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IMPULSIVE QUASIDIFFERENTIAL EQUATIONS IN SEMILINEAR METRIC SPACES

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Розглядаються питання неперервної залежності розв'язків імпульсних квазідиференціальних рівнянь і метод усереднення для цих рівнянь в полулінійних метричних просторах.

Рассматриваются вопросы непрерывной зависимости решений импульсных квазидифференциальных уравнений и метод усреднения для этих уравнений в полулинейных метрических пространствах.

In this paper we study some problems about continuous dependence of the solutions of impulsive quasidifferential equations and the method of averaging for these equations in semilinear metric spaces.

1. Introduction. The investigation of different problems, in which the solutions are multivalued trajectories, led to the creating of the theory of quasidifferential equations in metric spaces [1].

In the quasidifferential equations is not used the operation differentiation, and that permit us not to require linearity of the space of solutions, and from general point of view to consider differential equations in linear metric spaces, equations with multivalued solutions and dynamic systems in nonlinear metric spaces [1–5].

In this way, we can say that quasidifferential equations define a dynamic system in a metric space.

Let X be a semilinear metric space, i.e. it has a binary operation +, for which X is Abel semigroup, and multiplying operation by a positive number, moreover, $1\cdot x = x$; $\lambda x + \mu x = (\lambda + \mu)x$; $\lambda(x+y) = \lambda x + \lambda y$, $(\lambda\mu)x = \lambda(\mu x)$, where $\lambda,\ \mu > 0, x, y \in X$; the distance function $\delta(x,y)$ is invariant over shifting and have property of homogeneity, i.e. $\delta(x+z, y+z) = \delta(x,y)$; $\delta(\lambda x, \lambda y) = |\lambda|\delta(x,y) = \lambda\delta(x,y)$. Let us denote $\|x\| = \delta(x,\theta)$.

Definition 1.1 A quasidifferential equation in semilinear metric space is called the equation

$$\delta(x(t+h), x(t) + \int_t^{t+h} F(s, x(t))ds) = h\alpha(h), \quad x(t_0) = x_0, \tag{1.1}$$

where $x:[t_0, t_0+T] \to X$, $F:[t_0, t_0+T]\times X \to X$, $\alpha(h)$ is an infinitesimal function when $h \to 0$.

A solution of equation (1.1) is called a absolutely continuous map $x:[t_0, t_0+T] \to X$, which satisfies (1.1) for $t \in [t_0, T)$.

2. Continuous Dependence of the Solutions of Impulsive Quasidifferential Equations. In the domain $Q = \{h \in [0,\tau), t \in [t_0, t_0+T], P \subset X\}$ we consider the following impulsive quasidifferential equation:

$$\delta(x(t+h), x(t) + \int_t^{t+h} F(s, x(t))ds) = h\alpha(h) \quad t \neq \tau_i(x), \tag{2.1}$$

$$x(\tau_i + 0) = \psi_i(x(\tau_i)), \tag{2.2}$$

where $\psi_i : X \to X, x(\tau_i) = x(\tau_i - 0)$.

Theorem 2.1 *Let X be a semilinear metric space and in the domain Q are fulfilled the following conditions:*

1) *let the solution $x(t, y_0), \delta(x_0, y_0) \leq \delta_0$ of quasidifferential equation* (1) *exist, for $t \in [t_0, t_0 + T)$ and belongs with some its ρ-neighborhood to the domain P, and in addition $\tau_i^{y_0}$ denotes time when this solution intersects surface $t = \tau_i(x)$;*

2) *the map $F(t,x)$ is continuous and satisfies Lipschitz condition in x with constant γ, and $\|F(t,x)\| \leq \lambda$;*

3) *the maps $\psi_i(x)$ and $\tau_i(x)$ satisfy Lipschitz condition in x with constants ν and μ respectively;*

4) *for every $x \in P$ is correct the inequality*

$$\tau_i(x) \geq \tau_i(\psi_i(x)). \tag{2.3}$$

Then when $\mu\lambda < 1$ every solution of quasidifferential equation intersects every hypersurface $t = \tau_i(x)$ just once and

$$\delta(x(t,t_0,x_0), x(t,t_0,y_0)) \leq \left(\frac{\gamma\mu + \nu}{1 - \lambda\mu}\right)^p e^{\gamma T} \delta(x_0, y_0) \tag{2.4}$$

for $t \in [\tau_i^+, \tau_{i+1}^-]$, where $\tau_i^- = \min\{\tau_i^{x_0}, \tau_i^{y_0}\}$, $\tau_i^+ = \max\{\tau_i^{x_0}, \tau_i^{y_0}\}$, p is the number of impulses on the interval $[t_0, t_0 + T)$, that means that the solution $x(t,t_0,x_0)$ depends continuously on the initial conditions in the following sense: for every $\eta > 0$ there exists such a $\delta_0 = \delta(\eta) > 0$, that for any other solution $x(t,t_0,y_0)$ of the equation (1), *which satisfies the inequality $\delta(x_0, y_0) \leq \delta_0$, we have*

$$\delta(x(t,t_0,x_0), x(t,t_0,y_0)) < \eta$$

for all $t \in [t_0, t_0 + T)$, which satisfy inequalities $\left|t - \tau_i^{x_0}\right| > \eta$.

Proof. Let us denote

$$\delta_i^- = \delta(x(\tau_i^-, t_0, x_0), x(\tau_i^-, t_0, y_0)),\ \delta_i^+ = \delta(x(\tau_i^+ + 0, t_0, x_0), x(\tau_i^+ + 0, t_0, y_0)).$$

Let us divide the interval $[\tau_i^+, \tau_{i+1}^-]$ into m parts. In the moment

$$t \in [t_k, t_{k+1}] \subset [\tau_i^+, \tau_{i+1}^-],\ t_k = \tau_i^+ + k\Delta,\ \Delta = [\tau_{i+1}^- - \tau_i^+]/m,\ k = 0,1,\dots,m-1,$$

the error estimation due to the initial conditions is:

$$\begin{aligned}
&\delta(x(t,t_0,x_0), x(t,t_0,y_0)) \leq \delta(x(t_k,t_0,x_0) + \int_{t_k}^{t} F(s, x(t_k,t_0,x_0))ds, x(t_k,t_0,y_0) + \\
&+ \int_{t_k}^{t} F(s, x(t_k,t_0,y_0))ds) + o(\Delta) \leq (1 + \Delta\gamma)\delta(x(t_k,t_0,x_0), x(t_k,t_0,y_0)) + o(\Delta) \leq \\
&\leq \left(1 + ((\tau_{i+1}^- - \tau_i^+)/m)\cdot\gamma\right)^m \delta_i^+ + ((1+\Delta\gamma)^{k+1} - 1)/(\Delta\gamma)\cdot o(\Delta) \leq \\
&\leq e^{\gamma(\tau_{i+1}^- - \tau_i^+)}\delta_i^+ + o(\Delta)/\Delta \cdot e^{\gamma(\tau_{i+1}^- - \tau_i^+)}/\gamma.
\end{aligned} \tag{2.5}$$

From (2.5) when $\Delta \to 0$ we obtain

$$\delta_{i+1}^- \leq e^{\gamma(\tau_{i+1}^- - \tau_i^+)}\delta_i^+. \tag{2.6}$$

We will proof that every solution $x(t,t_0,x_0)$ of the equation (2.1), (2.2), belonging to domain P in $t_0 < t \leq t_0 + T$, intersects every hypersurface $t = \tau_i(x)$ in the interval $[t_0, t_0 + T)$ just once.

Suppose that the opposite is true. Let for some solution, which go out of the point $\psi_i(x_0)$ when $t=\tau_i(x_0)+0$, intersects the surface $t=\tau_i(x)$ in some point (t_i^*,x^*), $t_i^*=\tau_i(x^*)$. It is obvious, that $t^*>\tau_i(x_0)$ and the interval $\tau_i(x_0)<t<t_i^*$ is appear as an interval of continuity of the solution $x(t,\tau_i(x_0)+0,\psi_i(x_0))$ and therefore

$$t_i^*-\tau_i(x_0)=\tau_i(x^*)-\tau_i(x_0)=\tau_i(x^*)-\tau_i(\psi_i(x_0))+$$
$$+\tau_i(\psi_i(x_0))-\tau_i(x_0)\le\mu\delta(x^*,\psi_i(x_0))+\tau_i(\psi_i(x_0))-\tau_i(x_0)\le$$
$$\le\mu\lambda(t_i^*-\tau_i(x_0))+\tau_i(\psi_i(x_0))-\tau_i(x_0). \tag{2.7}$$

From (2.7) we obtain

$$(1-\mu\lambda)(t_i^*-\tau_i(x_0)\le\tau_i(\psi_i(x_0))-\tau_i(x_0). \tag{2.8}$$

Taking into account that $(1-\mu\lambda)>0$ and $t_i^*-\tau_i(x_0)>0$ we obtain that (2.8) contradict to (2.3).

Suppose that $\tau_i^-=\tau_i^{y_0}$, $\tau_i^+=\tau_i^{x_0}$. Then

$$x(\tau_i^++0,t_0,x_0)=\psi_i(x(\tau_i^++0,t_0,x_0)),\ x(\tau_i^++0,t_0,y_0)=x(\tau_i^+,\tau_i^-,\psi_i(x(\tau_i^-,t_0,y_0))),$$
$$\delta(\psi_i(x(\tau_i^-,t_0,x_0)),\psi_i(x(\tau_i^+,t_0,x_0))\le\nu\lambda\left|\tau_i^+-\tau_i^-\right|,$$
$$\delta(x(\tau_i^+,\tau_i^-,\psi_i(x(\tau_i^-,t_0,y_0)),\psi_i(x(\tau_i^-,t_0,y_0)))\le\lambda\left|\tau_i^+-\tau_i^-\right|,$$
$$\delta(\psi_i(x(\tau_i^-,t_0,y_0)),\psi_i(x(\tau_i^-,t_0,x_0)))\le\nu\delta_i^-,$$
$$\delta_i^+=\delta(\psi_i(x(\tau_i^+,t_0,x_0)),x(\tau_i^+,\tau_i^-,\psi_i(x(\tau_i^-,t_0,y_0))\le$$
$$\le\lambda\nu\left|\tau_i^+-\tau_i^-\right|+\nu\delta_i^-+\lambda\left|\tau_i^+-\tau_i^-\right|. \tag{2.9}$$

Let $z_0=x_0$, if $\tau_i^-=\tau_i^{y_0}$, and $z_0=y_0$, if $\tau_i^-=\tau_i^{x_0}$ then

$$\left|\tau_i^{x_0}-\tau_i^{y_0}\right|=\left|\tau_i(x(\tau_i^{x_0},t_0,x_0))-\tau_i(x(\tau_i^{y_0},t_0,y_0))\right|\le$$
$$\le\mu\delta(x(\tau_i^-,t_0,x_0),x(\tau_i^-,t_0,y_0))+\mu\delta(x(\tau_i^{x_0},t_0,z_0),x(\tau_i^{y_0},t_0,z_0))\le$$
$$\le\mu\delta(x(\tau_i^-,t_0,x_0),x(\tau_i^-,t_0,y_0))+\mu\lambda\left|\tau_i^{x_0}-\tau_i^{y_0}\right|. \tag{2.10}$$

From (2.10) we have

$$\left|\tau_i^{x_0}-\tau_i^{y_0}\right|\le\mu\delta_i^-/(1-\lambda\mu). \tag{2.11}$$

If we substitute (2.11) in (2.9), we obtain

$$\delta_i^+\le(\lambda\mu+\nu)/(1-\lambda\mu)\cdot\delta_i^-. \tag{2.12}$$

It is obviously that (2.12) is true also for $\tau_i^-=\tau_i^{x_0}$, $\tau_i^+=\tau_i^{y_0}$. From Lemma 1 of [5] follows that

$$\delta_1^-\le e^{\gamma(\tau_1-t_0)}\delta_0,$$
$$\delta_{i+1}^-\le e^{\gamma(\tau_{i+1}^--\tau_i^+)}\delta_i^+,$$
$$\delta(x(t_0+T,t_0,x_0),x(t_0+T,t_0,y_0))\le\delta_p^+e^{\gamma(t_0+T-\tau_p)}. \tag{2.13}$$

From (2.12), (2.13) we obtain that (2.4) is correct.

3. Method of Averaging for Impulsive Quasidifferential Equations. Now let us consider a quasidifferential equation depending on the parameter ε

$$\delta(x(t+h),x(t)+\varepsilon\int_t^{t+h}F(s,x(t))ds)=\varepsilon h\alpha(\varepsilon,h), \tag{3.1}$$

where the map $F(t,x)$ is 2π periodic in t, and $\alpha(\varepsilon,h)$ is infinitesimal function, when $h\to 0$.

The quasidifferential equation (3.1) is assigned to the following quasidifferential equation

$$\delta(y(t+h),y(t)+\varepsilon\int_t^{t+h}F_1(s,y(t))ds)=\varepsilon h\alpha(\varepsilon,h), \tag{3.2}$$

where $F_1(t,y)$ is 2π periodic in t,

$$\int_0^{2\pi}F(t,x)dt=\int_0^{2\pi}F_1(t,x)dt. \tag{3.3}$$

Theorem 3.1 *Let X be a semilinear local compact or a complete metric space, and in the domain Q are fulfilled the following conditions:*

1) *the maps $F(t,x)$ and $F_1(t,x)$ are continuous, satisfy Lipschitz condition in x with constant γ, and for arbitrary constant λ are valid the inequalities $\|F(t,x)\|\le\lambda$, $\|F_1(t,x)\|\le\lambda$;*

2) *the solutions of the quasidifferential equations* (3.1) *and* (3.2) *exist for $t\in[0,L\varepsilon^{-1}]$, $\varepsilon\in(0,\varepsilon_1]$, and together with their ρ-neighborhoods belong to the domain P,L and ε_1 are arbitrary constants.*

Then for every $\eta\in(0,\rho]$ exists $\varepsilon_0(\eta)\le\varepsilon_1$ such that when $\varepsilon\in(0,\varepsilon_0]$ and $t\in[0,L\varepsilon^{-1}]$ is valid the inequality

$$\delta(x(t,\varepsilon,x_0),y(t,\varepsilon,x_0))\le\eta \tag{3.4}$$

where $x(t,\varepsilon,x_0)$, $y(t,\varepsilon,x_0)$ are respectively solutions of the quasidifferential equations.

Proof. Let us divide the interval $[0,L\varepsilon^{-1}]$on parts with points $2k\pi, k=0,1,\ldots$.We construct generalized Euler's polygons

$$x^m(t,\varepsilon,x_0)=x^m(t_k,\varepsilon,x_0)+\varepsilon\int_{t_k}^{t}F(s,x^m(t_k,\varepsilon,x_0))ds,$$

$$y^m(t,\varepsilon,x_0)=y^m(t_k,\varepsilon,x_0)+\varepsilon\int_{t_k}^{t}F_1(s,y^m(t_k,\varepsilon,x_0))ds,\quad t\in[t_k,t_{k+1}].$$

Let us estimate

$$\begin{aligned}&\delta(x^m(t_k,\varepsilon,x_0),x(t_k,\varepsilon,x_0))\le\\ &\le(1+\varepsilon h\gamma)\delta(x^m(t_{k-1},\varepsilon,x_0),x(t_{k-1},\varepsilon,x_0))+\varepsilon h\alpha(\varepsilon,h)\le\ldots\le\\ &\le\frac{\alpha(\varepsilon,h)}{\gamma}[(1+\varepsilon h\gamma)^{k+1}-1]\le\frac{\alpha(\varepsilon,h)}{\gamma}(e^{\gamma L}-1).\end{aligned} \tag{3.5}$$

By analogy we obtain

$$\delta(y^m(t_k,\varepsilon,x_0),y(t_k,\varepsilon,x_0))\le\frac{\alpha(\varepsilon,h)}{\gamma}(e^{\gamma L}-1). \tag{3.6}$$

We consider

$$\delta(x^m(t,\varepsilon,x_0),x^m(t_k,\varepsilon,x_0))\le\delta(x^m(t_k,\varepsilon,x_0)+\varepsilon\int_{t_k}^{t}F(s,x^m(t_k,\varepsilon,x_0))ds,x^m(t_k,\varepsilon,x_0))=$$

$$=\left\|\varepsilon\int_{t_k}^{t}F(s,x^m(t_k,\varepsilon,x_0))ds\right\|\le\varepsilon\lambda(t-t_k)\le\varepsilon 2\lambda\pi,$$

$$\delta(y^m(t,\varepsilon,x_0),y^m(t_k,\varepsilon,x_0))\le\varepsilon 2\lambda\pi,\quad \delta(y(t,\varepsilon,x_0),y(t_k,\varepsilon,x_0))\le\varepsilon 2\lambda\pi,$$

$$\delta(x(t,\varepsilon,x_0),x(t_k,\varepsilon,x_0))\le\varepsilon 2\lambda\pi. \tag{3.7}$$

From (3.5)-(3.7) follows, that for every $\eta>0$ exists $\varepsilon_0(\eta)$ such that when $\varepsilon\le\varepsilon_0(\eta)$ we have

$$\delta(y(t,\varepsilon,x_0),y^m(t,\varepsilon,x_0))\le\eta/2,\quad \delta(x(t,\varepsilon,x_0),x^m(t,\varepsilon,x_0))\le\eta/2. \tag{3.8}$$

Now let us estimate

$$\delta(x^m(t_{k+1},\varepsilon,x_0),y^m(t_{k+1},\varepsilon,x_0))\le$$

$$\le\delta(x^m(t_k,\varepsilon,x_0)+\varepsilon\int_{t_k}^{t_{k+1}}F(s,x^m(t_k,\varepsilon,x_0))ds,y^m(t_k,\varepsilon,x_0)+\varepsilon\int_{t_k}^{t_{k+1}}F_1(s,y^m(t_k,\varepsilon,x_0))ds)\le$$

$$\le(1+\varepsilon\gamma h)\delta(x^m(t_k,\varepsilon,x_0),y^m(t_k,\varepsilon,x_0)).$$

Since $\delta(x^m(0,\varepsilon,x_0),y^m(0,\varepsilon,x_0))=0$ we have

$$\delta(x^m(t_k,\varepsilon,x_0),y^m(t_k,\varepsilon,x_0))=0. \tag{3.9}$$

From (3.8) and (3.9) follows the statement of the theorem.

Now let us consider a impulsive quasidifferential equation depending on the parameter ε

$$\delta(x(t+h),x(t)+\varepsilon\int_t^{t+h}F(s,x(s))ds)=\varepsilon h\alpha(\varepsilon,h),\quad t\ne\tau_i(x), \tag{3.10}$$

$$x(\tau_i+0)=x(\tau_i)+\varepsilon I_i(x(\tau_i)),\quad t=\tau_i(x), \tag{3.11}$$

where the map $F(t,x)$ is 2π periodic in t, and there is an integer p, such that

$$\tau_{i+p}(x)=\tau_i(x)+2\pi.$$

The quasidifferential equation (3.10),(3.11) is assigned to the following quasidifferential equation

$$\delta(y(t+h),y(t)+\varepsilon\int_t^{t+h}F_1(s,y(s))ds)=\varepsilon h\alpha(\varepsilon,h),\quad t\ne s_j(x), \tag{3.12}$$

$$y(s_j+0)=x(s_j)+\varepsilon K_j(y(s_j)),\quad t=s_j(x), \tag{3.13}$$

where the map $F_1(t,y)$ is 2π periodic in t, there is an integer l, such that

$$s_{j+l}(y)=s_j(y)+2\pi,$$

$$\int_0^{2\pi}F(t,x)dt+\sum_{0\le\tau_i(x)<2\pi}I_i(x)=\int_0^{2\pi}F_1(t,x)dt+\sum_{0\le s_j(x)<2\pi}K_j(x). \tag{3.14}$$

Theorem 3.2 *Let X be a semilinear local compact or a complete metric space, and in the domain $Q_1=\{\varepsilon\in(0,\varepsilon_1],h\in[0,r],t\in[0,L\varepsilon^{-1}],x\in P\subset X\}$ the following conditions are fulfilled:*

1) *the maps $F(t,x)$ and $F_1(t,x)$ are continuous and satisfy Lipschitz conditions in x with constant γ, and $\|F(t,x)\|\le\lambda$, $\|F_1(t,x)\|\le\lambda$;*

2) *the maps $I_i(x)$, $K_j(x)$, $\tau_i(x)$, $s_j(x)$ are continuous and satisfy Lipschitz conditions with constant ν, and $\|I_i(x)\|\le\lambda$, $\|K_j(x)\|\le\lambda$;*

3) *the solutions of quasidifferential equations* (3.10),(3.11) *and* (3.12), (3.13) *exist when $\varepsilon\in(0,\varepsilon_1]$, $t\in[0,L\varepsilon^{-1}]$ and together with their ρ-neighborhoods belong to the domain P.*

Then for every $\eta \in (0,\rho]$ *exists* $\varepsilon_0(\eta) \le \varepsilon_1$ *such that when* $\varepsilon \in (0,\varepsilon_0]$ *and* $t \in [0, L\varepsilon^{-1}]$ *for* $\left|t-\tau_i^{x_0}\right| > \eta$ *is valid the inequality*

$$\delta(x(t), y(t)) \le \eta, \tag{3.15}$$

where $x(t)$ *and* $y(t)$ *are respectively solutions of quasidifferential equations* (3.10), (3.11) *and* (3.12), (3.13) *with the initial conditions* $x(0) = y(0) = x_0$.

Proof. Let us divide the interval $[0, L\varepsilon^{-1}]$ on parts with points $2k\pi$, $k = 0,1,\dots$. We construct generalized Euler's polygons

$$x^m(t) = x^m(t_k) + \varepsilon\int_{t_k}^{t} F(s, x^m(t_k))ds + \varepsilon \sum_{t_k \le \tau_i(x^m(t_k)) < t} I_i(x^m(t_k)),$$

$$y^m(t) = y^m(t_k) + \varepsilon\int_{t_k}^{t} F_1(s, y^m(t_k))ds + \varepsilon \sum_{t_k \le s_j(y^m(t_k)) < t} K_j(y^m(t_k)), \quad t \in [t_k, t_{k+1}].$$

We estimate

$$\delta(x^m(t_k), x(t_k)) \le \delta(x^m(t_{k-1}) + \varepsilon\int_{t_{k-1}}^{t_k} F(s, x^m(t_{k-1}))ds + \varepsilon \sum_{t_{k-1} \le \tau_i(x^m(t_{k-1})) < t_k} I_i(x^m(t_{k-1})),$$

$$x(t_{k-1}) + \varepsilon\int_{t_{k-1}}^{t_k} F(s, x(t_{k-1}))ds + \varepsilon \sum_{t_{k-1} \le \tau_i(x(t_{k-1})) < t_k} I_i(x(t_{k-1}))) \le$$

$$\le (1 + \varepsilon h\gamma + \varepsilon \nu p)\delta(x^m(t_{k-1}), x(t_{k-1})) + \varepsilon h\alpha(\varepsilon h) \le \dots \le \frac{\alpha(\varepsilon 2\pi)}{\gamma + \frac{\nu p}{2\pi}} e^{L(\gamma + \frac{\nu p}{2\pi})}. \tag{3.16}$$

By analogy we have

$$\delta(y^m(t_k), y(t_k)) \le \frac{\alpha(\varepsilon 2\pi)}{\gamma + \frac{\nu p}{2\pi}} e^{L(\gamma + \frac{\nu p}{2\pi})}. \tag{3.17}$$

We consider

$$\delta(x^m(t), x^m(t_k)) \le \delta(x^m(t_k) + \varepsilon\int_{t_k}^{t} F(s, x^m(t_k))ds + \varepsilon \sum_{t_k \le \tau_i(x^m(t_k)) < t} I_i(x^m(t_k)), x^m(t_k)) =$$

$$= \left\| \varepsilon\int_{t_k}^{t} F(s, x^m(t_k))ds + \varepsilon \sum_{t_k \le \tau_i(x^m(t_k)) < t} I_i(x^m(t_k)) \right\| \le \varepsilon\lambda 2\pi + \varepsilon p\lambda, \tag{3.18}$$

$\delta(y^m(t), y^m(t_k)) \le \varepsilon\lambda(2\pi + p)$, $\delta(y(t), y(t_k)) \le \varepsilon\lambda(2\pi + p)$, $\delta(x(t), x(t_k)) \le \varepsilon\lambda(2\pi + p)$.

Then from (3.16)–(3.18), for each $\eta > 0$ there exist $\varepsilon_0(\eta)$, such that when $\varepsilon \le \varepsilon_0$ the following inequalities are correct

$$\delta(x(t), x^m(t_k)) \le \eta/2, \quad \delta(y(t), y^m(t_k)) \le \eta/2. \tag{3.19}$$

Now let us estimate $\delta(x^m(t_{k+1}), y^m(t_{k+1})) \le (1 + \varepsilon h\gamma + \varepsilon\nu p)\delta(x^m(t_k), y^m(t_k))$. Since $\delta(x^m(0), y^m(0)) = 0$, we have

$$\delta(x^m(t_k), y^m(t_k)) = 0. \tag{3.20}$$

From (3.19) and (3.20) follows validity of the theorem.

Now let us consider the following quasidifferential equation with impulses

$$\delta(x(t+h), x(t) + \varepsilon\int_t^{t+h} F(s, x(t))ds) = \varepsilon h\alpha(\varepsilon, h), \quad t \ne \tau_i(x), \quad t \ne \beta_l^1(x)/\varepsilon, \tag{3.21}$$

$$x(\tau_i + 0) = x(\tau_i) + \varepsilon I_i(x(\tau_i)), \tag{3.22}$$

$$x((\beta_l^1 + 0)/\varepsilon) = x(\beta_l^1/\varepsilon) + M_l(x(\beta_l^1/\varepsilon)). \tag{3.23}$$

The quasidifferential equation (3.21)–(3.23) is assigned to the following quasidifferential equation

$$\delta(y(t+h), y(t)+\varepsilon\int_t^{t+h} F(s, y(t))ds) = \varepsilon h\alpha(\varepsilon,h), \quad t \neq s_j(y), \quad t \neq \beta_l^2(y)/\varepsilon, \tag{3.24}$$

$$y(s_j+0) = y(s_j)+\varepsilon K_j(y(s_j)), \tag{3.25}$$

$$y((\beta_l^2+0)/\varepsilon) = y(\beta_l^2/\varepsilon)+N_l(x(\beta_l^2/\varepsilon)). \tag{3.26}$$

Theorem 3.3 *Let X be a semilinear local compact or a complete metric space, and in the domain Q are fulfilled the following conditions:*

1) *let conditions 1) and 2) from theorem 3.2 hold;*

2) *the maps $M_l(x)$, $N_l(x)$ and, $\beta_l^1(x)$, $\beta_l^2(x)$ satisfy Lipschitz condition with constant ν, $\|M_l(x)\| \leq \lambda$, $\|N_l(x)\| \leq \lambda$;*

3) *the solutions of quasidifferential equations* (3.21)–(3.23) *and* (3.24)–(3.26) *exist when $t \in [0, L\varepsilon^{-1}]$, and together with their ρ-neighborhoods belong to the domain P.*

Then for every $\eta \in (0,\rho]$ exist $\varepsilon^0(\eta) > 0$, $\delta^0(\eta) > 0$ such that when $\varepsilon \in (0,\varepsilon^0]$, $\delta \in (0,\delta^0]$ and $t \in [0, L\varepsilon^{-1}] \cap \{|t-\beta_i/\varepsilon| > \eta/\varepsilon\} \cap \{|t-\tau_i| > \eta\}$ is valid the inequality

$$\delta(x(t,\varepsilon,x_0), y(t,\varepsilon,x_0)) \leq \eta,$$

where $x(t,\varepsilon,x_0)$, $y(t,\varepsilon,x_0)$ are respectively solutions of the quasidifferential equations (3.21)–(3.23) *and* (3.24)–(3.26) *with the initial conditions $x(0,\varepsilon,x_0) = y(0,\varepsilon,x_0) = x_0$.*

Validity of this theorem follows from the theorems in [5] and from the theorem 3.2.

Remark 3.1. Let $\int_0^{2\pi} F(t,x)dt = 2\pi F_0(x)$, $\sum_{0\leq\tau_i(x)<2\pi} I_i(x) = 2\pi I_0(x)$, then according to (3.14) we have

$$\delta(y(t+h), y(t)+\varepsilon 2\pi[F_0(y(t))+I_0(y(t))] = \varepsilon h\alpha(\varepsilon,h), \tag{3.27}$$

or

$$\delta(y(t+h), y(t)+\varepsilon 2\pi F_0(y(t)) = \varepsilon h\alpha(\varepsilon,h), \quad t \neq 2\pi i, \tag{3.28}$$

$$y(2\pi i+0) = y(2\pi i)+\varepsilon I_0(y(2\pi i)).$$

We can show, that for the impulsive quasidifferential equation (3.21)–(3.23) there is a corresponding quasidifferential equation without the impulses (3.27), quasidifferential equation with impulses at $2\pi i$ (3.28) and the discrete equation

$$y(2\pi i) = y(2\pi(i-1))+\varepsilon(I_0(y(2\pi(i-1)))+F_0(y(2\pi(i-1))))\cdot 2\pi.$$

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