_2, k_2)$ are directing vectors for the given lines, set relatively Cartesian rectangular coordinate system. Angle between these vectors is equal to one of angles formed with the given straight lines. Hence, cosines of angles between the given lines are expressed with \ formula

$$\cos\varphi_{1,2} = \pm\frac{\ell_1\ell_2 + m_1m_2 + k_1k_2}{\sqrt{\ell_1^2 + m_1^2 + k_1^2}\cdot\sqrt{\ell_2^2 + m_2^2 + k_2^2}}.$$

There we obtain necessary and sufficient condition for perpendicularity of two lines:

$$\ell_1\ell_2 + m_1m_2 + k_1k_2 = 0;$$

to make two straight lines mutually perpendicular, we need necessary and sufficient condition: sum of products for corresponding coordinates, which relate to directing vectors of these lines, should be equal to zero.

§ 3. CONDITION FOR CONSIDERING TWO STRAIGHT LINES BELONGING TO THE SAME PLANE

Let us suppose that two straight lines are set with canonical equations

$$\frac{x - x_1}{\ell_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{k_1}$$
$$\frac{x - x_2}{\ell_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{k_2} \qquad (4.7)$$

If two straight lines belong to the same plane, their directing vectors $\vec{a}_1(\ell_1, m_1, k_1)$, $\vec{a}_2(\ell_2, m_2, k_2)$ and vector $\overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ are coplanar, i. e. there is mixed product:

$$[\vec{a}_1 \times \vec{a}_2]\cdot\overrightarrow{M_1M_2} = 0$$

There, condition for considering straight lines belonging to the same plane, which is written in coordinate form, has the form

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \ell_1 & m_1 & k_1 \\ \ell_2 & m_2 & k_2 \end{vmatrix} = 0$$

§ 4. DISTANCE BETWEEN POINT AND STRAIGHT LINE IN THREE-DIMENSIONAL SPACE

Let us suppose that point $M_1(x_1, y_1, z_1)$ and a straight line are set with canonical equation

$$\frac{x - x_0}{\ell} = \frac{y - y_0}{m} = \frac{z - z_0}{g}$$

relatively Cartesian rectangular coordinate system in space.

Distance d between point M_1 and straight line can be defined as height of a parallelogram, whose sides are vector $\overrightarrow{M_0M_1}$ and directing vector for straight line, which is marked off from point M_0 of this line. Therefore, to find out the distance d, we consider modulus of vector product:

$$\left|\left[\overrightarrow{M_0M_1} \times \vec{a}\right]\right| = \left|\overrightarrow{M_0M_1}\right| \left|\vec{a}\right| sin\varphi$$

although $|M_0M_1| \sin\varphi = d$ is height of a parallelogram, that is why we have

$$\left|\left[\overrightarrow{M_0M_1} \times \vec{a}\right]\right| = d\left|\vec{a}\right|, \text{ thus,}$$

$$d = \frac{\left|\left[\overrightarrow{M_0M_1} \times \vec{a}\right]\right|}{\left|\vec{a}\right|}$$

Since $\overrightarrow{M_0M_1}(x_1 - x_0, y_1 - y_0, z_1 - z_0)$, $\vec{a}(\ell, m, g)$, then we obtain that

$$d = \frac{\left\|\begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ x_1 - x_0 & y_1 - y_0 & z_1 - z_0 \\ \ell & m & g \end{matrix}\right\|}{\sqrt{\ell^2 + m^2 + g^2}} =$$

$$= \frac{\sqrt{\begin{vmatrix} y_1 - y_0 & z_1 - z_0 \\ m & g \end{vmatrix}^2 + \begin{vmatrix} z_1 - z_0 & x_1 - x_0 \\ g & \ell \end{vmatrix}^2 + \begin{vmatrix} x_1 - x_0 & y_1 - y_0 \\ \ell & m \end{vmatrix}^2}}{\sqrt{\ell^2 + m^2 + g^2}}$$

§ 5. ANGLE BETWEEN STRAIGHT LINE AND PLANE. CONDITION FOR PERPENDICULARITY OF STRAIGHT LINE AND PLANE

Angle between straight line and plane (provided they are not perpendicular) is called smaller angle among two angles between this straight line and its orthogonal projection onto this plane. If the straight line and the plane are perpendicular, then the angle between them is equal to $\pi/2$.

Orthogonal projection of a straight line onto a plane is a straight line formed by the intersection of the given plane with a plane passing through a given straight line perpendicular to the given plane.

Let us suppose that a plane is set with general equation

$$Ax + By + Cz + D = 0 \tag{4.8}$$

and a straight line is set with canonical equation

$$\frac{x - x_0}{\ell} = \frac{y - y_0}{m} = \frac{z - z_0}{g}. \tag{4.9}$$

relatively Cartesian rectangular coordinate system.

We designate angle between straight line and plane by φ, angle between normal vector $\vec{n}(A, B, C)$, perpendicular to the given plane, and directing vector $\vec{a}(\ell, m, g)$ for this straight line by α (fig. 3.7).

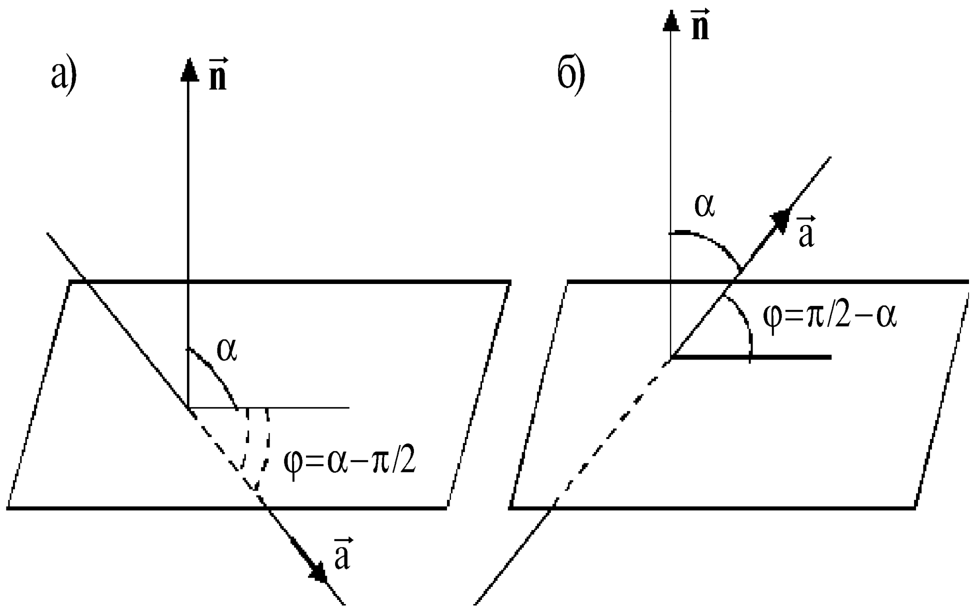


Fig. 3.7

Then $\varphi = \alpha - \pi/2$ (fig. 3.7, a) or $\varphi = \pi/2 - \alpha$ (fig. 3.7, b) and $\sin\varphi = |\cos\alpha|$. Cosine of angle α between vectors $\vec{n}$ and $\vec{a}$ is

$$\cos\alpha = \frac{\vec{n}\cdot\vec{a}}{|\vec{n}|\cdot|\vec{a}|} = \frac{A\ell + Bm + Cg}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{\ell^2+m^2+g^2}},$$

thus, sine of angle φ between the given straight line and the given plane is determined with

$$\sin\varphi = \frac{|A\ell + Bm + Cg|}{\sqrt{A^2+B^2+C^2}\cdot\sqrt{\ell^2+m^2+g^2}}.$$

If straight line (4.9) is perpendicular to plane (4.8), then directing vector $\vec{a}(\ell, m, g)$ for the straight line is collinear to vector $\vec{n}(A, B, C)$ which is perpendicular to the given plane. So, coordinates of these vectors are proportional, i.e. there is such a number λ which is not equal to zero that

$$A = \lambda\ell,\quad B = \lambda m,\quad C = \lambda g,$$

or

$$\frac{A}{\ell} = \frac{B}{m} = \frac{C}{g}.$$

On the contrary, if these ratios are accomplished, vectors $\vec{a}(\ell, m, g)$ and $\vec{n}(A, B, C)$ are collinear, i.e. directing vector for the given straight line is collinear to vector $\vec{n}(A, B, C)$, perpendicular to the given plane, hence, the given straight line and plane are perpendicular to each other.

So, to consider straight line and plane perpendicular, which are set relatively Cartesian rectangular coordinate system, we need necessary and sufficient condition according to which coordinates of directing vector for a straight line should be proportional to coefficients with *x, y, z* in the equation of a plane.

§ 6. THE SHORTEST DISTANCE BETWEEN TWO SKEW LINES

Two straight lines of three-dimensional space are called skew ones, if they do not intersect and are not parallel, i.e. do not lie on the same plane. Only two parallel planes can be drawn through skew lines. The distance between these planes is the shortest distance between the given lines which designates length of segment between common perpendicular and these two straight lines, whose ends lie on these straight lines.

Let us assume that two skew lines are set with equations (4.7) and we need to find the shortest distance *d* between them.

Vector product of directing vectors $\vec{a}_1(\ell_1, m_1, g_1)$ and $\vec{a}_2(\ell_2, m_2, g_2)$ for the given straight lines is vector, which is perpendicular to each of these straight lines: $\vec{n} = [\vec{a}_1 \times \vec{a}_2]$. Then the shortest distance *d* between them is equal to absolute value of vector projection

$$\overrightarrow{M_1M_2}(x_2 - x_1, y_2 - y_1, z_2 - z_1),$$

whose beginning $M_1(x_1, y_1, z_1)$ and end $M_2(x_2, y_2, z_2)$ lie on the 1st and 2nd straight lines reciprocally, on a straight line which is parallel to vector $\vec{n}$.

$$d = \left| \frac{\overrightarrow{M_1M_2} \cdot \vec{n}}{|\vec{n}|} \right|, \text{ or } d = \frac{\left| \overrightarrow{M_1M_2} \cdot [\vec{a}_1 \times \vec{a}_2] \right|}{\left| [\vec{a}_1 \times \vec{a}_2] \right|}$$

In the coordinates

$$d = \frac{\left\| \begin{matrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ \ell_1 & m_1 & g_1 \\ \ell_2 & m_2 & g_2 \end{matrix} \right\|}{\sqrt{\begin{vmatrix} m_1 & g_1 \\ m_2 & g_2 \end{vmatrix}^2 + \begin{vmatrix} g_1 & \ell_1 \\ g_2 & \ell_2 \end{vmatrix}^2 + \begin{vmatrix} \ell_1 & m_1 \\ \ell_2 & m_2 \end{vmatrix}^2}}$$

We should mention that formula also matches two straight lines, intersecting each other: numerator vanishes zero, the denominator is not equal to zero and that is why $d = 0$.

CHAPTER 5
LINES AND SURFACES OF THE 2nd ORDER

Let us remember equation of a surface of the 2nd order (1.6):

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{13}xz + 2a_{10}x + 2a_{20}y + 2a_{30}z + a_{00} = 0 \qquad (5.1)$$

If any plane (a surface of the 1st order) intersects a surface of the 2nd order, then the line obtained becomes a curve of the 2nd order. Without loss of generality, we can take any of the coordinate planes as a secant plane.

A system of equations consisting of equation (5.1) and one of the equations $x = 0,\ y = 0,\ z = 0$, determines a curve of the 2nd order which, in its turn, puts on coordinate planes *yOz, xOz* and *xOy* reciprocally. Next we will consider lines of the 2nd order which puts on *xOy*.

General equation of such a line takes the form (1.4)

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{10}x + 2a_{20}y + a_{00} = 0\,, \qquad (5.2)$$

where there is $a_{11}^2 + a_{12}^2 + a_{22}^2 \neq 0$, i.e. at least one of numbers a_{11}, a_{12}, a_{22} is not equal to zero.

§ 1. LINES OF THE 2nd ORDER, SET WITH CANONICAL EQUATIONS

1.1. Ellipse

Definition. An ellipse is the locus of points in the plane, whose sum of the distances for each of them between two points (which are focuses) of the same plane is a constant number 2*a* and it is bigger than the ***focal distance*** 2*c* between focuses $(a > 0, c > 0)$.

Let us suppose that *M* is an arbitrary number of an ellipse and F_1 and F_2 (and the lengths of segments r_1 and r_2) are its focuses. Segments MF_1 and MF_2 are called ***focal radiuses*** of point *M* for ellipse. According to this definition about ellipse we have

$$r_1 + r_2 = 2a = const \qquad (5.3)$$

We express focal radiuses r_1 and r_2 using coordinates of points *M,* F_1 and F_2. For this purpose, we introduce Cartesian rectangular coordinate system *xOy* to the plane, taking segment middle F_1F_2 as origin and line

F_1F_2 (distance between points F_1 and F_2) as axis Ox (fig. 3.8). In this coordinate system focus F_1 has coordinates -c, 0, and focus F_2 has coordinates c, 0. Expressing coordinates of points M by x and y, we obtain

$$r_1 = \sqrt{(x+c)^2 + y^2}, \quad r_2 = \sqrt{(x-c)^2 + y^2}$$

and ratio (5.3) takes the form

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a \qquad (5.4)$$

At first it can seem unclear whether equation (5.4) relates to equations of curves of the 2nd order as equation (5.2). Somewhat we transform equation (5.4), in particular, we dispose of irrationality. We transpose the first radical to the right side and square both parts of obtained equation:

$$(x-c)^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + (x+c)^2 + y^2.$$

After the transformation we find

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx.$$

Squaring both parts of this equation we obtain (according to condition $a > c$, то $a^2 - c^2 > 0$)

$$a^2\left[(x+c)^2 + y^2\right] = \left(a^2 + cx\right)^2,$$

or

$$\left(a^2 - c^2\right)x^2 + a^2y^2 = a^2\left(a^2 - c^2\right).$$

Expressing $a^2 - c^2$ by b^2 and taking $a^2\left(a^2 - c^2\right) = a^2b^2 > 0$ into account we find

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (5.5)$$

We proved that the coordinates of any point M (x, y) of the ellipse satisfy equation (5.5), hence, equation of curve of the 2nd order (5.2), provided that

$$a_{11} = \frac{1}{a^2}, \; a_{22} = \frac{1}{b^2}, \; a_{00} = -1, \; a_{12} = a_{10} = a_{20} = 0.$$

However, equation (5.5) can not be called equation of an ellipse yet, since reverse statement is not proved i. e. if the numbers x and y satisfy equation (5.5) and $a^2 - c^2 = b^2 > 0$ then point *M* with coordinates *x* and *y* satisfies ratio (5.4), i. e. it lies on the ellipse.

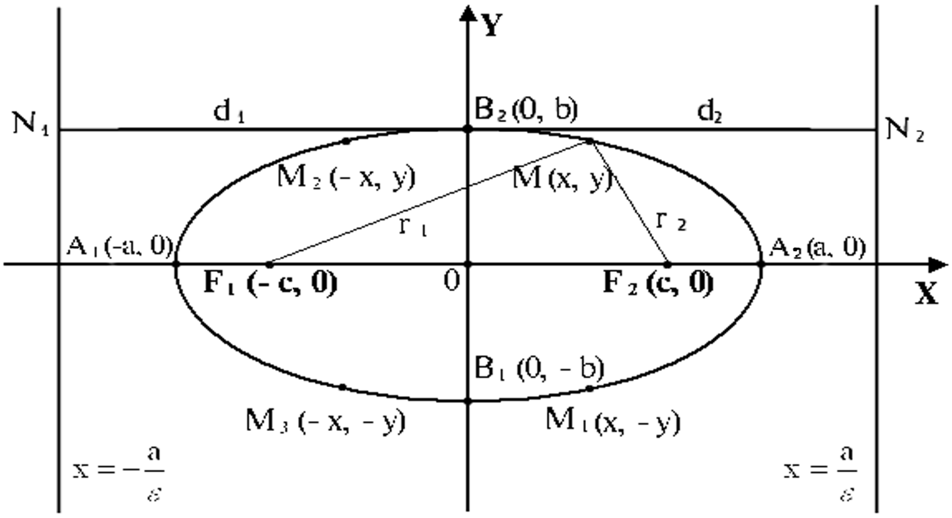


Fig. 3.8

We are to prove it. Let us assume that coordinates of point *M* (*x*, *y*) satisfy equation (5.5). Then using equation (5.5) we find $y^2 = \frac{b^2}{a^2}\left(a^2 - x^2\right)$.

We define focal radiuses of point *M* (*x*, *y*)

$$r_1 = \sqrt{(x+c)^2 + \frac{b^2}{a^2}\left(a^2 - x^2\right)} = \sqrt{(x+c)^2 + \frac{a^2 - c^2}{a^2}\left(a^2 - x^2\right)} =$$

$$= \sqrt{\frac{a^4 + 2a^2cx + c^2x^2}{a^2}} = \sqrt{\frac{\left(a^2 + cx\right)^2}{a^2}} = \left|a + \frac{cx}{a}\right|.$$

In the same way we can acquire value of r_2:

$$r_2 = \left|a - \frac{cx}{a}\right|.$$

Since

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

then we have $|x| \le a$, and since $0 < c < a$, then we have $a + \frac{cx}{a} > 0$ and $a - \frac{cx}{a} > 0$, hence,

$$r_1 = a + \frac{cx}{a}, r_2 = a - \frac{cx}{a}, \tag{5.6}$$

and thus

$$r_1 + r_2 = 2a.$$

So, equation (5.5) is equation of an ellipse and is called canonical.

Ellipse properties

1. According to canonical equation of an ellipse (5.5), an ellipse relates to curves of the 2[nd] order.

2. According to equation (5.5), $|x| \le a,\ |y| \le b$.

In geometrical sense, it means that an ellipse is located inside a rectangle whose sides are lines $x = a,\ x = -a,\ y = b,\ y = -b$, i.e. an ellipse is limited curve.

3. Since coordinates *x* and *y* raise into an even power (namely, into the second power), then if point *M* (*x*, *y*) lies on an ellipse, then points $M_1(x, -y)$ and $M_2(-x, y)$ lie on the same ellipse, which are symmetric (either point *M*) relatively axes *Ox* and *Oy*, and $M_3(-x, -y)$ is symmetric (either point *M*) relatively an origin (fig.3.8). Therefore, coordinate axes *Ox* and *Oy* for an ellipse set with canonical equation (5.5) are axes of symmetry, and the origin is a center of symmetry.

Points where an ellipse and its symmetry axes intersect each other are called ***vertices of an ellipse***. Thus, an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has four vertices: $A_1(-a, 0),\ A_2(a, 0),\ B_1(0, -b),\ B_2(0, b)$ (fig.3.8)

Values $2a$ and $2b$ are called major and minor axes of an ellipse reciprocally, *a* and *b* are called major and minor semi-axes.

Solving equation of ellipse relatively *y* and taking only a non-negative value for it

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \tag{5.7}$$

and considering $0 \le x \le a$, we obtain points of ellipse (5.5), which lie on the first quadrant. It follows from equation (5.7) that values *y* decrease when *x* increases from zero to $x = a$, moreover $y = b$ when $x = 0$ and $y = 0$ when $x = a$. By means of adding arcs which are symmetric relatively coordinate axes and origin to the arc set with equation (5.7), we obtain a closed line, which represents graph of an ellipse.

So, ellipse is a closed line with a single center of symmetry and only two axes of symmetry which are perpendicular to each other (если $a \ne b$).

A curve having a center of symmetry is called ***central***.

4. If in equation (5.5) there is $a = b$, then we obtain equation of a circle $x^2 + y^2 = a^2$ (5.8)

with an origin as a center and radius a.

Thus, circle is an ellipse whose focuses coincide with the center of symmetry, i.e. focal distance $2c = 2\sqrt{(a^2 - b^2)}$ is equal to zero.

To give characteristics to an ellipse, we introduce ratio $\xi = \frac{c}{a}$, which is called ***eccentricity of an ellipse***

$$\xi = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a} = \sqrt{1 - \frac{b^2}{a^2}} \qquad (5.9)$$

Eccentricity characterizes deviation of ellipse from circle, i.e. how much it is stretched. We have $\xi = 0$ for a circle and $0 < \xi < 1$ for an ellipse.

Lines $x = \pm\frac{a}{\xi}$ are called ***directrices*** of an ellipse. Since $0 < \xi < 1$, then $\frac{a}{\xi} > 0$ and, therefore, directrices of an ellipse are more remote than its vertices from the center (fig. 3.8). A circle (which has $\xi = 0$) does not possess directrices.

The peculiarity of directrices is that ratio of focal radius of any point for ellipse to the corresponding distance to a directrix is a constant value which is equal to eccentricity of an ellipse. Indeed (see fig.3.8),

$$d_1 = |MN_1| = \frac{a}{\xi} + x; \; d_2 = |MN_2| = \frac{a}{\xi} - x .$$

Then taking into account (5.6), we have

$$\frac{r_1}{d_1} = \frac{a + \xi x}{\frac{a}{\xi} + x} = \xi , \qquad \frac{r_2}{d_2} = \frac{a - \xi x}{\frac{a}{\xi} - x} = \xi .$$

Thus, $\frac{r_1}{d_1} = \frac{r_2}{d_2} = \xi$.

5. In case of $a < b$ a major semi-axis becomes b and focuses are located on axis Oy at a distance $\sqrt{b^2 - a^2}$ from center of an ellipse.

6. Concerning equation (5.5), ellipse is set if its semi-axes a and b or a and c are set. Ellipse can be plotted by means of these semi-axes.

There is another way to draw an ellipse. We take inextensible thread with length $2a$; its ends are established in focuses F_1 and F_2; thread is stretched with a pencil and an ellipse is drew.

In conclusion we consider parametric and polar equations of an ellipse.

Parametric equations of an ellipse are equations

$$x = a\cos\beta,\ \ y = b\sin\beta, \tag{5.10}$$

where there is $0 \le \beta < 2\pi$. It is easy to prove with substitution of these equations for canonical equation of the ellipse (5.5):

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{(a\cos\beta)^2}{a^2}+\frac{(b\sin\beta)^2}{b^2}=\cos^2\beta+\sin^2\beta=1.$$

Parameter β is an ***eccentric angle*** of a point for an ellipse. If point M for an ellipse is set, then finding β requires drawing a circle on the major axis of an ellipse (as on the diameter) and drawing a straight line which is parallel to a minor axis of an ellipse through point M. Point P where this line and the circle lying on the same side from a major axis of an ellipse intersect each other (as in case of point M), is called ***pre-image of point M*** (when establishing a one-to-one correspondence between points of a circle and points of an ellipse). Angle between axis Ox and beam OP is an eccentric angle β, which corresponds with the given point M for the ellipse (fig.3.9).

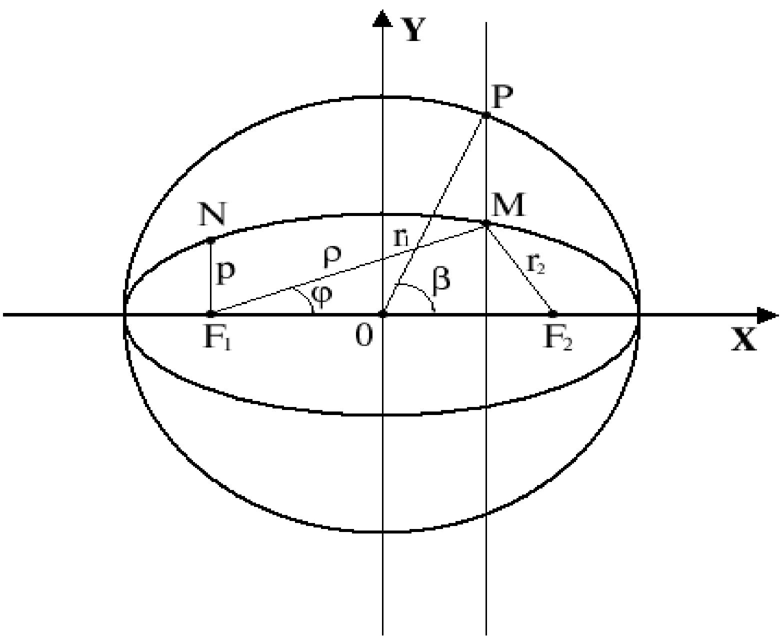


Fig.3.9

To establish polar equation, we introduce polar coordinate system in such a way that its pole would coincide with focus F_1, and polar axis would coincide with beam F_1x (fig.3.9).

According to the definition of an ellipse, $r_1 + r_2 = 2a$.

If to follow condition $r_1 = \rho$. then we have $r_2 = 2a - \rho$.

Using triangle MF_1F_2 and cosine formula we find equation

$$r_2^2 = \rho^2 + 4c^2 - 4\rho c \cos\varphi,$$

which leads to

$$a^2 - c^2 = a\rho - c\rho\cos\varphi.$$

Concerning $a^2 - c^2 = b^2$, we obtain $b^2 = \rho(a - c\cos\varphi)$. There we have

$$\rho = \frac{b^2}{a - c\cos\varphi}, \text{ or } \rho = \frac{\frac{b^2}{a}}{1 - \frac{c}{a}\cos\varphi}.$$

We designate $\frac{b^2}{a} = p$. Number p is called focal parameter of an ellipse; it is equal to length of perpendicular dropped from focus to focal axis before interception of this perpendicular and ellipse, i. e. $p = |F_1N|$ (fig. 3.9).

Thus, equation of an ellipse in polar coordinates has the form

$$\rho = \frac{p}{1 - \xi\cos\varphi},$$

where ξ and p are eccentricity and focal parameter of the ellipse reciprocally.

We also can state that if focus F_2 is taken as a pole of polar coordinate system, remaining a direction of polar axis without changes, then equation of an ellipse in polar coordinates takes the form

$$\rho = \frac{p}{1 + \xi\cos\varphi}.$$

1.2. Hyperbola

Definition. Hyperbola is a locus of points, whose absolute value of the distance difference to two fixed points of the plane (which are focuses) for each of them is a given positive number $2a$ and it is less than distance $2c$ between focuses.

Let us suppose that M is an arbitrary point of a hyperbola, and F_1 and F_2 are its focuses. Segments F_1M and F_2M (and their lengths r_1 and r_2 too) are called the ***focal radiuses of a hyperbola***. Therefore we have

$$|r_1 - r_2| = 2a = const\,. \qquad (5.11)$$

We introduce Cartesian coordinate system into a plane, taking segment middle F_1F_2 as the origin and line F_1F_2 as axis Ox which is orientated from point F_1 to point F_2. In this coordinate system, focus F_1 has coordinates -c, 0, and focus F_2 has coordinates c, 0. Expressing coordinates of point M of a hyperbola by x, y, we obtain

$$r_1 = \sqrt{(x+c)^2 + y^2}\,,\;\; r_2 = \sqrt{(x-c)^2 + y^2}\,,$$

and ratio (5.11) takes the form

$$\left|\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2}\right| = 2a\,.$$

Like equation of an ellipse, we transform this equation (par. 1.1) and obtain

$$\left(a^2 - c^2\right)x^2 + a^2y^2 = a^2\left(a^2 - c^2\right).$$

But we already have $a < c$. Expressing difference $a^2 - c^2$ by $-b^2$: $a^2 - c^2 = -b^2$ or $c^2 = a^2 + b^2$, we obtain

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \qquad (5.12)$$

So, the coordinates of any hyperbola point satisfy equation (5.12). Reverse statement is also true: if coordinates of a point satisfy this equation, then this point lies on the hyperbola considered. The proof is similar to the proof done during case of deriving equation of an ellipse.

Consequently, equation (5.12) is the hyperbola equation: it is called ***canonical equation of a hyperbola***.

A hyperbola has following properties:

1. Hyperbola is a curve of the 2nd order.

2. Canonical equation of a hyperbola (5.12) contains current coordinates to even powers, hence, a hyperbola has two axes of symmetry which are coordinate axes and a center of symmetry as the origin (similar case of an ellipse).

3. Concerning equation (5.12), we have $|x| \geq a$, i.e. solution can be either $x \geq a$, or $x \leq -a$. Therefore, a hyperbola consists of two branches. The left branch lies in the half-plane $x < -a$, and the right branch lies in the half-plane $x > a$.There are no hyperbola points between lines $x = a$ and $x = -a$.

4. Axis of symmetry *Oy* does not intersect the hyperbola set with equation (5.12), and is called the ***imaginary axis***; axis *Ox* intersects a hyperbola at two points: $A_1(-a,0)$ and $A_2(a,0)$. This axis is called a **real axis** of a hyperbola. The points where a real axis intersects a hyperbola are called ***vertices of a hyperbola***.

Numbers a and b in canonical equation of a hyperbola (5.12) is called ***real and imaginary semi-axes of a hyperbola*** respectively.

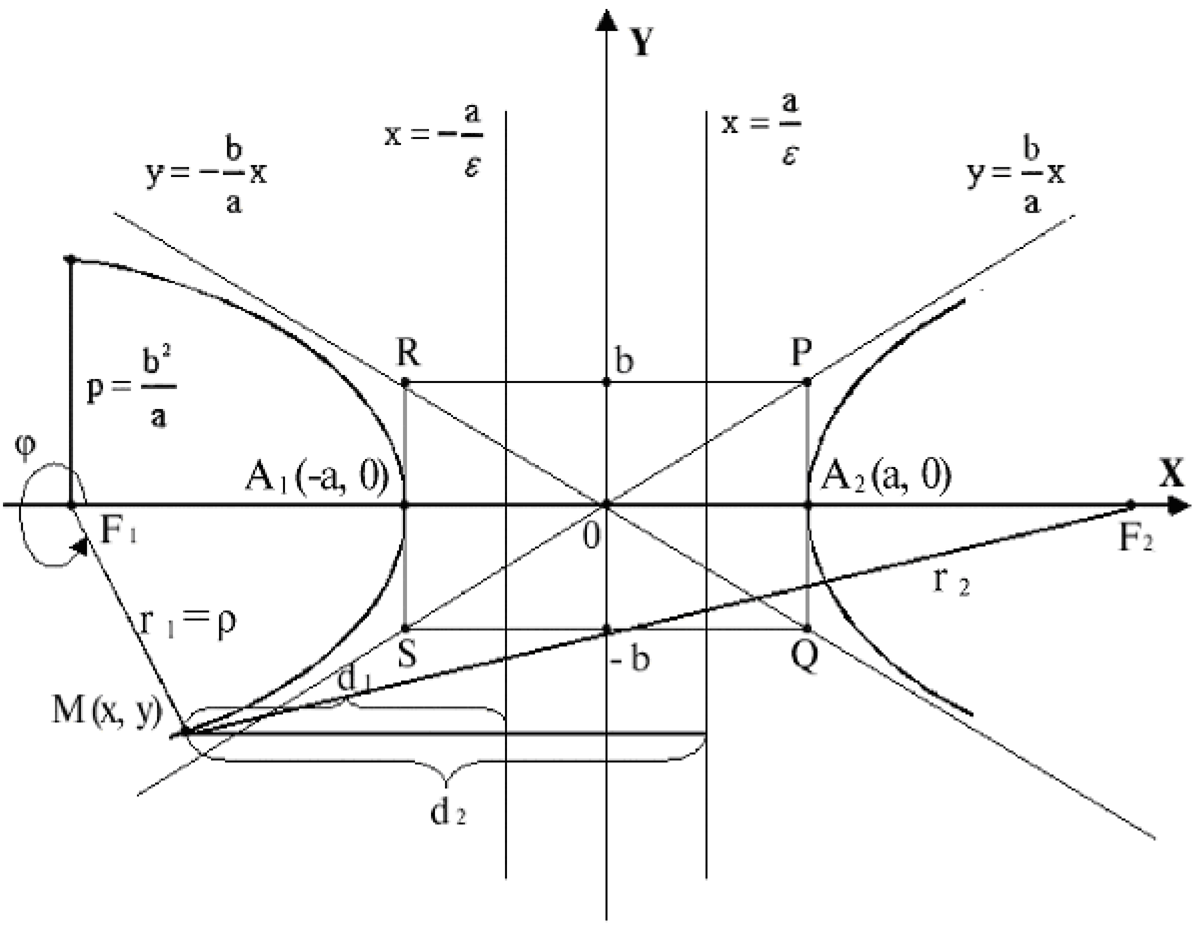


Fig. 3.10

5. Solving equation (5.12) for y, taking only a positive value for it

$$y = \frac{b}{a}\sqrt{x^2 - a^2} \qquad (5.13)$$

and considering $x \geq a$, we obtain hyperbola points which lie in the first quadrant. Concerning equation (5.13), we have that values of y increase without limit either when x increases from a without limit. Then, taking into account the fact, that a hyperbola is symmetrical relatively coordinate axes, we obtain points of a hyperbola which lie in other (the second, the third and the fourth) quadrants (fig.3.10).

6. Straight lines determined by equations $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$, are called ***asymptotes of a hyperbola***. Asymptote of a hyperbola is a line which possesses the same property, as a hyperbola point which withdraws from the origin into infinity and approaches its asymptote (fig. 3.10).

Asymptotes of a hyperbola coincide with diagonals of a rectangle which has vertices $P(a,b)$, $Q(a,-b)$, $R(-a,b)$, $S(-a,-b)$ (fig. 3.10).

Now we consider a way how to graph a hyperbola with usage of asymptotes. We plot a rectangle *SRPQ* (its sides are $2a$ and $2b$); draw lines coinciding with diagonals of this rectangle, i.e. asymptote; then graph a hyperbola with points A_1 and A_2 as its vertices.

7. Ratio of distance between a center of hyperbola and its focus to a real semi-axis of a hyperbola is called ***eccentricity of a hyperbola***:

$$\xi = \frac{c}{a}$$

Since there is $0 < a < c$ for a hyperbola, then we have $\xi > 1$ and eccentricity of a hyperbola is more than 1.

Two lines set with equations $x = \frac{a}{\xi}$ and $x = -\frac{a}{\xi}$, are called ***directrices of a hyperbola***.

Since $\xi > 1$, then directrices of a hyperbola are located from its center at distance which is less than a real semi-axis (fig. 3.10). Directrices of a hyperbola has the same properties as directrices of an ellipse, i.e.

$$\frac{r_1}{d_1} = \frac{r_2}{d_2} = \xi \quad \text{(fig. 3.10).}$$

Finally, we consider parametric and polar equations of a hyperbola. We re-write equation of a hyperbola (5.12) as

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=1.$$

There we see

$$\left(\frac{x}{a}-\frac{y}{b}\right)\neq 0,\quad \left(\frac{x}{a}+\frac{y}{b}\right)\neq 0.$$

Let us assume that $\left(\frac{x}{a}+\frac{y}{b}\right)=t$,then $t\neq 0$ and $\left(\frac{x}{a}-\frac{y}{b}\right)=\frac{1}{t}$, thus

$$x=\frac{a}{2}\left(t+\frac{1}{t}\right),\ y=\frac{b}{2}\left(t-\frac{1}{t}\right). \tag{5.14}$$

We proved that coordinates of any point for a hyperbola could be presented as equation (5.14), where $t\neq 0$. Inversely, taking into account the fact $t\neq 0$, a point with coordinate (5.14) lies on a hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

and it is easy to prove by means of substituting expressions of x and y instead of them (formula (5.14)). Consequently, equations (5.14) are called parametric equations of a hyperbola.

To derive polar equation of a hyperbola, we introduce polar coordinate system in such a way, that its pole would coincide with focus F_1 and polar axis would coincide with a positive direction of axis Ox. Coordinates of any point M for a hyperbola are designated as ρ and φ, i. e. $M(\rho,\varphi)$ (see fig. 3.10).

We write equation of a hyperbola in the form $|r_1-r_2|=2a$,

for the right branch:

$$r_1-r_2=2a, \tag{5.15}$$

for the left one:

$$r_2-r_1=2a. \tag{5.16}$$

We write equation (5.16) in polar coordinates. There is $r_1=\rho$ for any point M on the left side of a hyperbola. Then we have

$$r_2=\rho+2a.$$

According to cosine theorem for triangle MF_1F_2 (fig. 3.10) we find

$$r_2^2=\rho^2+4c^2-4\rho c\cos\varphi,$$

or

$$(\rho + 2a)^2 = \rho^2 + 4c^2 - 4\rho c\cos\varphi,$$

where we have

$$a^2 - c^2 = -\rho(a + c\cos\varphi).$$

Considering $a^2 - c^2 = -b^2$, a $\frac{c}{a} = \xi$, we obtain

$$-b^2 = -\rho a(1 + \xi\cos\varphi).$$

Designating focal parameter of a hyperbola as $p = \frac{b^2}{a}$ (see fig. 3.10), we derive ***polar equation of the left branch of a hyperbola***:

$$\rho = \frac{p}{1 + \xi\cos\varphi}.$$

In the same way we derive polar equation of the right branch of a hyperbola:

$$\rho = \frac{-p}{1 - \xi\cos\varphi}.$$

If we put a pole into the right focus F_2 and retain a direction of polar axis, then polar equation of the right branch of a hyperbola takes the form

$$\rho = \frac{p}{1 - \xi\cos\varphi}, \qquad (5.17)$$

and polar equation of the left branch of a hyperbola takes the form

$$\rho = \frac{-p}{1 + \xi\cos\varphi}. \qquad (5.18)$$

1.3. Parabola

Definition. Parabola is a locus of points whose distance to some fixed point in a plane (which is called ***focus***) for each of them is equal to a distance to some fixed line which does not pass through a focus and is called ***directrix***.

Distance between focus of a parabola and its directrix is called ***parameter of a parabola***.

Eccentricity of a parabola which stands for a ratio of distance between any point of a parabola and focus to distance between this point to a directrix, is a constant number, and half of directrix (fig.3.11) is equal to 1.

Let us find equation of a parabola. We choose such a coordinate system *xOy* that abscissa axis would intersect focus *F* perpendicularly to directrix *d* of a parabola, and ordinate axis would divide distance between focus and directrix in half (fig. 3.11).

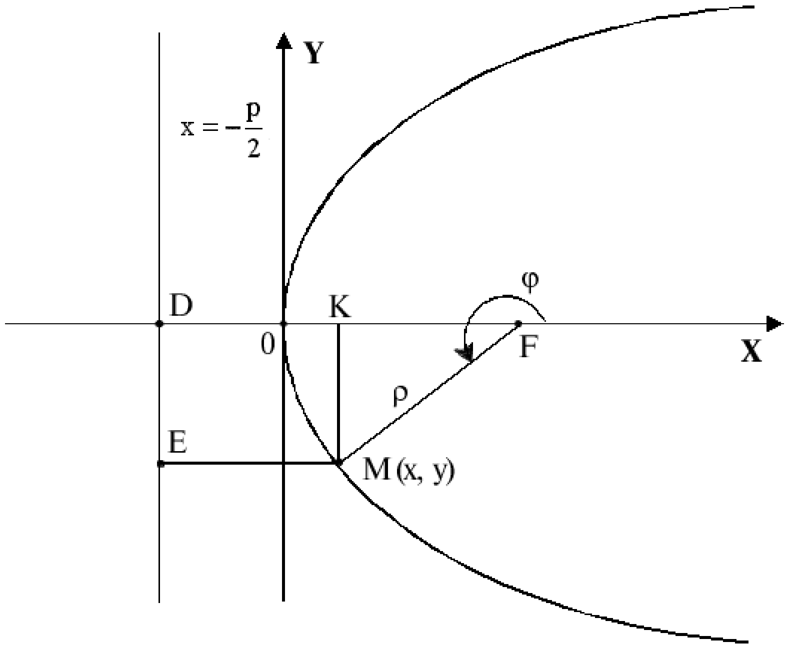


Fig. 3.11

Distance *FD* between focus and directrix of a parabola is designated as *p* (parameter of a parabola). In this coordinate system focus *F* has coordinates $\frac{p}{2}, 0$, and equations of a directrix looks like $x = -\frac{p}{2}$.

Let us suppose $M(x,y)$ is an arbitrary point of a plane. Then *M*, according to the definition, is a point of parabola only in case $|ME| = |MF|$.

Since

$$|ME| = \left|x + \frac{p}{2}\right|, \text{ a } |MF| = \sqrt{\left(x - \frac{p}{2}\right)^2 + y^2},$$

then equation of a parabola looks like $\sqrt{\left(x - \frac{p}{2}\right)^2 + y^2} = \left|x + \frac{p}{2}\right|$.

This equation is equivalent to the following one:

$$\left(x - \frac{p}{2}\right)^2 + y^2 = \left(x + \frac{p}{2}\right)^2,$$

or

$$y^2 = 2px. \qquad (5.19)$$

Equation (5.19) is called ***canonical equation of a parabola***.

Parabola properties:

1. When comparing equations (5.19) and (5.2), we make sure that a parabola is a curve of the 2nd order.

2. Since $p > 0$, then taking into account equation (5.19), we have $x \geq 0$, $y = \pm\sqrt{2px}$. Consequently, a parabola is a boundless curve which is located in the right semi-plane relatively axis *Oy* and axis *Ox* is axis for symmetry of a parabola (fig. 3.11). It is the one axis for symmetry of a parabola.

Parabola does not have a center of symmetry, it is not a central curve.

Point where a parabola and its axis of symmetry intersects each other, is called ***vertex of a parabola***. Parabola (5.19) has only one vertex which lies on
origin *O*(0, 0).

3. Equation $x^2 = 2py$, where $p > 0$, determines a parabola with origin as its vertex and axis of symmetry *Oy*. Parabola is located in the high semi-plane relatively axis *Ox*.

Equation $x^2 = 2py$ is often written as which is possible relatively ordinate y:

$$y = ax^2, \text{ where } a = \frac{1}{2p}.$$

4. Equation $y^2 = -2px$, where $p > 0$, determines a parabola which symmetric to parabola $y^2 = 2px$ relatively axis *Oy*, and equation $x^2 = -2py$ determines a parabola which is symmetric to parabola $x^2 = 2py$ relatively axis *Ox*.

5. We find polar equation of a parabola. Let us suppose that pole of polar coordinate system coincides with focus of parabola $F\left(\frac{p}{2}, 0\right)$, and polar axis coincides with a positive direction of axis *Ox* (fig. 3.11). Polar

coordinates of point $M(x, y)$ for a parabola are designated as ρ and φ, i. e. $M(\rho,\varphi)$. Using triangle *FMK* we find

$$y = \rho\sin\varphi,\ x = |OF| - |FK| = \frac{p}{2} + \rho\cos\varphi.$$

When substituting values *x* and *y* in equation (5.19), we obtain

$$(\rho\sin\varphi)^2 = 2\mathrm{p}\left(\frac{\mathrm{p}}{2} + \rho\cos\varphi\right),$$

and as a result $\rho^2 = (p + \rho\cos\varphi)^2$.

Concerning $\rho > 0$ and $p + \rho\cos\varphi > 0$, we have $\rho = p + \rho\cos\varphi$.

Then polar equation of a parabola is

$$\rho = \frac{p}{1 - \cos\varphi}. \tag{5.20}$$

§ 2. REDUCTION OF GENERAL EQUATION OF A LINE OF THE 2nd ORDER TO THE SIMPLEST (CANONICAL) FORM

To reduce general equation of the 2nd to the simplest form

$$a_{11}x^2 + a_{22}y^2 + 2a_{12}xy + 2a_{10}x + 2a_{20}y + a_{00} = 0 \tag{5.21}$$

is to transform it into such a form which allows us to define easily whether this equation sets a curve and which kind of this curve is (a circle, a hyperbola, a parabola, a line or a point).

Theorem 1. Using rotation and transposition of coordinate axes, general equation of a line of the 2nd order set relatively Cartesian coordinate system *xOy*, can be reduced to one of the following types:

$$b_{11}x^2 + b_{22}y_1^2 + d = 0, \quad \text{where } b_{11} \neq 0,\ b_{22} \neq 0,$$

$$b_{11}x^2 + 2b_{20}y_1 = 0, \quad \text{where } b_{11} \neq 0,\ b_{20} \neq 0,$$

$$b_{11}x^2 + d = 0, \quad \text{where } b_{11} \neq 0.$$

These equations are called the simplest equations of a line of the 2nd order.

Proof. We prove that axes *xOy* can be rotated to such an angle α, that in a transposed equation the coefficient would vanish zero when producing new coordinates $x'y'$. Thus, we suppose that there is $a_{12} \neq 0$ (if $a_{12} = 0$, then we can miss this part of the proof) and rotate axes *xOy* to some arbitrate angle α. Then coordinates x and y for point M in system xOy with usage of coordinates $x'$and y' for the same point M in system $x'Oy'$ are expressed with ratios (see book 2, ch. 8, §1, par. 1.1).

$$X = TX' \text{ or } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} x' \\ y' \end{pmatrix},$$

where $T = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ is a transformation matrix, $\alpha = \angle(\vec{i} \wedge \vec{i}') = \angle(\vec{j} \wedge \vec{j}')$, and equation (5.21) takes the form

$$a_{11}(x'\cos\alpha - y'\sin\alpha)^2 + 2a_{12}(x'\cos\alpha - y'\sin\alpha)(x'\sin\alpha + y'\cos\alpha) +$$
$$+ a_{22}(x'\sin\alpha + y'\cos\alpha)^2 + 2a_{10}(x'\cos\alpha - y'\sin\alpha) + 2a_{20}(x'\sin\alpha + y'\cos\alpha) +$$
$$+ a_{00} = 0$$

or

$$b_{11}x'^2 + 2b_{12}x'y' + b_{22}y'^2 + 2b'_{10}x' + 2b_{20}y' + a_{00} = 0,$$

where

$$b_{11} = a_{11}\cos^2\alpha + 2a_{12}\cos\alpha\sin\alpha + a_{22}\sin^2\alpha$$
$$b_{12} = a_{12}(\cos^2\alpha - \sin^2\alpha) + (a_{22} - a_{11})\sin\alpha\cos\alpha,$$
$$b_{22} = a_{11}\sin^2\alpha - 2a_{12}\cos\alpha\sin\alpha + a_{22}\cos^2\alpha,$$
$$b_{10} = a_{10}\cos\alpha + a_{20}\sin\alpha,$$
$$b_{20} = -a_{10}\sin\alpha + a_{20}\cos\alpha.$$

Condition $b_{12} = 0$ takes the form

$$a_{12}(\cos^2\alpha - \sin^2\alpha) + (a_{22} - a_{11})\sin\alpha\cos\alpha = 0,$$

and as a result

$$ctg\,2\alpha = \frac{a_{11} - a_{22}}{2a_{12}} \qquad (5.22)$$

In a transposed equation coefficient b_{12} vanishes zero when rotating to angle α, determined with this ratio, and it takes the form

$$b_{11}x'^2 + b_{22}y'^2 + 2b_{10}x' + 2b_{20}y' + a_{00} = 0. \qquad (5.23)$$

We should mention that a quadratic form

$$\varpi(\vec{x}) = a_{11}x^2 + 2a_{12}xy + a_{22}y^2, \text{ где } \vec{x} = (x, y),$$

in general equation (5.21), is reduced to canonical form when rotating coordinate system to angle α, determined with ratio (5.22) (see Book 2, Ch.8, § 3, p. 3.1):

$$a_{11}x^2 + 2a_{12}xy + a_{22}y^2 = b_{11}x'^2 + b_{22}y'^2.$$

In this case coefficients b_{11} and b_{22} are characteristic numbers ρ of this quadratic form and matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, where $a_{11} = a_{21}$. Consequently, they can also be determined with usage of characteristic equation of matrix A:

$$\begin{vmatrix} a_{11} - \rho & a_{12} \\ a_{21} & a_{22} - \rho \end{vmatrix} = 0,$$

or

$$\rho^2 - (a_{11} + a_{22})\rho + (a_{11}a_{22} - a_{12}^2) = 0.$$

Here we find

$$\rho_1 = b_{11} = \frac{a_{11} + a_{22}}{2} + \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + (a_{12})^2},$$

$$\rho_2 = b_{22} = \frac{a_{11} + a_{22}}{2} - \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + (a_{12})^2}.$$

To reduce equation to the simplest form (5.23), we are enough to make translation of coordinate system.

The 1st case: $b_{11} \neq 0$, $b_{22} \neq 0$.

We reduce equation (5.23) to the form

$$b_{11}\left(x' + \frac{b_{10}}{b_{11}}\right)^2 + b_{22}\left(y' + \frac{b_{20}}{b_{22}}\right)^2 + a_{00} - \frac{b_{10}^2}{b_{11}} - \frac{b_{20}^2}{b_{22}} = 0.$$

When making translation of axes $x'Oy'$ in such a way that point $O_1\left(-\frac{b_{10}}{b_{11}}, -\frac{b_{20}}{b_{22}}\right)$ would become a new origin (coordinates of this point are set relatively system $x'Oy'$) and expressing a new coordinate system through $x_1O_1y_1$, we have $x_1 = x' + \frac{b_{10}}{b_{11}}$, $y_1 = y' + \frac{b_{20}}{b_{22}}$, hence, equation (5.23) takes the form

$$b_{11}x_1 + b_{22}y_1 + d = 0, \qquad (5.24)$$

where

$$d = a_{00} - \frac{b_{10}^2}{b_{11}} - \frac{b_{20}^2}{b_{22}}.$$

The 2nd case: $b_{22} = 0$ and $b_{20} \neq 0$, or $b_{11} = 0$ and $b_{10} \neq 0$.

Let us assume that $b_{22} = 0$, $b_{20} \neq 0$. Then equation (5.23) has the form

$$b_{11}x'^2 + 2b_{10}x' + 2b_{20}y' + a_{00} = 0,$$

or

$$b_{11}\left(x' + \frac{b_{10}}{b_{11}}\right)^2 + 2b_{20}y' + a_{00} - \frac{b_{10}^2}{b_{11}} = 0,$$

or

$$b_{11}\left(x' + \frac{b_{10}}{b_{11}}\right)^2 + 2b_{20}\left(y' + \frac{a_{00} - \frac{b_{10}^2}{b_{11}}}{2b_{20}}\right) = 0.$$

When making translation of axes $x'Oy'$ in such a way that point $0_1\left(-\frac{b_{10}}{b_{11}},\ -\frac{a_{00} - \frac{b_{10}^2}{b_{11}}}{2b_{20}}\right)$ would become a new origin (coordinates of this point are set relatively system $x'Oy'$) and expressing a new coordinate system through $x_1O_1y_1$, we have $x_1 = x' + \frac{b_{10}}{b_{11}}$, $y_1 = y' + \frac{a_{00} - \frac{b_{10}^2}{b_{11}}}{2b_{20}}$, hence, equation (5.23) takes the form

$$b_{11}x_1^2 + 2b_{20}y_1 = 0.$$

It is equation of a parabola.

The 3rd case: $b_{22} = b_{20} = 0,$ or $b_{11} = b_{10} = 0.$

Let us suppose that $b_{22} = b_{20} = 0.$ Then equation (5.23) has the form

$$b_{11}x'^2 + 2b_{10}x' + a_{00} = 0, \text{ or } b_{11}\left(x' + \frac{b_{10}}{b_{11}}\right)^2 + a_{00} - \frac{b_{10}^2}{b_{11}} = 0.$$

When making translation of axes $x'Oy'$ in such a way that point $O_1\left(-\frac{b_{10}}{b_{11}},\ 0\right)$, would become a new origin (coordinates of this point are set relatively system $x'Oy'$) and expressing a new coordinate system through $x_1O_1y_1$, we have $x_1 = x' + \frac{b_{10}}{b_{11}}$, $y_1 = y'$, hence, equation (5.23) takes the form

$$b_{11}x_1^2 + d = 0, \text{ where } d = a_{00} - \frac{b_{10}^2}{b_{11}}.$$

Theorem 2. General equation of a line of the 2^{nd} order (5.21), set relatively Cartesian coordinate system, determines one of the following 9 lines (see the table).

*Proof***.** In the previous theorem we proved that if general equation of a line of the 2^{nd} order (5.21) is set relatively Cartesian coordinate system, then using transformation of Cartesian coordinate system into Cartesian one it can be reduced to one of the following simplest forms:

$$b_{11}x^2 + b_{22}y^2 + d = 0, \quad b_{11} \neq 0,\ b_{22} \neq 0, \qquad (\mathrm{I})$$

$$b_{11}x^2 + 2b_{20}y = 0, \quad b_{11} \neq 0,\ b_{20} \neq 0, \qquad (\mathrm{II})$$

$$b_{11}x^2 + d = 0, \quad b_{11} \neq 0. \qquad (\mathrm{III})$$

Here we express point coordinates through x and y in that coordinate system where equation of a line is the simplest one.

Table

Group	№ par.	Equation of a line	Title of a line
I	1	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	Ellipse
	2	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$	Imaginary ellipse
	3	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$	Two imaginary intersecting ellipses
	4	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	Hyperbola
	5	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$	Two intersecting lines
II	6	$x^2 = 2py$	Parabola
III	7	$x^2 = a^2\ (a \neq 0)$	Two parallel lines
	8	$x^2 = -a^2\ (a \neq 0)$	Two imaginary parallel lines
	9	$x^2 = 0$	Two coincident lines

Let us consider which form the simplest equations of a line of the 2nd order $(\mathrm{I}),(\mathrm{II}),(\mathrm{III})$ can be reduced into depending on coefficient signs in these equations.

(I): 1. If b_{11} and b_{22} have the same sign and d has a positive sign, then we reduce equation (I) to the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

multiplying the both parts of equation (I) by – d and supposing $-\frac{d}{b_{11}}=a^2,\ -\frac{d}{b_{22}}=b^2$. This is a canonical equation of an ellipse.

2. If b_{11}, b_{22} and d have the same sign, then equation (I) is reduced to the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$$

and determines an imaginary ellipse (there are no points (real ones) for an imaginary ellipse, since x and y are real numbers, so, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\geq 0$).

3. If b_{11} and b_{22} have the same sign and d =0, then equation (I) is reduced to the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0.$$

This equation is satisfied only in case x = y = 0. But taking into account

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{y}{b}+i\frac{x}{a}\right)\left(\frac{y}{b}-i\frac{x}{a}\right),$$

then we say that this equation divides into pair of imaginary lines $\frac{y}{b}\pm i\frac{x}{a}=0$, which intersect each other in real point $O_1(0, 0)$.

4. If b_{11} and b_{22} have different signs, $d\neq 0$, then equation (I) is reduced to the form

$$\frac{x^2}{\frac{-d}{b_{11}}}-\frac{y^2}{\frac{d}{b_{22}}}=1$$

Considering $-\frac{d}{b_{11}}>0,\ \frac{d}{b_{22}}>0$ and supposing $-\frac{d}{b_{11}}=a^2,\ \frac{d}{b_{22}}=b^2$, we obtain canonical equation of a hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

(if $-\frac{d}{b_{11}}<0,\ \frac{d}{b_{22}}<0$, then we obtain $-\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and when rotating axes to angle 90°, i. e. supposing $x=-y'$, $y=x'$, we have $\frac{x'^2}{a^2}-\frac{y'^2}{b^2}=1$).

5. If b_{11} and b_{22} have different signs and $d=0$, then equation (I) is reduced to the form $\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$ and determines two intersecting lines:

$$\frac{x}{a}+\frac{y}{b}=0,\quad \frac{x}{a}-\frac{y}{b}=0.$$

(II) Equation (II) can be reduced to the form $x^2=2py$, where $p=-\frac{b_{20}}{b_{22}}\neq 0.$ Number p can be considered positive, since otherwise we are enough to change a positive direction of axis Oy into negative one.

(III) Equation (III) is reduced to the form

$$x^2=-\frac{d}{b_{11}},\ \text{or}\quad x^2=a^2,\ x^2=-a^2,\ x^2=0$$

depending on whether there is $-\frac{d}{b_{11}}>0,\ -\frac{d}{b_{11}}<0$ or $d=0$.

Equation $x^2=a^2$ determines two parallel lines $x=a$ and $x=-a$.

Equation $x^2=-a^2$ on the set of real numbers corresponds with an empty set of points and it determines two imaginary parallel lines $x=ia$ and $x=-ia$.

Equation $x^2=0$ determines two coinciding lines – abscissa axis.

Example. Reduce equation of a curve to canonical form and draw a curve $2x^2+6xy+2y^2+2x-2y+3=0$,
which is determined with the given equation.

Solution. Let us define to which angle α we should rotate coordinate system to obtain coefficient $b_{12}=0.$ in transformed equation. We use condition (5.12)

$$ctg2\alpha = \frac{a_{11} - a_{22}}{2a_{12}} = \frac{2-2}{6} = 0.$$

where we have $\alpha = \pm\pi/4.$ as a result.

Choosing any angle, both $\alpha = \pi/4$ and $\alpha = -\pi/4$, we obtain the same curve. In our case we choose $\alpha = \pi/4$. Then we get

$$b_{11} = a_{11}\cos^2\alpha + 2a_{12}\cos\alpha\sin\alpha + a_{22}\sin^2\alpha = 5,$$

$$b_{22} = a_{11}\sin^2\alpha - 2a_{12}\cos\alpha\sin\alpha + a_{22}\cos^2\alpha = -1,$$

$$b_{10} = a_{10}\cos\alpha + a_{20}\sin\alpha = 0,$$

$$b_{20} = -a_{10}\sin\alpha + a_{20}\cos\alpha = -\frac{2}{\sqrt{2}}$$

and equation of curve in coordinate system $x'Oy'$ takes the form

$$5x'^2 - y'^2 - \frac{4}{\sqrt{2}}y' + 3 = 0.$$

In the left side of this equation we complete a perfect square:

$$5x'^2 - \left(y' + \frac{2}{\sqrt{2}}\right)^2 + 5 = 0.$$

Next we make translation of coordinate system $x'Oy'$ using formulas:

$$\begin{cases} x_1 = x'; \\ y_1 = y' + \sqrt{2} \end{cases} \quad \text{or} \quad \begin{cases} x' = x_1; \\ y' = y_1 - \sqrt{2}. \end{cases}$$

Then we obtain

$$5x_1^2 - y_1^2 = -5,$$

in coordinate system $x_1O_1y_1$ therefore

$$\frac{x_1^2}{1} - \frac{y_1^2}{5} = -1.$$

Rotating axes $x_1O_1y_1$ to angle $-\pi/2$ (or $\pi/2$), i.e. supposing $x_1 = y_2$, $y_1 = -x_2$, we obtain $\frac{x_2^2}{5} - \frac{y_2^2}{1} = 1$. This equation determines hyperbola with semi-axes $a = \sqrt{5}$ and $b = 1$ (fig.3.12).

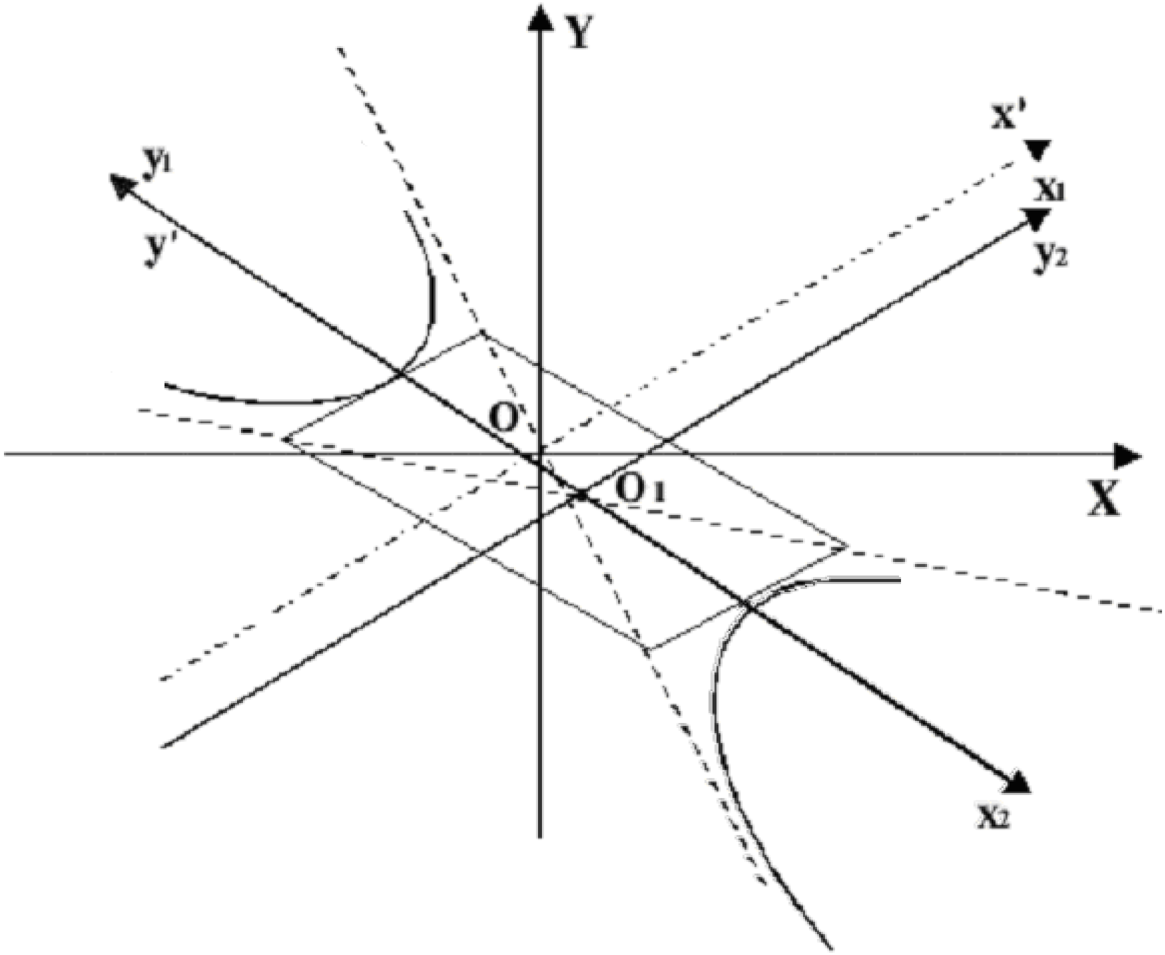


Fig.3.12

§ 3. SURFACES OF THE 2nd ORDER SET WITH CANONICAL EQUATIONS

3.1. Ellipsoid

Definition. Ellipsoid is called a surface whose equation has the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad (5.25)$$

in some particularly chosen Cartesian coordinate system.

Let us assume that $a \geq b \geq c$. If point (x, y, z) lies on ellipsoid (5.25), then points $(\pm x, \pm y, \pm z)$ with any sets of positive and negative signs also lie there. Therefore, ellipsoid (5.25) has an origin of coordinates as its center of symmetry and this origin is called a center of an ellipsoid; coordinate axes are axes of symmetry and called principal axes; coordinate planes are planes of symmetry and called principal planes.

If $a > b > c$, then ellipsoid (5.25) is called three-axial.

If $a > b = c$, then ellipsoid (5.25) is called a prolate spheroid; it is created by rotating ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ around its major axis (fig.3.13,a).

If $a = b > c$, then ellipsoid (5.25) is called an oblate spheroid; it is created by rotating ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ around its minor axis (fig.3.13,b).

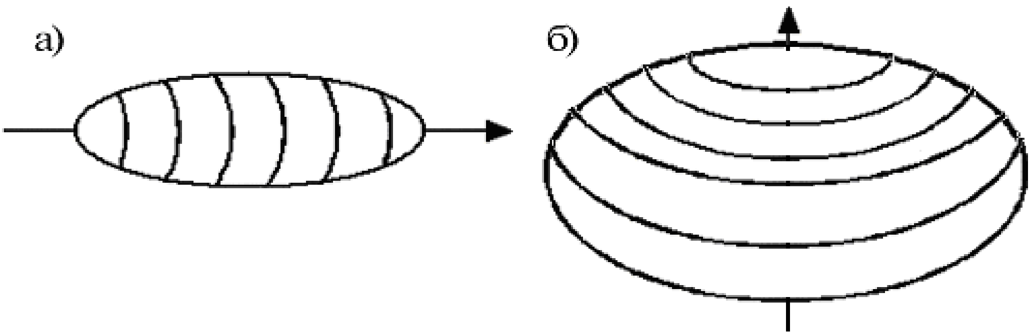


Fig. 3.13

If $a = b = c$, then ellipsoid (5.25) is called is a sphere of radius a which has a center at origin of coordinates.

Vertices of three-axial ellipsoid are points of interception of ellipsoid and its principal axes. Three-axial ellipsoid has six vertices $(\pm a, 0, 0), (0, \pm b, 0), (0, 0, \pm c)$.

Considering equation (5.52), we obtain $|x| \le a, |y| \le b, |z| \le c$.

It means that ellipsoid (5.25) lies inside rectangular parallelepiped with vertices $(\pm a, \pm b, \pm c)$. Each face of this parallelepiped and ellipsoid (5.25) has only one mutual point – its vertex.

Plane xOy intercepts ellipsoid (5.25) in a line expressed by equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \ \ z = 0$$

or its equivalent

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ \ z = 0 \qquad (5.26)$$

In the same way plane yOz intercepts ellipsoid (5.25) in a line whose equation is

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \ \ x = 0 \qquad (5.27)$$

and plane xOz intercepts it in the line

$$\frac{x^2}{a^2}+\frac{z^2}{c^2}=1,\quad y=0 \tag{5.28}$$

Lines (5.26), (5.27), (5.28) form ellipses. These ellipses, i.e. cross sections of ellipsoid (5.25) created by its principal planes, are called ***principal cross sections***.

Let us consider cross sections of ellipsoid (5.25) made by planes which are parallel to some coordinate plane, for instance, by planes which are parallel to plane *xOy*, i.e. planes expressed by equation $z=h$, where h is an arbitrary real number.

Equations of section line have the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\quad z=h$$

or

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{h^2}{c^2},\, z=h. \tag{5.29}$$

If $|h|>c$, then the first equation of this system is not satisfied by any pairs of real numbers x, y, i.e. system (5.29) does not have solution. It means that plane $z=h$ with $|h|>c$ does not intercept ellipsoid (5.25).

Considering $h=\pm c$ the first equation of system (5.29) has the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0,$$

where we obtain $x=y=0$. Thus, planes $z=\pm c$ meet ellipsoid (5.25) in its vertices $(0,0,\pm c)$. Finally, if $|h|<c$, then system of equations which express section line can be re-written in the following way:

$$\frac{x^2}{\left(a\sqrt{1-\frac{h^2}{c^2}}\right)^2}+\frac{y^2}{\left(b\sqrt{1-\frac{h^2}{c^2}}\right)^2}=1,\quad z=h$$

These equations are equations of an ellipse which lies in plane section $z=h$; a center of this ellipse is point $(0,0,h)$, axes of symmetry are parallel to axes *Ox* and *Oy*, and semi-axes are equal to

$$a'=a\sqrt{1-\frac{h^2}{c^2}},\quad b'=b\sqrt{1-\frac{h^2}{c^2}}.$$

Cross sections considered give an idea of how an ellipsoid looks. Such a way to consider surfaces is called method of parallel sections; we will use it when considering other surfaces.

3.2. One-sheeted hyperboloid

Definition. One-sheeted hyperboloid is a surface whose equation takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad (5.30)$$

in some particularly chosen Cartesian coordinate system.

Let us think that $a \geq b$. Like in a previous paragraph we prove that origin is a center of symmetry, coordinate axes are axes of symmetry (principal axes) and coordinate planes are planes of symmetry (principal planes) for one-sheet hyperboloid (5.30).

If $a = b$ in equation (5.30), then one-sheeted hyperboloid (5.30) is called hyperboloid of revolution (5.30), since it can be generated by rotating a hyperbola

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$$

around its imaginary axis (fig. 3.14).

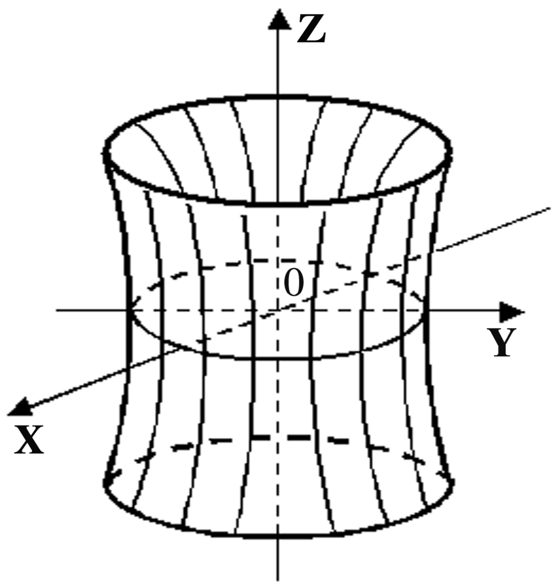


Fig. 3.14

Vertices of one-sheeted hyperboloid are points of interception of hyperboloid and its principal axes. In case of $a \neq b$ a hyperbola has four vertices $(\pm a, 0, 0)$, $(0, \pm b, 0)$.

Plane *xOy* intercepts one-sheeted hyperboloid (5.30) in an ellipse expressed by equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ z = 0$$

and called throat ellipse of one-sheet hyperbola. Plane *yOz* intercepts one-sheeted hyperboloid (5.30) in a hyperbola expressed by equations

$$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \ \ x = 0$$

and plane *xOz* intercepts it in a hyperbola expressed by equations

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, \ \ y = 0$$

Let us consider cross-section of one-sheeted hyperboloid (5.30) made by planes which are parallel to coordinate plane *xOy*, i.e. by planes $z = h$.

Equations of section line are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \ \ z = h$$

or

$$\frac{x^2}{\left(a\sqrt{1+\frac{h^2}{c^2}}\right)^2} + \frac{y^2}{\left(b\sqrt{1+\frac{h^2}{c^2}}\right)^2} = 1, \ \ z = h$$

These equations express an ellipse with semi-axes

$$a_1 = a\sqrt{1+\frac{h^2}{c^2}}, \quad b_1 = b\sqrt{1+\frac{h^2}{c^2}} \tag{5.31}$$

and center at point $(0,0,h)$ on axis *Oz* and axes which are parallel to axes *Ox* and *Oy* respectively. Considering equations (5.31) we conclude that $a_1 \geq a$, $b_1 \geq b$, i. e. throat ellipse is the least among all ellipses in which one-sheeted hyperboloid (5.30) is cut by planes which are parallel to plane *xOy*.

Plane $x = h$ which is parallel to plane *yOz*, intercepts one-sheeted hyperboloid (5.30) in a line expressed by

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{h^2}{a^2}, \quad x=h.$$

If $|h|<a$, then these equations determine a hyperbola with a center at point $(h,0,0)$ which lies on plane $x=h$, whose real axis is parallel to axis *Oy* and imaginary axis is parallel to axis *Oz*. Semi-axes of this hyperbola are $b\sqrt{1-\frac{h^2}{a^2}}$ (real semi-axis) and $c\sqrt{1-\frac{h^2}{a^2}}$ (imaginary semi-axis).

If $|h|=a$, then equations of section line has the form

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=0, \quad x=\pm a.$$

Equations

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=0, \quad x=a$$

are equations of two intersecting straight lines:

$$\frac{y}{b}+\frac{z}{c}=0, \quad x=a \text{ – the first line;}$$

$$\frac{y}{b}-\frac{z}{c}=0, \quad x=a \text{ – the second line.}$$

In the same way equations $\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$, $x=-a$ are equations of two lines:

$$\frac{y}{b}+\frac{z}{c}=0, \quad x=-a \text{ and } \frac{y}{b}-\frac{z}{c}=0, \quad x=-a.$$

If $|h|>a$, then section forms a hyperbola whose equations are

$$\frac{z^2}{\left(c\sqrt{\frac{h^2}{c^2}-1}\right)^2}-\frac{y^2}{\left(b\sqrt{\frac{h^2}{c^2}-1}\right)^2}=1, \quad x=h.$$

A real axis of this hyperbola is parallel to axis *Oz*, an imaginary axis is parallel to axis *Oy*; a center lies at point $(h,0,0)$. Asymptotes of all hyperbolas obtained when one-sheeted hyperboloid (5.30) is intercepted by planes $x=h$ $(h\neq\pm a)$, are parallel to lines obtained when hyperboloid is intercepted by planes $x=\pm a$. Sections made by planes $y=h$ which are parallel to plane *xOz*, are similar to considered sections. All these sections give an idea of how one-sheeted hyperboloid (5.30) looks (fig. 3.14).

3.3. Two-sheeted hyperboloid

Definition. ***Two-sheeted hyperboloid*** is a surface whose equation takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1. \qquad (5.32)$$

in some particularly chosen Cartesian coordinate system.

An origin is a center of symmetry (***center***), coordinate axes are axes of symmetry (***principal axes***) and coordinate planes are planes of symmetry (***principal planes***) for two-sheet hyperboloid.

If $a=b$ in equation (5.32), then two-sheeted hyperboloid (5.32) is called ***two-sheeted hyperboloid of revolution*** since it is generated by rotating a hyperbola

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=-1$$

around its real axis *Oz* (fig. 3.15).

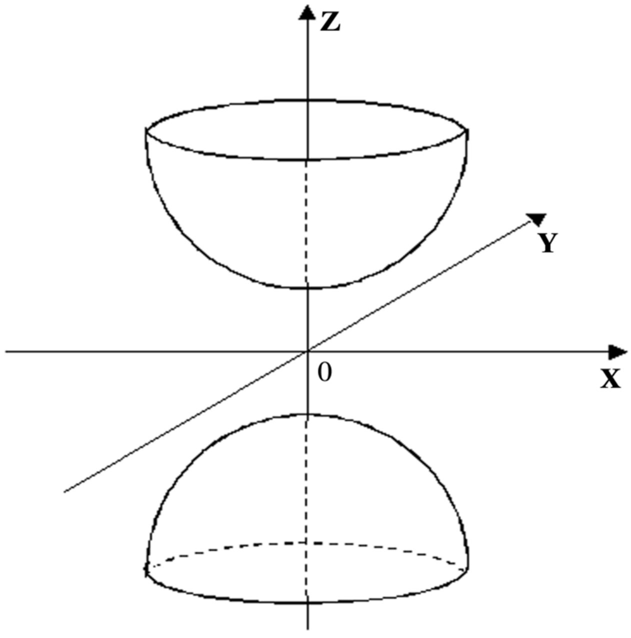


Fig. 3.15

Vertices of two-sheeted hyperboloid are points of intersection of this hyperboloid and major axis *Oz*.

Two-sheeted hyperboloid (5.32) has two vertices $(0,0,\pm c)$.

Planes xOz and yOz intercept two-sheeted hyperboloid (5.32) in hyperbolas

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=-1,\quad y=0 \text{ and } \frac{y^2}{b^2}-\frac{z^2}{c^2}=-1,\quad x=0.$$

Sections of two-sheeted hyperboloid made by plane $z=h$ is expressed by equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{h^2}{c^2}-1,\quad z=h.$$

If $|h|<c$, the first equation does not have solution – plane $z=h$ does not intercept the surface.

If $h=\pm c$, then

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0, \text{ where we obtain } x=y=0,$$

and planes $z=\pm c$ meet surface of two-sheeted hyperboloid in its vertices $(0,0,\pm c)$.

If $|h|>c$, then equations of section line can be re-written as

$$\frac{x^2}{\left(a\sqrt{\frac{h^2}{c^2}-1}\right)^2}+\frac{y^2}{\left(b\sqrt{\frac{h}{c^2}-1}\right)^2}=1,\quad z=h.$$

These equations express an ellipse with semi-axes

$$a_1=a\sqrt{\frac{h^2}{c^2}-1},\quad b_1=b\sqrt{\frac{h^2}{c^2}-1}$$

and center at point $(0,0,h)$ and axes which are parallel to axes *Ox* and *Oy* respectively. Plane $\mathrm{x}=\mathrm{h}$ intercepts surface of two-sheeted hyperboloid in a line expressed by equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1,\; x=h,$$

or

$$\frac{z^2}{\left(c\sqrt{1+\frac{h^2}{a^2}}\right)^2}-\frac{y^2}{\left(b\sqrt{1+\frac{h^2}{a^2}}\right)^2}=1,\quad x=h,$$

i. e. in hyperbola with a center at point $(h,0,0)$, which lies on plane $x=h$. A real axis of this hyperbola is parallel to axis *Oz*, an imaginary one is parallel to axis *Oy*.

In the same way we consider sections of surface (5.32) made by planes $y = h$.

3.4. Cone of the 2nd order

Definition 1. ***Cone of the 2nd order*** is called a surface whose equation has the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \qquad (5.33)$$

in some particularly chosen Cartesian coordinate system (we believe that $a \geq b$ in this equation).

An origin is a center of symmetry (vertex), coordinate axes are axes of symmetry (principal axes) and coordinate planes are planes of symmetry (principal planes). Axis of cone (5.33) is usually called *Oz*.

The main feature of a cone: if point $M_0(x_0, y_0, z_0)$ (which does not coincide with a vertex) lies on a cone, then all points of straight line OM_0, which passes through vertex *O* and point M_0, also lie there.

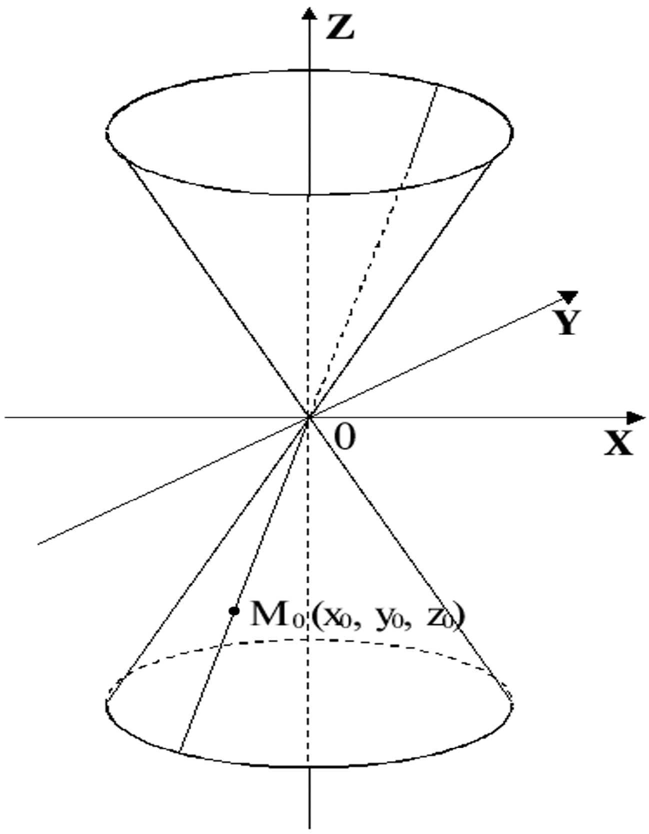


Fig. 3.16

Indeed, if $M(x,y,z)$ is an arbitrary point, lying on straight line OM_0, then $x=\lambda x_0$, $y=\lambda y_0$, $z=\lambda z_0$ and, therefore, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\lambda^2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}\right)=0,$$

and the point belongs to the cone.

Thus, surface (5.33) is generated by lines which pass through an origin. So when presenting the form of this surface, it is enough to consider its section made by some plane $z=h$ which is parallel to plane xOy. When cutting it, we obtain an ellipse whose equation is

$$\frac{x^2}{\left(\frac{ah}{c}\right)^2}+\frac{y^2}{\left(\frac{bh}{c}\right)^2}=1,\ z=h.$$

A center of this ellipse lies at point $(0,0,h)$ on axis Oz and, consequently, surface (5.33) is generated by lines which connect an origin and all points of this ellipse (fig. 3.16).

On the basis of the main feature of a cone of the 2nd order, we will give a definition for an arbitrary conical surface.

Definition 2. ***Conical surface*** *(or a* ***cone****)* is a surface generated by moving a line which passes through the same point and set curve.

Straight line moved is called ***generatrix of a cone***, the given point is a ***vertex*** and line set is a ***directrix*** of a cone.

According to the definition, equation of conical surface $F(x,y,z)=0$ should be satisfied by coordinates of all points for generatrix of a cone, i.e. points with coordinates $(\lambda x,\lambda y,\lambda z)$, where λ is any real number; $F(x,y,z)=F(\lambda x,\lambda y,\lambda z)=0$, consequently, function $F(x,y,z)$ should be homogeneous in a equation which sets a conical surface.

Definition 3. Function $F(x,y,z)$ is called homogeneous if it possesses the following features:

1. If point (x,y,z) belongs to domain of function $F(x,y,z)$, then point $(\lambda x,\lambda y,\lambda z)$ (where λ is arbitrary real number) also belongs to domain of this function;
2. There is such a number *k*, that ratio

$$F(\lambda x, \lambda y, \lambda z) = \lambda^k F(x, y, z).$$

is done for any point (x, y, z) of domain of function $F(x, y, z)$.

Number k is called a ***degree of homogeneity***. For cone of the 2nd order the degree of homogeneity $k = 2$.

3.5. Elliptic paraboloid

Definition. ***Elliptic paraboloid*** is called a surface whose equation has the form

$$\frac{x^2}{p} + \frac{y^2}{q} = 2z, \text{ where } p > 0, q > 0 \qquad (5.34)$$

in some particularly chosen Cartesian coordinate system.

Let us think that $p \geq q$. If $p = q$, then elliptic paraboloid (5.34) is a paraboloid of revolution, since it is generated by rotating parabola $y^2 = 2pz$ around axis *Oz*, which is parabola axis (fig. 3.17).

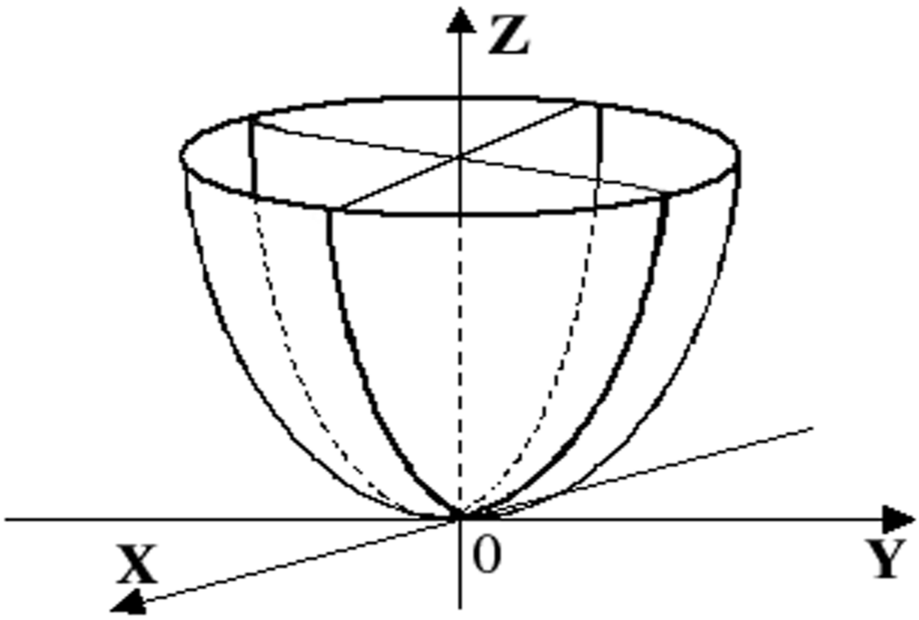


Fig. 3.17

Axis *Oz* is axis of symmetry for elliptic paraboloid (5.34) (it is paraboloid axis) and axes *xOz* and *yOz* are planes of symmetry (principal planes). For an elliptic paraboloid an origin is point of interception of this surface and its axis and called a vertex.

Plane $z = h$ intercepts elliptic paraboloid (5.34) in the line

$$\frac{x^2}{p}+\frac{y^2}{q}=2z,\ z=h. \qquad (5.35)$$

If $h<0$, then the first equation does not have solution since $p>0, q>0$; consequently, plane $z=h$ with $h<0$ does not intercepts an elliptic paraboloid. If $z=0$, then $x=y=0$, i.e. plane *xOy* and an elliptic paraboloid have only one mutual point – vertex $(0,0,0)$.

If $h>0$, then after re-writing equation (5.35) as

$\frac{x^2}{\left(\sqrt{2ph}\right)^2}+\frac{y^2}{\left(\sqrt{2qh}\right)^2}=1$, z=h, we notice that the section is an ellipse with center at point $(0,0,h)$ and semi-axes $a=\sqrt{2ph}$ and $b=\sqrt{2qh}$.

Plane *xOz* intercepts elliptic paraboloid (5.34) in parabola $x^2=2pz$, $y=0$, and plane *yOz* intercepts it in parabola $y^2=2qz$, $x=0$.

Thus, numbers *p* and *q* are parameters of parabolas acquired when cutting off paraboloid by its planes of symmetry (fig. 3.17).

Let us consider sections of an elliptic paraboloid made by planes which are parallel to plane *xOz*, i.e. planes expressed by equation $y=t$.

Equations of section line are $\frac{x^2}{p}+\frac{y^2}{q}=2z$, $y=t$, or

$$x^2=2p\left(z-\frac{t^2}{2q}\right),\ y=t. \qquad (5.36)$$

These equations express parabola with vertex at point $\left(0,t,\frac{t^2}{2q}\right)$, whose axis of symmetry has the same direction as axis *Oz*. Parabola parameter (5.36) is equal to *p*, i.e. parameter of the main section of an elliptic paraboloid made by plane *xOz* (while $t=0$).

Thus, an elliptic paraboloid can be generated by translation of parabola (5.36), where vertex of this parabola moves in parabola
$y^2=2qz$, $x=0$, acquired by intercepting an elliptic paraboloid by plane *yOz*. Consequently, planes of these parabolas are parallel and the axes are parallel too and have the same direction.

We have the same case when considering sections of elliptic paraboloid (5.34) made by planes which are parallel to plane *yOz*.

3.6. Hyperbolic paraboloid

Definition. ***Hyperbolic paraboloid*** is called a surface whose equation has the form

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z, \text{ where } \quad p > 0,\ q > 0. \qquad (5.37)$$

in some particularly chosen Cartesian coordinate system.

Planes *xOz* and *yOz* are planes of symmetry and axis *Oz* is axis of symmetry for hyperbolic paraboloid (5.37).

Axis of symmetry of a hyperbolic paraboloid is called an ***axis***. The point where axis of a hyperbolic paraboloid intercepts this surface is called a ***vertex***. Hyperbolic paraboloid (5.37) has an origin as its vertex.

Planes *xOz* и *yOz*, which are planes of symmetry for hyperbolic paraboloid (5.37) are called principal planes of a hyperbolic paraboloid.

In case of $p \neq q$ hyperbolic paraboloid (5.37) has only one axis of symmetry (axis *Ox*), in case of $p = q$ a paraboloid has two more axes of symmetry: $y = x,\ z = 0$ and $y = -x,\ z = 0$.

Indeed, if coordinates of point $M(x, y, z)$ satisfy equation $x^2 - y^2 = 2pz$, then coordinates of point $M_1(y, x, -z)$, which is symmetric to point $M(x, y, z)$ relatively lines $y = x,\ z = 0$, also satisfy this equation. In the same way we prove that straight line $y = -x,\ z = 0$ is axis of symmetry.

Plane *xOy* intercepts a hyperbolic paraboloid in two lines:

$$\frac{x^2}{p} - \frac{y^2}{q} = 0, \text{ or } \frac{z}{\sqrt{p}} + \frac{y}{\sqrt{q}} = 0,\ z = 0,$$

and

$$\frac{z}{\sqrt{p}} - \frac{y}{\sqrt{q}} = 0,\ z = 0.$$

Plane $z = h$, which is parallel to plane *xOy*, intercepts a hyperbolic paraboloid in a hyperbola (fig. 3.18, a)

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z,\ z = h. \qquad (5.38)$$

If $h > 0$, then these equations can be re-written as

$$\frac{x^2}{\left(\sqrt{2ph}\right)^2} - \frac{y^2}{\left(\sqrt{2qh}\right)^2} = 1, \ z = h.$$

This is a hyperbola which is located in plane $z = h$ with a center at point $(0,0,h)$, whose real axis is parallel to axis *Ox* and imaginary axis is parallel to axis *Oy*.

If $h < 0$, then equations of section line can be presented as

$$\frac{y^2}{\left(\sqrt{-2qh}\right)^2} - \frac{x^2}{\left(\sqrt{-2ph}\right)^2} = 1, \ z = h.$$

This is a hyperbola which is located in plane $z = h$ with a center at point $(0,0,h)$, whose real axis is parallel to axis *Oy* and imaginary axis is parallel to axis *Ox*. Asymptotes of all hyperbolas we obtain when intercepting hyperbolic paraboloid (5.37) by planes $z = h$, $h \neq 0$, which are parallel to lines, and this paraboloid intercepts plane $z = 0$ in these lines.

Plane *xOz* intercepts a hyperbolic paraboloid in parabola (fig. 3.18,b)

$$x^2 = 2pz, \ y = 0 \qquad (5.39)$$

and plane yOz intercepts it in parabola

$$y^2 = -2qz, \ x = 0. \qquad (5.40)$$

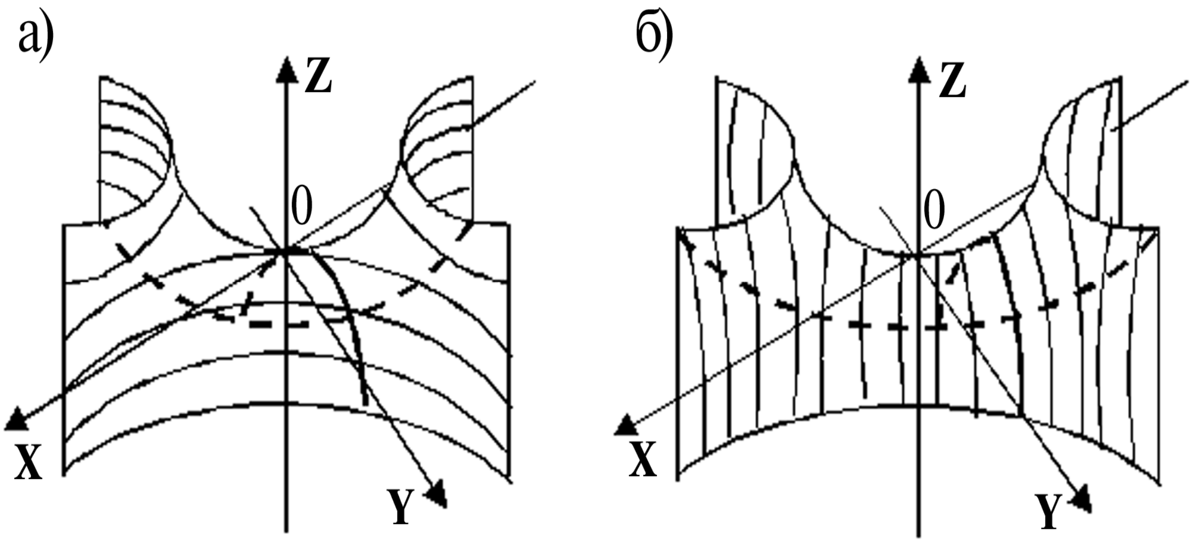


Fig. 3.18

Thus, numbers *p* and *q* are parameters of parabolas acquired when cutting off hyperbolic paraboloid (5.37) by its principal planes.

Let us consider sections of hyperbolic paraboloid (5.37) made by planes which are parallel to plane yOz (fig. 3.18, b), i.e. by planes expressed by equation $x = t$.

Equations of section line have the form

$$\frac{x^2}{p} - \frac{y^2}{q} = 2z,\ x = t, \text{ or } y^2 = -2q\left(z - \frac{t^2}{2p}\right),\ x = t.$$

These equations express a parabola with vertex at point $\left(t, 0, \frac{t^2}{2p}\right)$, whose axis is expressed by equations $x = t$, $y = 0$, and direction of the axis coincides with negative direction of axis Oz. Parameter of parabola

$$y^2 = -2q\left(z - \frac{t^2}{2p}\right),\ x = t \tag{5.41}$$

is equal to q, i.e. parameter of the main section (5.40) of hyperbolic paraboloid made by plane yOz ($t = 0$).

Thus, hyperbolic paraboloid can be generated by translation of parabola (5.41), where vertex of a parabola moves in parabola (5.39); plane of parabola (5.39) is perpendicular to plane of parabola (5.41), axes of these parabolas are parallel and have opposite direction (fig. 3.18, b).

We have the same case when considering sections of a hyperbolic paraboloid made by planes which are parallel to plane xOz.

Hyperbolic paraboloid is sometimes called a saddle-shaped surface.

3.7. Cylinders of the 2nd order

Definition 1. ***Cylindrical surface*** is a surface created by lines which are parallel to each other and called its ***generatrices***.

If some plane, intercepting all generatrices of cylindrical surface, also intercepts it in line P, then this line is called a ***directrix*** of this cylindrical surface.

Theorem. If Cartesian coordinate system is introduced in a space and equation $F(x, y) = 0$ is equation of some line P in plane xOy, then in a space this equation is equation of cylindrical surface L with directrix P, and generatrices are parallel to axis Oz (fig. 3.19, a).

Proof. Point $M(x,y,z)$ lies on cylindrical surface L only in case when projection $M_1(x,y,0)$ of point M onto plane xOy in parallel to axis Oz lies on line P, i.e. only in case when equation $F(x,y)=0$ is accomplished.

We make the same conclusions for equations kind of $F(y,z)=0$ (fig. 3.19, b) and $F(z,x)=0$ (fig. 3.19, c).

Definition 2. Cylindrical surfaces whose directrices are lines of the 2nd order are called ***cylindrical surfaces of the 2nd order***.

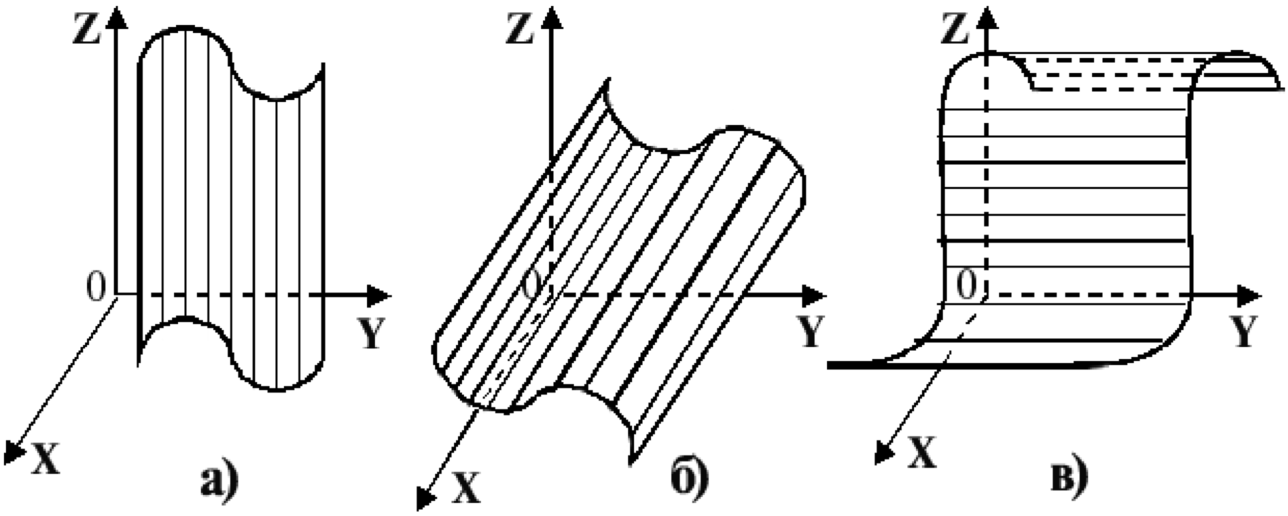


Fig. 3.19

There are three types of cylinders of the 2nd order:

- ***elliptic*** (fig. 3.20)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad (5.42)$$

- ***hyperbolic*** (fig. 3.21)

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad (5.43)$$

- ***parabolic*** (fig. 3.22)

$$y^2=2px. \qquad (5.44)$$

For cylinders set by equations (5.42), (5.43) and (5.44), directrices are ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0,$$

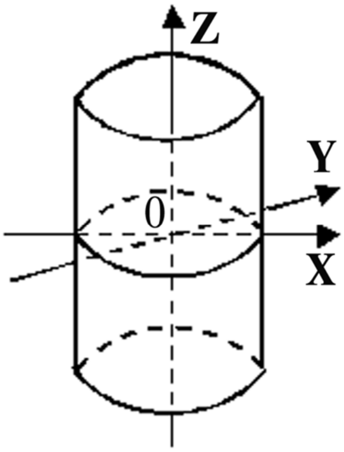
hyperbola

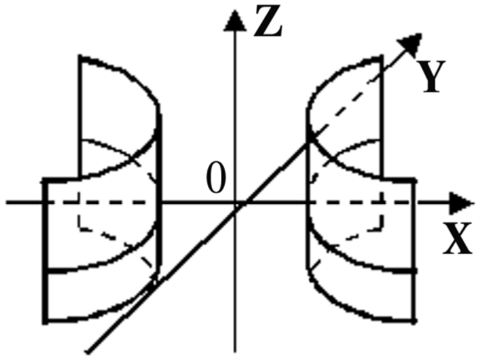
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\ z=0,$$

parabola

$$y^2=2px,\ z=0$$

respectively and generatrices are parallel to axis *Oz*.





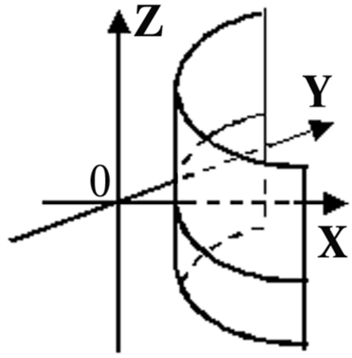


Fig. 3.20 **Fig. 3.21** **Fig. 3.22**

Remark. As we saw previously, conical and cylindrical surfaces of the 2^{nd} order have rectilinear generatrices, besides, each of these surfaces can be generated by motion of a straight line in a space.

It turns out, that one-sheeted hyperboloid and hyperbolic paraboloid also have rectilinear generatrices among all surfaces of the 2^{nd} order (except a cylinder and a cone), besides, both these surfaces as well as a cylinder and a cone can be generated by motion of a straight line in a space (see the literature on specialized subject).

§ 4. REDUCTION OF GENERAL EQUATION OF SURFACE OF THE 2^{nd} ORDER TO CANONICAL FORM

General equation of surface of the 2^{nd} order

$$a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{12}xy+2a_{13}xz+2a_{23}yz+2a_{10}x++2a_{20}y+2a_{30}z+a_{00}=0 \qquad (5.45)$$

contains the following terms:

- quadratic form

$$\omega(\vec{r})=a_{11}x^2+a_{22}y^2+a_{33}z^2+2a_{12}xy+2a_{13}xz+2a_{23}yz, \qquad (5.46)$$

where $\vec{r}=(x,y,z)$;

- linear form

$$\varphi(\vec{r}) = 2a_{10}x + 2a_{20}y + 2a_{30}z\,, \qquad (5.47)$$

where $\vec{r} = (x, y, z)$;

- absolute term a_{00} .

To reduce equation (5.45) to canonical form, we need to transform coordinates x, y, z and, consequently, related orthonormal basis $\vec{i}, \vec{j}, \vec{k}$ which transforms quadratic form (5.46) to canonical form (see Book II, Ch. 8, § 3, p. 3.1).

The matrix of this quadratic form has the form

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix},$$

where $a_{12} = a_{21}, a_{13} = a_{31}, a_{23} = a_{32}$, i.e. matrix A is symmetric. We designate characteristic numbers as $\lambda_1, \lambda_2, \lambda_3$ and orthonormal basis as $\vec{i}_1, \vec{j}_1, \vec{k}_1$ which consists of characteristic vectors of matrix A. Let us assume that

$$T = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix}$$

is transformation matrix from basis $\vec{i}, \vec{j}, \vec{k}$ to $\vec{i}_1, \vec{j}_1, \vec{k}_1$, and x', y', z' is a new coordinate system related to this basis.

When transforming coordinates

$$\begin{cases} x = \ell_1 x' + m_1 y' + n_1 z' \\ y = \ell_2 x' + m_2 y' + n_2 z' \\ z = \ell_3 x' + m_3 y' + n_3 z' \end{cases} \qquad (5.48)$$

quadratic form (5.46) takes canonical form

$$\omega(\vec{r}) = \lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2,$$

where $\vec{r} = (x', y', z')$.

Here, using transformation of coordinates (5.48) to linear form (5.47), we obtain

$$\varphi(\vec{r}) = 2\mu_1 x' + 2\mu_2 y' + 2\mu_3 z',$$

where $\vec{r} = (x', y', z')$, μ_1, μ_2, μ_3 are new coefficients of the form (5.47).

Thus, equation (5.45) takes the form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + 2\mu_1 x' + 2\mu_2 y' + 2\mu_3 z' + a_{00} = 0\,.$$

This equation can be reduced to canonical form by means of translation of coordinate system according to formulas

$$\begin{cases} x_1 = x' + \dfrac{\mu_1}{\lambda_1} \\ y_1 = y' + \dfrac{\mu_2}{\lambda_2} \\ z_1 = z' + \dfrac{\mu_3}{\lambda_3} \end{cases} \text{ or } \begin{cases} x' = x_1 - \dfrac{\mu_1}{\lambda_1} \\ y' = y_1 - \dfrac{\mu_2}{\lambda_2} \\ z' = z_1 - \dfrac{\mu_3}{\lambda_3} \end{cases} \qquad (5.49)$$

After making transformation of coordinate system by means of translation (5.49), general equation of surface of the 2nd order (5.45) relatively Cartesian coordinate system x_1, y_1, z_1, expresses one of the following 17 surfaces:

1) ellipsoid $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1;$

2) imaginary ellipsoid $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=-1;$

3) one-sheeted hyperboloid $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=1;$

4) two-sheeted hyperboloid $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=-1;$

5) cone $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=0;$

6) imaginary cone $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=0;$

7) elliptic paraboloid

$$\frac{x_1^2}{p}+\frac{y_1^2}{q}=2z,\ p>0,\ q>0$$

8) hyperbolic paraboloid

$$\frac{x_1^2}{p}-\frac{y_1^2}{q}=2z,\ p>0,\ q>0$$

9) elliptic cylinder

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1;$$

10) imaginary elliptic cylinder

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=-1;$$

11) two imaginary intersecting planes

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=0;$$

12) hyperbolic cylinder

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1;$$

13) two intersecting planes

$$\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=0;$$

14) parabolic cylinder

$$x_1^2=2py_1,\ p>0;$$

15) two parallel planes

$$x^2=a^2,\quad a\neq 0;$$

16) two imaginary parallel planes

$$x^2=-a^2,\quad a\neq 0;$$

17) two coinciding planes

$$x^2=0.$$

Example. Find out type and location of the surface which is set by equation

$$2x^2+y^2+2z^2-2xy-2yz+4x-2y=0 \qquad (5.50)$$

relatively Cartesian coordinate system x, y, z and related orthonormal basis $\vec{i},\vec{j},\vec{k}$

Solution. We reduce quadratic form

$$\omega(\vec{r})=2x^2+y^2+2z^2-2xy-2yz \qquad (5.51)$$

to canonical form. Matrix of this form looks like

$$A=\begin{pmatrix} 2 & -1 & 0\\ -1 & 1 & 1\\ 0 & 1 & 2\end{pmatrix}.$$

We find out characteristic numbers of this matrix using characteristic equation

$$\begin{vmatrix} (2-\lambda) & -1 & 0 \\ -1 & (1-\lambda) & 1 \\ 0 & 1 & (2-\lambda) \end{vmatrix} = (2-\lambda)\begin{vmatrix} (1-\lambda) & 1 \\ 1 & (2-\lambda) \end{vmatrix} + \begin{vmatrix} -1 & 1 \\ 0 & (2-\lambda) \end{vmatrix} = (2-\lambda)\cdot\lambda\cdot(\lambda-3) = 0.$$

There we obtain $\lambda_1 = 2,\ \lambda_2 = 0,\ \lambda_3 = 3$.

Now we find characteristic vectors of matrix A:

1. Let us suppose that $\lambda = \lambda_1 = 2$, then using equation $A(\vec{U}_1') = \lambda_1 \vec{U}_1'$ or coordinate form

$$\begin{cases} (2-\lambda_1)\ell_1' - m_1' & = 0 \\ -\ell_1' + (1-\lambda_1)m_1' + n_1' & = 0 \\ m_1' + (2-\lambda_1)n_1' = 0 \end{cases} \Rightarrow \begin{cases} -m_1' = 0 \\ -\ell_1' + n_1' = 0 \\ (2-\lambda_1)n_1' = 0 \end{cases}$$

we find $\ell_1' = \alpha,\ m_1' = 0,\ n_1' = \alpha$, where α is any number and, as a result, $\vec{U}_1' = (\alpha, 0, \alpha)$, $\vec{U}_1 = \alpha\vec{i} + \alpha\vec{k}$. Then we choose vector $\vec{U}_1 = \vec{i}_1$, whose module is $|\vec{U}_1| = |\vec{i}_1| = 1$, i. e. we normalize vector $\vec{U}_1$ from whole set of collinear vectors $\vec{U}_1$.

$$\vec{i}_1 = \frac{\vec{U}_1}{|\vec{U}_1|} = \frac{\ell_1'}{|\vec{U}_1|}\vec{i} + \frac{m_1'}{|\vec{U}_1|}\vec{j} + \frac{n_1'}{|\vec{U}_1|}\vec{k} = \ell_1\vec{i} + m_1\vec{j} + n_1\vec{k} =$$

$$= \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{k}, \quad \text{ãäå} \ |\vec{U}_1| = \alpha\sqrt{2}$$

2. We have for $\lambda = \lambda_2 = 0$

$$\begin{cases} (2-\lambda_2)\ell_2' - m_2' & = 0 \\ -\ell_2' + (1-\lambda_2)m_2' + n_2' & = 0 \\ m_2' + (2-\lambda_2)n_2' = 0 \end{cases} \Rightarrow \begin{cases} 2\ell_2' - m_1' = 0 \\ -\ell_2' - m_2' + n_2' = 0. \\ m_2' + 2n_2' = 0 \end{cases}$$

There we obtain $\ell_2' = \frac{1}{2}\alpha,\ m_2' = \alpha,\ n_2' = -\frac{1}{2}\alpha$, where α is any number. Then $\vec{U}_2' = \left(\frac{1}{2}\alpha, \alpha, -\frac{1}{2}\alpha\right)$, and $\vec{U}_2 = \frac{1}{2}\alpha\vec{i} + \alpha\vec{j} - \frac{1}{2}\alpha\vec{k}$. After normalizing vector $\vec{U}_2$, we find unit vector $\vec{j}_1$ for direction set by vector $\vec{U}_2$:

$$\vec{j}_1 = \frac{\vec{U}_2}{|\vec{U}_2|} = \frac{1}{\sqrt{6}}\vec{i} + \sqrt{\frac{2}{3}}\vec{j} - \frac{1}{\sqrt{6}}\vec{k},$$

where $|\vec{U}_2| = \alpha\sqrt{\frac{3}{2}}$.

3. $\lambda = \lambda_3 = 3$, then we have system for elements $\ell'_3,\ m'_3,\ n'_3$ of vector $\vec{U}'_3$

$$\begin{cases} -\ell'_3 - m'_3 = 0 \\ -\ell'_3 - 2m'_3 + n'_3 = 0 \\ m'_3 - n'_3 = 0 \end{cases}$$

There we obtain $\ell'_3 = -\alpha,\ m'_3 = \alpha,\ n'_3 = \alpha$, where α is any number and, as a result, $\vec{U}'_3 = (-\alpha, \alpha, \alpha)$ and $\vec{U}_3 = -\alpha\vec{i} + \alpha\vec{j} + \alpha\vec{k}$. After normalizing vector $\vec{U}_3$ we find unit vector $\vec{k}_1$ for direction set by vector $\vec{U}_3$:

$$\vec{k}_1 = \frac{\vec{U}_3}{\left|\vec{U}_3\right|} = \ell_3\vec{i} + m_3\vec{j} + n_3\vec{k} = -\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k},$$

where $\left|\vec{U}_3\right| = \alpha\sqrt{3}$.

Now we proceed to orthonormal basis $\vec{i}_1, \vec{j}_1, \vec{k}_1$, consisting of characteristic vectors of matrix A, and connect it with new Cartesian coordinate system x', y', z'. In case of such a change, transformation matrix has the form

$$T = \begin{pmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ 0 & \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix},$$

and coordinates are changed according to formulas

$$\begin{cases} x = \frac{1}{\sqrt{2}}x' + \frac{1}{\sqrt{6}}y' - \frac{1}{\sqrt{3}}z' \\ y = \sqrt{\frac{2}{3}}y' + \frac{1}{\sqrt{3}}z' \\ z = \frac{1}{\sqrt{2}}x' - \frac{1}{\sqrt{6}}y' + \frac{1}{\sqrt{3}}z' \end{cases} \tag{5.52}$$

Appling the given transformation of coordinates to quadratic form (5.51), we reduce it to canonical form
$\omega(\vec{r}) = 2x'^2 + 3z'^2$, where $\vec{r} = (x', y', z')$.

Now we find out what form linear formula

$$\varphi(\vec{r}) = 4x - 2y\text{, где } \vec{r} = (x, y, z),$$

has, if coordinates are transformed by formulas (5.52). We acquire

$$\varphi(\vec{r}) = 4\left(\frac{1}{2}x' + \frac{1}{\sqrt{6}}y' - \frac{1}{\sqrt{3}}z'\right) - 2\left(\sqrt{\frac{2}{3}}y' + \frac{1}{\sqrt{3}}z'\right) = 2\sqrt{2}x' - 2\sqrt{3}z'.$$

Thus, if we transform coordinate system x, y, z according to formulas (5.52), then surface of the 2nd order is set by equation

$$2x'^2 + 3z'^2 + 2\sqrt{2}x' - 2\sqrt{3}z' = 0. \qquad (5.53)$$

relatively new Cartesian coordinate system.

We reduce equation (5.53) to canonical form by means of translation of coordinate system according to formulas

$$\begin{cases} x' = x_1 - \dfrac{\sqrt{2}}{2} = x_1 - \dfrac{1}{\sqrt{2}} \\ z' = z_1 + \dfrac{\sqrt{3}}{3} = z_1 + \dfrac{1}{\sqrt{3}}, \end{cases}$$

whereupon equation of a surface takes the form

$$2\left(x_1 - \frac{1}{\sqrt{2}}\right)^2 + 3\left(z_1 + \frac{1}{\sqrt{3}}\right)^2 + 2\sqrt{2}\left(x_1 - \frac{1}{\sqrt{2}}\right) - 2\sqrt{3}\left(z_1 + \frac{1}{\sqrt{3}}\right) = 2x_1^2 + 3z_1^2 - 2 = 0$$

or

$$\frac{x_1^2}{1} + \frac{z_1^2}{\left(\sqrt{\frac{2}{3}}\right)^2} = 1.$$

relatively coordinate system x_1, y_1, z_1.

This equation expresses an elliptic cylinder whose leading ellipse is located in coordinate plane x_1, O_1, z_1, and generatrices are parallel to axis $O_1 y_1$.

Remark. The outline of reducing general equation of surface of the 2nd order to canonical form, which we considered in this paragraph, can be used when reducing general equation of curve of the 2nd order to canonical form.

EXERCISES

1. Make up an equation of a line which passes through point *M*(-2,5) in parallel and perpendicular way to straight line $2x+5y+1=0.$

2. Write equations of triangle sides with vertices *P*(-4,3), *Q*(2,5), *R*(6,-2).

3. Find a distance between parallel lines:
$5x-12y-26=0;\ 5x-12y-65=0.$

4. Make up an equation of a line which passes through point *M* (1,2) and forms (so does line, set by equation $3x-5y+3=0,$) such an acute angle φ, that $tg\varphi=\frac{1}{2}.$

5. Find out a distance between point *M* (2,-1) and line $3x+4y-1=0.$

6. Make up an equation of a plane which passes through point точку (-1,2,3) and line $\frac{x-1}{3}=\frac{y+2}{1}=\frac{z+1}{2}.$

7. Make up an equation of a plane which passes through three points M_1(1,2,3), M_2(3,-2,1), M_3(1,1,1).

8. Find a point which is symmetric to point (2,7,1) relatively plane

$$x-4y+z+7=0.$$

9. Make up canonical equation of an ellipse provided its major semi-axis is equal to 10 and eccentricity is equal to 0,8.

10. Hyperbola $\frac{x^2}{9}-\frac{y^2}{1}=1.$ is set. Find coordinates of focuses and vertices, eccentricity and equations of asymptotes.

11. Parabola with vertex located in an origin, passes through point *A*(-1,2) and is symmetric relatively axis *Oy*. Write its equation, find a focus and a directrix.

12. Parabola with vertex located in an origin, passes through point *A*(-2,-3) and is symmetric relatively axis *Ox*. Write its equation, find a focus and a directrix.

13. Reduce general equation of a curve to canonical form:

a) $x^2-4xy+4y^2+2x-4y-3=0;$

b) $3x^2+2xy+3y^2+8x+8y+4=0;$

c) $3x^2 - 10xy + 3y^2 - 16x + 16y + 24 = 0$;

d) $6xy + 8y^2 - 12x - 26y + 11 = 0$;

e) $x^2 + 4x - y + 5 = 0$.

14. Reduce general equation of a surface to canonical form:

a) $x^2 + 5y^2 + z^2 + 2xy + 6xz + 2yz - 2x + 6y + 2z = 0$;

b) $y^2 + 2xy + 4xz + 2yz - 4x - 2y = 0$;

c) $5x^2 - y^2 + z^2 + 4xy + 6xz + 2x + 4y + 6z - 8 = 0$;

d) $x^2 + y^2 + 4z^2 + 2xy + 4xz + 4yz - 6z + 1 = 0$;

e) $4x^2 + y^2 - z^2 - 24x - 4y + 2z + 35 = 0$;

f) $x^2 + 2y^2 + 2z^2 - 2xy - 2xz + 2x + y + z = 0$;

g) $xy + 2xz - 1 = 0$;

h) $7x^2 + 6y^2 + 5z^2 - 4xy - 4yz - 6x - 24y + 18z + 30 = 0$.

Навчальне видання

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