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THE DYNAMICAL PROBLEM ON ACTING CONCENTRATED LOAD ON THE ELASTIC QUARTER SPACE

The wave field of an elastic quarter space is constructed when one face is rigidly fixed and a dynamic normal compressive load acts on the other along a rectangular section at the initial moment of time. Integral Laplace and Fourier transforms are applied sequentially to the equations of motion and boundary conditions in contrast to traditional approaches when integral transforms are applied to solutions' representations through harmonic functions. This leads to a one-dimensional vector homogeneous boundary value problem with respect to unknown displacement's transformants. The problem was solved using matrix differential calculus. The original displacement field was found after applying inverse integral transforms. For the case of stationary vibrations a method of calculating integrals in the solution in the near loading zone was indicated. For the analysis of oscillations in a remote zone the asymptotic formulas were constructed. The amplitude of vertical vibrations was investigated depending on the shape of the load section, natural frequencies of vibrations and the material of the medium.
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1. Introduction

During the construction and analysis of structures when dynamic or static loads appear, stress arise and concentrate in elastic bodies. These stresses can deform and even break the structure. Therefore they must be taken into account during design calculation. Because of this, problems of the elasticity theory appear in mathematical physics.

These problems were considered in a static and dynamic statements by many authors for different objects under different initial and boundary conditions [1–4]. An object such as a quarter space can be considered as a model before solving a similar problems for an infinite or semi-infinite layer and then for a plate. A quarter space is a special case of a spatial wedge. In particular for the second boundary value problem for a spatial wedge the exact solution

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was constructed by Ya. S. Uflyand [5]. In another work [6], the exact solution for the case, when normal displacements and tangential stress are given, was constructed. The exact solution of the mixed problem of the elasticity theory for a quarter-space in the static statement was found by G. Ya. Popov in [1]. It is essential that a new method was used in the solving of this problem, based on representation of new functions which are the sum of displacements' derivatives [7]. This method was successfully applied to solving Lamb problem [8]. Also using this method, homogeneous and inhomogeneous problems of the elasticity theory for a semifinite layer were solved [7]. The development of methods for problems of the elasticity theory for various objects, in particular for a quarter space, was also carried out by A. M. Alexandrov in [9]. A general solution for an elastic quarter space contact problem was presented in [10]. Dynamical stresses in elastic half-space were analysed in [11]. Plane contact problem on the pressure of a stamp with a rectangular base on a rough elastic halfspace was considered in [12].

Based on the results of [1; 8], as well as the method of representing the equations of motion in terms of two jointly and one independently solvable equations, proposed in [7], the aim of this work is to obtain the exact formulas for displacements that appear in a quarter space when a dynamic compressive load acts on one of its faces.

2. Main Results

2.1. Statement of the problem.

An elastic quarter space $x > 0,\ -\infty < y < \infty,\ 0 < z < \infty,$ is considered. At the moment of time $t = 0$ dynamic normal load

$$\sigma_z\ (x, y, z, t)|_{z=0} = -p(x, y)P(t)$$

is applied to the boundary of the quarter space $z = 0$ across the rectangular area $0 \leq x \leq A,\ -B \leq y \leq B$ the tangential stresses over the entire XOY plane are zero. The face $x = 0$ is rigidly fixed. The nonstationary points' displacements of the quarter space $u(x, y, z, t),\ v(x, y, z, t),\ w(x, y, z, t)$ are required to be determined with zero initial conditions. The statement leads to the following boundary conditions

$$\sigma_z(x, y, 0, t) = -p(x, y)P(t),\ 0 \leq x \leq A;\ -B \leq y \leq B \qquad (1)$$

$$\sigma_z(x,y,0,t)=0,\ x>A;\ |y|>B$$
$$\tau_{zx}(x,y,0,t)=0,\ \tau_{zy}(x,y,0,t)=0$$
$$u(0,y,z,t)=v(0,y,z,t)=w(0,y,z,t)=0$$

The equations of motion in vector form are [8]

$$\Delta(u,v,w)+\frac{2}{\kappa-1}\left(\frac{\partial\Theta}{\partial x},\frac{\partial\Theta}{\partial y},\frac{\partial\Theta}{\partial z}\right)=\frac{\rho}{G}\left(\frac{\partial^2 u}{\partial t^2},\frac{\partial^2 v}{\partial t^2},\frac{\partial^2 w}{\partial t^2}\right), \tag{2}$$

where Δ is the Laplace operator, $\kappa=3-4\mu$, μ — Poisson's ratio, $\Theta=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$ — volume expansion, ρ — density of the medium material, G — shear modulus; $\frac{\rho}{G}=\frac{1}{c^2}$, c — wave propagation speed.

To obtain a solution to the posed problem, it is enough firstly to obtain a solution when the dynamic force concentrated at an arbitrary point (a,b) of the face $z=0$, and then distribute it over the required section, i.e.

$$p(x,y)=\delta(x-a)\delta(y-b)$$

Let's set up a dimensionless coordinate system

$$\tilde{x}=\frac{x}{a},\ \tilde{y}=\frac{(y-b)}{a},\ \tilde{z}=\frac{z}{a},\ \tilde{t}=\left(\frac{1}{c^2}\right)t \tag{3}$$

Further, the "waves" are omitted, implying the replacement (3), introduce the new functions [7]

$$\begin{aligned} Z(x,y,z)&=\frac{\partial}{\partial x}u(x,y,z)+\frac{\partial}{\partial y}v(x,y,z)\\ \widetilde{Z}(x,y,z)&=\frac{\partial}{\partial x}v(x,y,z)-\frac{\partial}{\partial y}u(x,y,z)\end{aligned} \tag{4}$$

Then the system of equations of motion (2) and the boundary conditions (1) are rewritten in the form relatively new functions.

$$\begin{cases}\Delta W+\dfrac{2}{\kappa-1}\dfrac{\partial}{\partial z}\left(Z+\dfrac{\partial W}{\partial z}\right)=\dfrac{\partial^2 W}{\partial t^2}\\ \Delta Z+\dfrac{2}{\kappa-1}\nabla_{xy}\left(Z+\dfrac{\partial W}{\partial z}\right)=\dfrac{\partial^2 Z}{\partial t^2}\end{cases} \tag{5}$$

$$\Delta\widetilde{Z}=\frac{\partial^2\widetilde{Z}}{\partial t^2} \tag{6}$$

$$
\begin{gathered}
\mu Z(x,y,0,t) + (1-\mu)\frac{\partial}{\partial z}W(x,y,0,t) = -\frac{\kappa-1}{4Ga}\delta(x-1)\delta(y)P(t) \\
\nabla_{xy}W(x,y,0,t) + \frac{\partial}{\partial z}Z(x,y,0,t) = 0 \\
\frac{\partial}{\partial z}\tilde{Z}(x,y,0,t) = 0 \\
u(0,y,z,t) = v(0,y,z,t) = w(0,y,z,t) = 0
\end{gathered} \tag{7}
$$

where $\nabla_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

The original initial boundary value problem takes the form (5)-(7) under the initial conditions

$$
\left[W, Z, \widetilde{Z}\right]\Big|_{t=0} = 0 \quad \frac{\partial}{\partial t}\left[W, Z, \widetilde{Z}\right]\Big|_{t=0} = 0 \tag{8}
$$

After finding the unknown functions $W, Z, \widetilde{Z}$ the Poisson equations should be solved in order to determine the displacements u and v

$$
\nabla_{xy}u = \frac{\partial}{\partial x}Z - \frac{\partial}{\partial y}\widetilde{Z},\ \nabla_{xy}v = \frac{\partial}{\partial y}Z + \frac{\partial}{\partial x}\widetilde{Z} \tag{9}
$$

2.2. Reducing the problem to a vector one-dimensional problem

The Fourier transform with respect to the variable y, sin - transform with respect to the variable x and the Laplace transform with respect to the variable t, with parameters β, α and p respectively are applied successively to (5), (6).

$$
\begin{bmatrix} W_{\alpha\beta p}(z) \\ Z_{\alpha\beta p}(z) \end{bmatrix} = \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{0}^{\infty} \begin{bmatrix} W(x,y,z,t) \\ Z(x,y,z,t) \end{bmatrix} e^{i\beta y}\sin\alpha x\, e^{-pt}\, dy\, dx\, dt \tag{10}
$$

The following conditions are assumed to be additionally satisfied [1]

$$
Z_{\beta}(0,z) = 0,\ \widetilde{Z}_{\beta}(0,z) = 0 \tag{11}
$$

The function $\widetilde{Z}_{\alpha\beta p}(z)$ satisfies the homogeneous problem

$$
\widetilde{Z}''_{\alpha\beta p}(z) - (N^2 + p^2)\widetilde{Z}_{\alpha\beta p}(z) = 0,\ 0 < z < \infty,\ \widetilde{Z}'_{\alpha\beta p}(0) = 0 \tag{12}
$$

and therefore $\widetilde{Z}(x,y,z,t) \equiv 0$. The system of equations (5) and the boundary conditions (7) take the form

$$
\begin{cases}
W''_{\alpha\beta p}(z) + \dfrac{2}{\kappa+1}Z'_{\alpha\beta p}(z) - N^2\dfrac{\kappa-1}{\kappa+1}W_{\alpha\beta p}(z) - \dfrac{\kappa-1}{\kappa+1}p^2 W_{\alpha\beta p} = 0 \\
Z''_{\alpha\beta p}(z) - \dfrac{2}{\kappa-1}N^2 W'_{\alpha\beta p}(z) - N^2\dfrac{\kappa+1}{\kappa-1}Z_{\alpha\beta p}(z) - p^2 Z_{\alpha\beta p}(z) = 0
\end{cases} \tag{13}
$$

$$
\begin{gathered}
-N^2 W_{\alpha\beta p}(0) + Z'_{\alpha\beta p} = 0 \\
(3-\kappa)\mu Z_{\alpha\beta p}(0) + (1-\mu) W'_{\alpha\beta p}(0) = -\frac{\kappa-1}{4Ga}\cdot \sin\alpha \cdot P_p \\
P_p = \int\limits_0^{\infty} P(t) e^{-pt} dt;\ N^2 = \alpha^2 + \beta^2;
\end{gathered}
\tag{14}
$$

To rewrite the system (13) in vector form, the unknown vector of the displacement's transformant is introduced

$$
\vec{\mathbf{y}}(z) = \begin{pmatrix} W_{\alpha\beta p}(z) \\ Z_{\alpha\beta p}(z) \end{pmatrix}
$$

as well matrices

$$
\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \mathbf{Q} = \begin{pmatrix} 0 & \frac{1}{\kappa+1} \\ \frac{-N^2}{\kappa-1} & 0 \end{pmatrix},\ \mathbf{P} = \begin{pmatrix} \frac{\kappa-1}{\kappa+1} & 0 \\ 0 & \frac{\kappa-1}{\kappa+1} \end{pmatrix},\ \mathbf{T} = \begin{pmatrix} \frac{\kappa-1}{\kappa+1} & 0 \\ 0 & 1 \end{pmatrix}
$$

So, the system (13) takes form

$$
\mathrm{L}_2 \vec{\mathbf{y}}(z) = 0,\ 0 < z < \infty \tag{15}
$$

where the differential operator L_2 has a form

$$
\mathrm{L}_2 \vec{\mathbf{y}}(z) = \mathbf{I}\vec{\mathbf{y}}''(z) + 2\mathbf{Q}\vec{\mathbf{y}}'(z) - N^2 \mathbf{P}\vec{\mathbf{y}}(z) - p^2 \mathbf{T}\vec{\mathbf{y}}(z)
$$

The solution of the vector equation (15) is constructed on the basis of the matrix equation's solution $\mathrm{L}_2[\mathbf{Y}(z)] = 0$. The substitution $\mathbf{Y}(z) = e^{Nz}\mathbf{I}$ is made to form the characteristic matrix $\mathbf{M}(s) = \mathbf{I}s^2 + 2\mathbf{Q}s - N^2\mathbf{P} - p^2\mathbf{T}$. Inverse matrix has a form

$$
\mathbf{M^{-1}}(\mathbf{s}) = \frac{1}{\prod_{i=1}^4 (s - s_i)} \begin{pmatrix} s^2 - \frac{\kappa+1}{\kappa-1}N^2 - p^2 & -\frac{2s}{\kappa+1} \\ \frac{2s}{\kappa-1}N^2 & s^2 - N^2\frac{\kappa-1}{\kappa+1} - p^2\frac{\kappa-1}{\kappa+1} \end{pmatrix}
$$

$$
\begin{aligned}
s_1 &= -\sqrt{N^2 + \frac{\kappa-1}{\kappa+1}p^2}, & s_2 &= -\sqrt{N^2 + p^2}, \\
s_3 &= \sqrt{N^2 + \frac{\kappa-1}{\kappa+1}p^2}, & s_4 &= \sqrt{N^2 + p^2}.
\end{aligned}
$$

Here s_i $(i = \overline{1,4})$ are roots of the characteristic equation $\det[\mathbf{M}(s)] = 0$. The solution of the matrix equation is constructed by the formula [13]

$$
\mathbf{Y}_-(z) = \frac{1}{2\pi i} \oint\limits_C e^{sz} \mathbf{M}^{-1}(s) ds
$$

where C is a closed contour encompassing all zeros of the matrix's determinant $\mathbf{M}(s)$. The residues at the poles s_3 and s_4 give a growing solution at infinity and are therefore discarded. The residues at the poles s_1 and s_2 give a solution decreasing at infinity. After calculation a decreasing solution takes a form

$$\mathbf{Y}_-(z) = \frac{1}{2p^2} e^{-\Delta_1 z} \begin{pmatrix} \frac{(\kappa+1)N^2}{(\kappa-1)\Delta_1} & -1 \\ \frac{(\kappa+1)N^2}{(\kappa-1)} & -\Delta_1 \end{pmatrix} + \frac{1}{2p^2} e^{-\Delta_2 z} \begin{pmatrix} -\frac{(\kappa+1)}{(\kappa-1)}\Delta_2 & 1 \\ -\frac{(\kappa+1)}{(\kappa-1)}N^2 & \frac{N^2}{\Delta_2} \end{pmatrix} \tag{16}$$

where $\Delta_1 = \sqrt{N^2 + p^2},\ \Delta_2 = \sqrt{N^2 + \frac{p^2(\kappa-1)}{(\kappa+1)}}$

The solution of the vector equation (15) is constructed in the form

$$\vec{\mathbf{y}}(z) = \mathbf{Y}_-(z) \cdot \begin{pmatrix} C_0 \\ C_1 \end{pmatrix}$$

where constants $C_i,\ i = 0, 1$ are found by satisfying the boundary conditions (14). Thus, a system of linear algebraic equations is obtained

$$\begin{cases} \frac{\kappa+1}{\kappa-1} \frac{2N^2}{\Delta_1} \left[\Delta_1\Delta_2 - N^2 - \frac{p^2}{2}\right] C_0 + p^2 C_1 = 0 \\ p^2 C_0 + 2\frac{\kappa-1}{\kappa+1} \frac{1}{\Delta_2} \left[\Delta_1\Delta_2 - N^2 - \frac{p^2}{2}\right] C_1 = -\frac{\kappa-1}{\kappa+1} \frac{2p^2}{Ga} \sin\alpha \cdot P_p \end{cases}$$

after solving it the expressions for the transformants were found

$$\begin{aligned} W_{\alpha\beta p}(z) &= \frac{\sin\alpha}{Ga} \cdot P_p \frac{\Delta_2}{\widetilde{\Delta}} \left[-2N^2 e^{-\Delta_1 z} + (2N^2 + p^2) e^{-\Delta_2 z}\right] \\ Z_{\alpha\beta p}(z) &= \frac{\sin\alpha}{Ga} \cdot P_p \frac{N^2}{\widetilde{\Delta}} \left[-2\Delta_1\Delta_2 e^{-\Delta_1 z} + (2N^2 + p^2) e^{-\Delta_2 z}\right] \end{aligned} \tag{17}$$

$$\widetilde{\Delta} = 4N^4 + 4N^2p^2 + p^4 - 4N^2\Delta_1\Delta_2 \tag{18}$$

Based on formulas (9), (12), the transformants of the remaining displacements are found

$$u_{\alpha\beta p}(z) = -\frac{\alpha}{N^2} Z_{\alpha\beta p}(z),\ v_{\alpha\beta p}(z) = \frac{i\beta}{N^2} Z_{\alpha\beta p}(z) \tag{19}$$

Thus, an exact solution to the posed vector problem (13) (14) in the transform space was obtained.

2.3. Construction of the original solutions

After applying the inverse integral transformations to the solution (17), the original vertical displacement was obtained

$$W(x,y,z,t)=\frac{1}{2\pi^2}\frac{1}{Ga}\frac{1}{2\pi i}\int\limits_{l}\int\limits_{-\infty}^{\infty}\int\limits_{0}^{\infty}\frac{\Delta_2}{\widetilde{\Delta}}P_p\left[-2N^2e^{-\Delta_1 z}+\right.$$
$$\left.+(2N^2+p^2)e^{-\Delta_2 z}\right]\sin\alpha\, e^{i\beta y}\sin\alpha x\, e^{pt}dp\, d\beta\, d\alpha$$

$l=(\lambda-i\infty,\ \lambda+i\infty)$

Using the parity of the integrand and applying Euler's formula, displacement is rewritten in the form

$$W(x,y,z,t)=\frac{1}{4\pi^2}\frac{1}{Ga}\frac{1}{2\pi i}\int\limits_{l}P_p\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{\Delta_2}{\widetilde{\Delta}}\left[-2N^2e^{-\Delta_1\cdot z}+\right.$$
$$\left.+(2N^2+p^2)e^{-\Delta_2\cdot z}\right]e^{i\beta y}\left[e^{-i(x-1)\alpha}-e^{-i(x+1)\alpha}\right]e^{pt}dp\, d\beta\, d\alpha$$

In order to get rid of the double integral over the parameters of the Fourier transforms, the relation connecting the Fourier and Hankel transforms [14] was used

$$\frac{1}{4\pi}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}F\left(\sqrt{\alpha^2+\beta^2+\chi_i^2}\right)e^{-i\alpha x-i\beta y}d\alpha d\beta=\int\limits_{0}^{\infty}sF(\sqrt{s^2+\chi_i^2})\times$$
$$\times J_0(s\sqrt{x^2+y^2})ds$$

where $J_0(s)$ is the Bessel function, $\chi_1=p,\ \chi_2=\sqrt{\frac{\kappa-1}{\kappa+1}}p$. After simplifications, formula for displacement takes a form

$$W(x,y,z,t)=\frac{1}{\pi Ga}\frac{1}{2\pi i}\int\limits_{l}P_p\int\limits_{0}^{\infty}\frac{F(s)}{\Delta_s}\cdot s\left[J_0(s\sqrt{(x-1)^2+y^2})-\right.$$
$$\left.-J_0(s\sqrt{(x+1)^2+y^2})\right]e^{pt}ds\, dp$$

$$F(s)=\sqrt{s^2+\frac{\kappa-1}{\kappa+1}p^2}\cdot\left[-4s^2e^{-\sqrt{s^2+p^2}}+(2s^2+p^2)e^{-\sqrt{s^2+\frac{\kappa-1}{\kappa+1}p^2}}\right]$$
$$\Delta_s=4s^4+4s^2p^2+p^4-4s^2\sqrt{s^2+p^2}\sqrt{s^2+\frac{\kappa-1}{\kappa+1}p^2}$$

Using the parity of the Bessel function $J_0(s)$, continue the integrand in an odd way to the interval $(-\infty,\ 0)$

$$W(x,y,z,t) = \frac{1}{\pi G a}\frac{1}{2\pi i}\int\limits_{l} P_p \int\limits_{-\infty}^{\infty} \frac{F(s)}{\Delta_s}\cdot s\left[J_0(s\sqrt{(x-1)^2+y^2})-\right.$$
$$\left.-J_0(s\sqrt{(x+1)^2+y^2}\right] e^{pt} ds\, dp$$

According to the obtained solution, the displacement from the distributed over a rectangular area load can be found

$$W^{AB}(x,y,z,t) = \frac{1}{\pi G a}\frac{1}{2\pi i}\int\limits_{0}^{A}\int\limits_{-B}^{B}\int\limits_{l} P_p \int\limits_{-\infty}^{\infty} \frac{F(s)}{\Delta_s}\cdot s\left[J_0(s\sqrt{(x-a)^2+(y-b)^2})-\right.$$
$$\left.-J_0(s\sqrt{(x+a)^2+(y-b)^2}\right] e^{pt} ds\, dp\, da\, db \quad (20)$$

Formula was written in the initial coordinate system.

2.4. Steady-state oscillation case

Suppose that the load applied across the area $0 < x < A;\ -B < y < B$ over the plane $X0Y$ changes according to the harmonic law $P(t) = e^{i\omega t}$ and $p(x,y) = P$, where P — constant intensity of the load, ω — is a natural frequency of vibrations. In this case, substituting into the constructed solution (20) $p = i\omega$, the displacement is written in the form

$$W^{AB}(x,y,z;\omega) = \frac{P}{\pi G a}\int\limits_{0}^{A}\int\limits_{-B}^{B}\int\limits_{-\infty}^{\infty} \frac{F(s;\omega)}{\Delta_{s\omega}}\cdot s\left[J_0(s\sqrt{(x-a)^2+(y-b)^2})-\right.$$
$$\left.-J_0(s\sqrt{(x+a)^2+(y-b)^2})\right] ds\, da\, db \quad (21)$$

$$F(s;\omega) = \delta_2\left[-2s^2 e^{-\delta_1 z} + (2s^2-\omega^2)e^{-\delta_2 z}\right]$$

$$\Delta_{s\omega} = 4s^4 - 4s^2\omega^2 + \omega^4 - 4s^2\delta_1\delta_2 = (2s^2-\omega^2)^2 - 4s^2\delta_1\delta_2. \quad (22)$$

$$\delta_1 = \sqrt{s^2-\omega^2},\ \delta_2 = \sqrt{s^2 - \tfrac{\kappa-1}{\kappa+1}\omega^2} \quad (23)$$

Since the expression (23) includes the multivalued functions [3], they have to be fixed. And after making cuts, using the contour integration methods, the displacement is calculated. It is necessary that from the loaded rectangle on the quarter space's face where the load is applied, the energy is carried away to infinity by each of the two types of possible waves. These requirements make it possible to fix multivalued functions $\sqrt{s^2-\omega^2}$ и $\sqrt{s^2-\frac{\kappa-1}{\kappa+1}\omega^2}$ [3; 8]

$$\begin{aligned}
&\text{when } |s|>\omega;\ |s|>\tfrac{\kappa-1}{\kappa+1}\omega:\ \delta_1=\sqrt{s^2-\omega^2};\ \delta_2=\sqrt{s^2-\tfrac{\kappa-1}{\kappa+1}\omega^2}\\
&\text{when } |s|<\omega;\ |s|<\tfrac{\kappa-1}{\kappa+1}\omega:\ \delta_1=-i\sqrt{\omega^2-s^2};\ \delta_2=-i\sqrt{\tfrac{\kappa-1}{\kappa+1}\omega^2-s^2}
\end{aligned} \tag{24}$$

Damping into the environment was introduced. The energy flow must be directed away from the place where the load is applied. The root of the equation (22), [3], is the number $s=\pm k_R$ — the wavenumber related to the propagation velocity of the Rayleigh wave. The denominator has no other roots for such a fixation of δ_1 and δ_2. Going around the branch points in the corresponding loops, choosing δ_1 and δ_2 on the corresponding sections of the loop, so that the requirements (24) are satisfied. Also taking into account the residue in the Rayleigh root, the solution for plane $z=0$ is obtained

$$\begin{aligned}
&\frac{G}{P}W^{AB}(x,y,0;\omega)=-\frac{2i\omega^2\sqrt{k_R^2-\frac{\kappa-1}{\kappa+1}\omega^2}}{F'(k_R)}J^{A,B}_{k_R,1}(x,y)+\\
&+\frac{2i}{\pi}\omega^2\int\limits_0^{\sqrt{\frac{\kappa-1}{\kappa+1}}\omega}\frac{s\sqrt{\frac{\kappa-1}{\kappa+1}\omega^2-s^2}}{(2s^2-\omega^2)^2+4s^2\sqrt{\omega^2-s^2}\sqrt{\frac{\kappa-1}{\kappa+1}\omega^2-s^2}}J^{A,B}_{s,1}(x,y)ds+\\
&+\frac{8i}{\pi}\omega^2\int\limits_{\sqrt{\frac{\kappa-1}{\kappa+1}}\omega}^{\omega}\frac{s^2\left(s^2-\frac{\kappa-1}{\kappa+1}\omega^2\right)\sqrt{\omega^2-s^2}}{(2s^2-\omega^2)^4+16s^4\left(s^2-\frac{\kappa-1}{\kappa+1}\omega^2\right)(\omega^2-s^2)}J^{A,B}_{s,1}(x,y)ds
\end{aligned} \tag{25}$$

$$\begin{aligned}
\text{Where } J^{A,B}_{s,1}(x,y)=\int\limits_0^A\int\limits_{-B}^{B}\Bigg[&J_0\left(s\sqrt{(x-a)^2+(y-b)^2}\right)-\\
&-J_0\left(s\sqrt{(x+a)^2+(y-b)^2}\right)\Bigg]da\,db
\end{aligned} \tag{26}$$

$$F'(s)=8s\left(2s^2-\omega^2\right)-\frac{4s^3\sqrt{s^2-w^2}}{\sqrt{s^2-\frac{\kappa-1}{\kappa+1}\omega^2}}-\frac{12s^3-8s\omega}{\sqrt{s^2-\omega^2}}$$

$k_R=\frac{7-\kappa}{6.84-1.12\kappa}\omega$, where the approximate formula from [3] was used.

If the formula (25) is being rewritten in terms of wavenumbers

$$k_2 = \frac{\omega}{c_2},\ k_1 = \sqrt{\frac{\kappa-1}{\kappa+1}}\omega = \frac{\omega}{c_1}$$

c_1 is longitudinal wave velocity; c_2 is shear wave velocity. The value of the integrand in (21) $\frac{F(s;\omega)\cdot s}{\Delta_{s\omega}}$ coincides with that one in Lamb's problem [3]. The difference with the work [8] is in the form of the function $J^{A,B}_{s,1}(x,y)$. Thus, under the assumption (11) that the functions $Z_\beta(0,z)$ and $\tilde{Z}_\beta(0,z)$ are equal to zero, the solution turned out to be practically identical to the solution of the Lamb problem.

2.5. Displacement for large values vibration frequency

For large values of frequency ω, using the expansion

$$(1-x)^{\frac{1}{2}} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4 - \dots \quad x^2 \le 1$$

and based on formulas (21) – (23), a calculation formula for the displacement was obtained in the form

$$\frac{G}{P}W^{AB}(x,y,0;\omega) = -\frac{i}{\pi}\int\limits_0^\infty F(s;\omega)J^{A,B}_{s,1}(x,y)ds, \tag{27}$$

where $F(s;\omega) =$

$$\frac{\omega^3 s - \frac{1}{2}\kappa_0\omega s^3 - \frac{1}{8}\kappa_0^2\frac{s^5}{\omega} - \frac{1}{16}\kappa_0^3\frac{s^7}{\omega^3} - \frac{5}{128}\kappa_0^4\frac{s^9}{\omega^5}}{4s^4\left(\sqrt{\kappa_0} - \frac{\kappa}{\kappa-1}\right) + \omega^4\sqrt{\kappa_0} + 4s^2\omega^2\left(\sqrt{\kappa_0}-1\right) - 4s^2\left\{\frac{s^4}{\omega^2}\kappa_1 + \frac{s^6}{\omega^4}\kappa_2 + \frac{s^8}{\omega^6}\kappa_3\right\}}$$

$$\kappa_0 = \frac{\kappa+1}{\kappa-1},\ \kappa_1 = \frac{1}{8}\kappa_0^2 + \frac{1}{4}\kappa_0,\ \kappa_2 = \frac{1}{16}\kappa_0^2 + \frac{1}{16}\kappa_0,\ \kappa_3 = \frac{1}{64}\kappa_0^2 + \frac{1}{32}\kappa_0$$

2.6. Transformation of the integral $J^{A,B}_{s,1}(x,y)$ from (26)

According to the scheme [8], consider the integral $J^{A,B}_{s,1}(x,y)$. Using the integral representation for the Bessel function [15]

$$J_0(s\sqrt{(x\mp a)^2+(y-b)^2}) = \frac{2}{\pi}\int\limits_0^{\frac{\pi}{2}} \cos\left[s(x\mp a)\cos\psi\right]\cdot\cos\left[s(y-b)\sin\psi\right]d\psi$$

which should be substituted into formula (26). After changing the order of integration and calculating the integrals as repeated, the procedure was detailed in [2], formula (26) was rewritten in the form

$$J_{s,1}^{A,B}(x,y)=\frac{8AB}{\pi}\int\limits_{0}^{\frac{\pi}{2}}S_s^{A,B}(\psi)\sin\left[sx\cos\psi\right]\cdot\cos\left[sy\sin\psi\right]d\psi, \qquad (28)$$

$$\text{where } S_s^{A,B}(\psi)=\frac{\sin\left[sB\sin\psi\right]}{sB\sin\psi}\cdot\frac{1-\cos\left[sA\cos\psi\right]}{sA\cos\psi}$$

the function $S_s^{A,B}(\psi)$ is infinitely differentiable with respect to ψ and also even, therefore, the integration path can be taken equal to $[-\pi/2,\ \pi/2]$. Subsequent change of variables $\sin\psi=\tau$ allows to rewrite (28) as

$$J_{s,1}^{A,B}(x,y)=\frac{4AB}{\pi}\int\limits_{-1}^{1}F_{s,\tau}^{A,B}(x,y)\frac{d\tau}{\sqrt{1-\tau^2}}, \qquad (29)$$

$$\text{where } F_{s,\tau}^{A,B}(x,y)=\frac{\sin\left[sB\tau\right]}{sB\tau}\cdot\frac{1-\cos\left[sA\sqrt{1-\tau^2}\right]}{sA\sqrt{1-\tau^2}}\cdot\sin\left[sx\sqrt{1-\tau^2}\right]\cdot\cos\left[sy\tau\right]$$

The quadrature formula of the highest degree of accuracy [16] was applied to the integral (29)

$$J_{s,1}^{A,B}(x,y)=\frac{4AB}{N}\sum_{i=1}^{N}F_{s,\tau_i}^{A,B}(x,y) \qquad (30)$$

where $\tau_i=\cos\frac{2i-1}{2N}\pi,\ i=\overline{1,N}$ are the zeros of the Chebyshev polynomial of the 1st kind.

$$F_{s,\tau_i}^{A,B}(x,y)=\frac{\sin\left[sB\tau_i\right]}{sB\tau_i}\cdot\frac{1-\cos\left[sA\sqrt{1-\tau_i^2}\right]}{sA\sqrt{1-\tau_i^2}}\cdot\sin\left[sx\sqrt{1-\tau_i^2}\right]\cdot\cos\left[sy\tau_i\right] \qquad (31)$$

Substituting the expression into the displacement formula (25) and (27) the final expression was constructed

$$\frac{G}{P}W^{AB}(x,y,0;\omega) = \frac{4AB}{N}\left[-\frac{2i\omega^2\sqrt{k_R^2-\frac{\kappa-1}{\kappa+1}\omega^2}}{F'(k_R)}\sum_{i=1}^{N}F^{A,B}_{k_R,\tau_i}(x,y)+\right.$$
$$+\frac{2i}{\pi}\omega^2\sum_{i=1}^{N}\int\limits_0^{\sqrt{\frac{\kappa-1}{\kappa+1}}\omega}\frac{s\sqrt{\frac{\kappa-1}{\kappa+1}\omega^2-s^2}}{(2s^2-\omega^2)^2+4s^2\sqrt{\omega^2-s^2}\sqrt{\frac{\kappa-1}{\kappa+1}\omega^2-s^2}}F^{A,B}_{s,\tau_i}(x,y)ds+$$
$$\left.+\frac{8i}{\pi}\omega^2\sum_{i=1}^{N}\int\limits_{\sqrt{\frac{\kappa-1}{\kappa+1}}\omega}^{\omega}\frac{s^2\left(s^2-\frac{\kappa-1}{\kappa+1}\omega^2\right)\sqrt{\omega^2-s^2}}{(2s^2-\omega^2)^4+16s^4\left(s^2-\frac{\kappa-1}{\kappa+1}\omega^2\right)(\omega^2-s^2)}F^{A,B}_{s,\tau_i}(x,y)ds\right] \tag{32}$$

where $F'(s)$ is defined in (26) and $F^{A,B}_{s,\tau_i}(x,y)$ – in (31). For large values of frequency ω the formula takes the form

$$\frac{G}{P}W^{AB}(x,y,0;\omega) = -\frac{4ABi}{N\pi}\sum_{i=1}^{N}\int\limits_0^{\infty}F(s;\omega)F^{A,B}_{s,\tau_i}(x,y)ds, \tag{33}$$

where the function $F(s;\omega)$ is defined in (27)

Thus, the formula has been simplified to the calculation of single integrals of continuous functions, which is not difficult if oscillations in the near zone are of interest.

2.7. Expressions for far field displacements

The calculation of integrals in (32), (33) for large values of x and y is difficult due to the presence of an oscillating function in the integrand. To eliminate this difficulty for large values of $r=\sqrt{x^2+y^2}$, the asymptotic expressions for analyzing the far field is advisable to obtained. In the integral (28) the change of variables $x=r\cos\phi,\ y=r\sin\phi,\ \lambda=tr$ was done

$$J^{A,B}_{s,1}(r\cos\phi, r\sin\phi) = \frac{4AB}{\pi}\,\mathrm{Im}\left\{\int\limits_0^{\frac{\pi}{2}}S^{A,B}_s(\psi)e^{i\lambda\cos(\phi-\psi)}d\psi+\right.$$

$$+\int_{0}^{\frac{\pi}{2}} S_s^{A,B}(\psi)e^{i\lambda\cos(\phi+\psi)}d\psi\Bigg\}\; 0\le\phi\le\frac{\pi}{2} \quad (34)$$

The stationary phase method was used for the analysis of asymptotics [8; 17], where the role of the function for the analysis of asymptotics, $f(\psi)$ is played by $\cos(\phi\mp\psi)$, and the role function $\phi(\psi)$ is an infinitely differentiable function $S_s^{A,B}(\psi)$. The first integral has a stationary point and the second has not, therefore, its contribution to the asymptotics of (34) can be neglected. The first integral in (34) can be represented as the sum

$$J_{s,1}^{A,B}(r\cos\phi, r\sin\phi) = \frac{4AB}{\pi}\,\mathrm{Im}\left(\int_{0}^{\phi} + \int_{\phi}^{\frac{\pi}{2}}\right) S_s^{A,B}(\psi)e^{i\lambda\cos(\phi-\psi)}d\psi.$$

where in the first integral the stationary point is at the end of the integration path $f'(\psi) = \frac{\partial}{\partial\psi}\cos(\psi-\phi) = 0$ for $\psi = \phi$ and $f''(\psi) = -1 < 0$, a in the second integral — at the beginning of the integration path. After application of theorems 2 and 3 [17], formula (34) was rewritten

$$J_{s,1}^{A,B}(r\cos\phi, r\sin\phi) = \frac{2AB}{\sqrt{\pi s r}}\left[\sin sr - \cos sr\right]\cdot S_s^{A,B}(\phi) + O\left(\frac{1}{r}\right)\; 0\le\phi\le\frac{\pi}{2}$$
$$S_s^{A,B}(\phi) = \frac{\sin\left[sB\sin\psi\right]}{sB\sin\psi}\cdot\frac{1-\cos\left[sA\cos\psi\right]}{sA\cos\psi} \quad (35)$$

Substitution of (35) into formulas (25) and (27) makes it possible to determine the displacement $W(x, y, 0;\omega)$ in the far field $r\to\infty$. As in the work [3; 8] only the Rayleigh term makes the main contribution to the asymptotic behavior of the displacement in the far field, the highest values are achieved with the angles $\phi = 0$ and $\phi = \frac{\pi}{2}$

$$J_{k_R,1}^{A,B}(r\cos\phi, r\sin\phi)\Big|_{\phi=0;\phi=\frac{\pi}{2}} = \sqrt{\frac{\pi}{2k_R}}\left(\sin k_R r - \cos k_R r\right)\times$$
$$\times\left[-\frac{\cos k_R A}{k_R A};\; \frac{\sin k_R B}{k_R B}\right] + O\left(\frac{1}{r}\right) \quad (36)$$

2.8. Discussion and numerical results

For numerical implementation, the displacement should be multiplied by $e^{i\omega t}$ and the real or imaginary part should be separated. The graphs are given for the function $\frac{G}{\rho}\operatorname{Im} W^{AB}(x, y, 0; \omega)$ from (32) for values of Poisson's ratio $\mu = \frac{1}{3}$ and $\mu = \frac{1}{4}$ for frequencies $\omega = 0.3;\ 1;\ 3$. For large values of frequencies formula (33) was used. Three forms of the load distribution section across the face $z = 0$ were considered

1. $B = A/2$ - the load is distributed over a square;

2. $B = A$ - the load is distributed over a rectangle extended along the Oy axis;

3. $B = A/4$ - the load is distributed over a rectangle extended along the Ox axis.

To analyze the far-field $r \to \infty$, the asymptotic equalities (35), (36) were used, substituted into the expressions for the displacement (25), (27)

Comparing the graphs of vertical displacements for the same frequency $\omega = 0.3$ and Poisson's ratio $\mu = 1/3$ under different sections of the load distribution (Fig. 1, Fig. 2, Fig. 3), it can be seen that the maximum absolute values equal to 2.5 achieved with the shape of the section $B = A$, which corresponds to a rectangle elongated along the y-axis. At the same time, the displacement has a maximum amplitude which is approximately 2 units. In the case when the load is distributed over a rectangle elongated along the x-axis, the displacement has a minimum amplitude 0.6 and its maximum displacement is about 0.7 units.

In the case when the load is distributed over the square $B = A/2$, with an increase in the vibration frequency (Fig. 1, Fig. 4, Fig. 7), the amplitude of displacement grows. In addition, in the case when the oscillation frequency is equal to 3, negative displacements are observed, which means the lifting of the face of the quarter space. Also growing of the amplitude with increasing frequency can be seen from Fig. 2 and Fig. 5, which corresponds to the case $B = A$, where the amplitude increased from 2 units ($\omega = 0.3$) to 4 units ($\omega = 1$). There is also the effect of raising the edge of a quarter space due to the presence of negative amplitudes' zones (Fig. 5).

Comparing the value of vertical displacements for different values of Poisson's ratio (Fig. 5, 6), it can be seen that the behavior of the graphs is similar, but for values of $\mu = 1/3$ the amplitude of oscillations is greater.

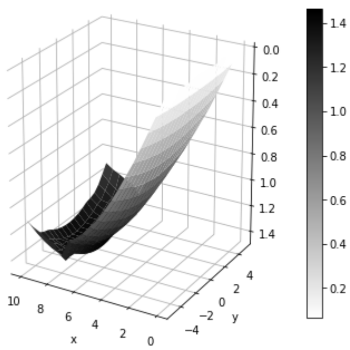

Figure 1: $B = A/2,\ \omega = 0.3,\ \mu = 1/3$

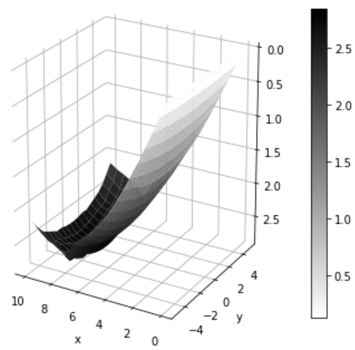

Figure 2: $B = A,\ \omega = 0.3,\ \mu = 1/3$

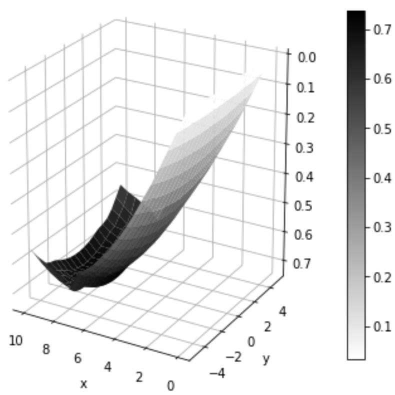

Figure 3: $B = A/4,\ \omega = 0.3,\ \mu = 1/3$

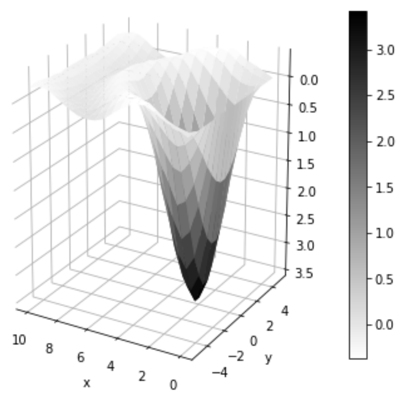

Figure 4: $B = A/2,\ \omega = 1,\ \mu = 1/3$

The vertical displacements' graphs in the remote zone of the load application area, depending on the vibration frequency with Poisson's ratio equal to 1/3 and load section $B = A$, represented in the Figure 8. As the distance from the load distribution section increases, the oscillations decay. Similar to the results for the near load zone, the maximum displacements occur in the case of the load section' shape B = A. The amplitude is greater for large values of vibration frequencies. With a decrease in the frequency of oscillations, the amplitudes are practically equal to zero.

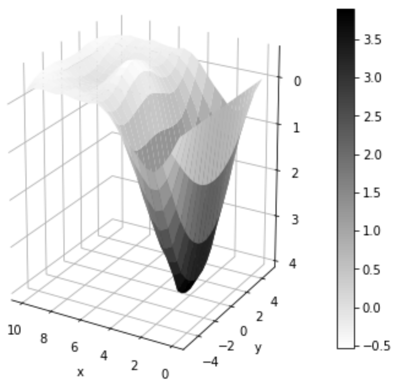

Figure 5: $B = A,\ \omega = 1,\ \mu = 1/3$

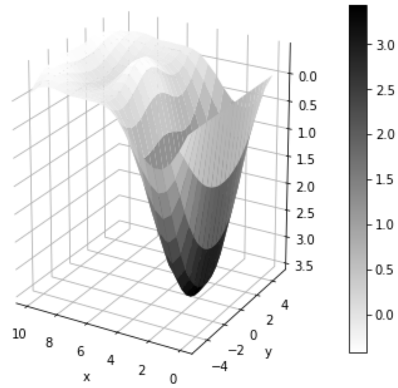

Figure 6: $B = A,\ \omega = 1,\ \mu = 1/4$

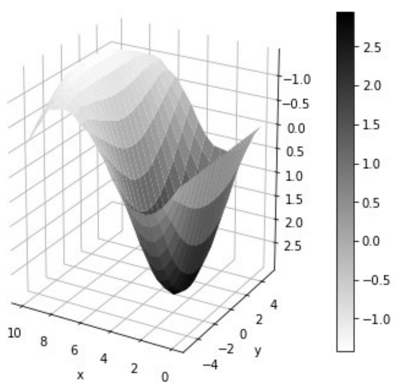

Figure 7: $B = A/2,\ \omega = 3,\ \mu = 1/3$

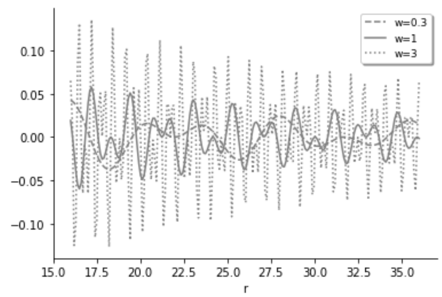

Figure 8: $B = A,\ \mu = 1/3$.

3. Conclusion

The dynamical problem's solution of the elasticity for the quarter space was derived, when one the faces is rigidly fixed and another is under the influence of the normal dynamic compressive load, applied at the initial moment of time and distributed across a rectangular section. Application of the integral transform method directly to the movement equations reduced the initial problem to the one-dimensional vector problem. The last one was solved exactly using the matrix differential calculus. The proposed approach makes it possible to obtain an exact solution of the problem in the transform's space. The case of steady state oscillations was investigated and vertical amplitude was analyzed in near loading and remote zone, for which asymptotic formulas were derived.

At the same time, it is also possible to construct and study the normal stress arising in a quarter space and compare the amplitudes of all three displacements. Using the proposed approach, the similar dynamical problem for the elastic semi-infinite layer, when different boundary conditions are set on the bottom face is under consideration.

Фесенко Г. О., Бондаренко К. С.

Динамічна задача про дію зосередженого навантаження на пружний чверть простір

Резюме

Побудовано хвильове поле пружного чверть простору, коли одну границю жорстко закріплено, а на іншій по прямокутній ділянці діє нестаціонарне нормальне стискаюче навантаження в початковий момент часу. Інтегральні перетворення Лапласа та Фур'є застосовано послідовно до рівнянь руху та до граничних умов, на відміну від традиційних підходів, коли інтегральні перетворення застосовуються до подання розв'язків через гармонічні функції. Це приводить до одновимірної векторної однорідної крайової задачі відносно невідомих трансформант переміщень. Задачу розв'язано за допомогою матричного диференціального числення. Поле вихідних переміщень знайдено після застосування обернених інтегральних перетворень. Для випадку стаціонарних коливань вказано спосіб обчислення у розв'язку квадратур в ближній зоні навантаження. Для аналізу коливань у віддаленій зоні побудовано асимптотичні формули. Досліджено амплітуду вертикальних коливань в залежності від форми ділянки навантаження, власних частот коливань та матеріалу середовища.
Ключові слова: точний розв'язок, пружний чвертьпростір, динамічне навантаження, інтегральні перетворення.

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Динамическая задача о действии сосредоточенной нагрузки на упругое четверть пространства

Резюме

Построено волновое поле упругого четверть пространства, когда одна грань жестко закреплена, а на другой по прямоугольному участку действует нестационарная нормальная сжимающая нагрузка в начальный момент времени. Интегральные преобразования Лапласа и Фурье применены последовательно к уравнениям движения и граничным условиям, в отличие от традиционных подходов, когда интегральные преобразования применяются к представлениям решений через гармонические функции. Это приводит

к одномерной векторной однородной краевой задаче относительно неизвестных трансформант перемещений. Задача решена с помощью матричного дифференциального исчисления. Поле исходных перемещений найдено после применения обратных интегральных преобразований. Для случая стационарных колебаний указан способ вычисления в решении квадратур в ближней зоне нагружения. Для анализа колебаний в отдаленной зоне построены асимптотические формулы. Исследована амплитуда вертикальных колебаний в зависимости от формы участка нагрузки, собственных частот колебаний и материала среды.
Ключевые слова: точное решение, упругое чтвертьпространство, динамическая нагрузка, интегральные преобразования.

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