A WAVE FIELD OF A SEMI-STRIP UNDER A NONSTATIONARY LOAD

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The plane problems of elasticity for a semi-strip in a static statement were investigated by many authors. However many unresolved issues remain especially for a dynamic statement of the problem. As for the static statements, for example, the problem for a symmetrically loaded semi-strip fixed by its short edge was reduced to the Fredholm integral equation of the first kind in [1]. The static problem for an elastic semi-strip loaded by its short edge in three configurations was solved in [2]. The solving of the dynamic problems is usually done with the help of Laplace’s transformation. However, the inversion of this transformation is enough complicated, so some authors use a numerical inversion or an asymptotic analysis of the derived solution in the transformation’s domain. The Laplace’s transform was used for the stress state evaluation of an elastic half-strip under a nonstationary load applied to its boundary and the solution is expanded into a Fourier series in [3]. Dynamic stress in an infinite elastic strip, containing two circular cylindrical cavities, of equal radii, were explored under the assumption of plane strain in [4]. In the Laplace’s transform domain, boundary conditions at the plane surfaces and those at the circular cavity were satisfied with the Fourier transformation and the Schmidt method respectively. The application of an asymptotic method for the investigation of the non-stationary stress-deformable state under the impact at the semi-strip’s edge was studied in [5]. The analysis of the solution of the approximate asymptotic equations derived by the symbolic Lurie method and the exact solution in the Fourier-Laplace’s transform domain was conducted there.

In the proposed work the new approach for the solving of the dynamic problem for an elastic semi-strip is proposed. It is based on the apparatuses of matrix differential calculation and matrix Green function. The analytical solution is derived in Laplace’s transform domain. The case of steady-state oscillations is investigated.

The plane elastic semi-strip (G is a share module, μ is a Poisson ratio) occupying an area $0<x<a, 0<y<\infty$ is loaded by its short edge by a non-stationary load

$$\begin{aligned} &\sigma_y\big|_{y=0} = p(x,t), \qquad \tau_{xy}\big|_{y=0} = 0, \qquad 0<a_0<x<a_1<a,\ t>0, \\ &v\big|_{y=0} = 0, \qquad \tau_{xy}\big|_{y=0} = 0, \qquad 0<x<a_0, a_1<x<a,\ t>0. \end{aligned} \tag{1}$$

The lateral sides of the semi-strip are fixed

$$u(0,y,t)=0, \quad v(0,y,t)=0, \quad 0<y<\infty, t>0\,, \tag{2}$$

$$u(a,y,t)=0, \quad v(a,y,t)=0, \quad 0<y<\infty, t>0\,. \tag{3}$$

Here displacement’s functions are denoted as $u_x(x,y,t)=u(x,y,t),\ u_y(x,y,t)=v(x,y,t)$.
The motion equations have the following form

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\kappa-1}{\kappa+1}\dfrac{\partial^2 u}{\partial y^2}+\dfrac{2}{\kappa+1}\dfrac{\partial^2 v}{\partial x\partial y}-\dfrac{\rho}{G}\dfrac{\kappa-1}{\kappa+1}\dfrac{\partial^2 u}{\partial t^2}=0, \\ \dfrac{\partial^2 v}{\partial x^2}+\dfrac{\kappa+1}{\kappa-1}\dfrac{\partial^2 v}{\partial y^2}+\dfrac{2}{\kappa-1}\dfrac{\partial^2 u}{\partial x\partial y}-\dfrac{\rho}{G}\dfrac{\partial^2 v}{\partial t^2}=0, \end{cases} \tag{4}$$

where ρ is semi-strip's density, $\kappa=3-4\mu$ is Muskhelishvili's constant. It is supposed that initial conditions of this problem are null.

It is necessary to evaluate the wave field of the semi-strip, to derive the analytical formulas for the displacements and stresses, and investigate them depending on the strip's geometrical parameters, size of segment where the load is applied and load's behavior.

To reduce the stated boundary initial problem to one dimensional boundary value problem the Laplace's transform is applied to the correspondences (1)-(4)

$$\begin{cases} \dfrac{\partial^2 u_s}{\partial x^2}+\dfrac{\kappa-1}{\kappa+1}\dfrac{\partial^2 u_s}{\partial y^2}+\dfrac{2}{\kappa+1}\dfrac{\partial^2 v_s}{\partial x\partial y}-q^2\dfrac{\kappa-1}{\kappa+1}u_s=0, \\ \dfrac{\partial^2 v_s}{\partial x^2}+\dfrac{\kappa+1}{\kappa-1}\dfrac{\partial^2 v_s}{\partial y^2}+\dfrac{2}{\kappa-1}\dfrac{\partial^2 u_s}{\partial x\partial y}-q^2 v_s=0, \\ u_s(0,y)=0,\quad v_s(0,y)=0,\quad 0<y<\infty, \\ u_s(a,y)=0,\quad v_s(a,y)=0,\quad 0<y<\infty, \\ \mu\dfrac{\partial u_s}{\partial x}(x,0)+(1-\mu)\dfrac{\partial v_s}{\partial y}(x,0)=p_s(x), a_0<x<a_1, \\ v_s(x,0)=0,\quad 0<x<a_0, a_1<x<a, \\ \dfrac{\partial u_s}{\partial y}(x,0)+\dfrac{\partial v_s}{\partial x}(x,0)=0,\quad 0<x<a, \end{cases} \tag{5}$$

here $q^2=\rho/G\cdot s^2$, s is the parameter of Laplace's transform.

The semi-infinite sin-, cos- integral Fourier transformation is applied to the boundary problem (5) with respect to variable y by the scheme

$$\begin{bmatrix} u_{s\beta}(x) \\ v_{s\beta}(x) \end{bmatrix}=\int_0^\infty \begin{bmatrix} u_s(x,y) \\ v_s(x,y) \end{bmatrix}\begin{bmatrix} \cos\beta y \\ \sin\beta y \end{bmatrix}dy$$

The one-dimensional boundary problem in transformations' domain is formulated as the vector boundary problem [6]

$$\begin{cases} L_2\vec{y}_{s\beta}(x)=\vec{f}_s(x), \\ \vec{y}_{s\beta}(0)=0, \vec{y}_{s\beta}(a)=0 \end{cases} \tag{6}$$

where $L_2\vec{y}_{s\beta}(x) = I\vec{y}_{s\beta}''(x) + 2\beta Q\vec{y}_{s\beta}'(x) - P_{s\beta}\vec{y}_{s\beta}(x)$, I is a unit matrix, $Q = \begin{pmatrix} 0 & \frac{1}{\kappa+1} \\ -\frac{1}{\kappa-1} & 0 \end{pmatrix}$, $P_{s\beta} = \begin{pmatrix} (\beta^2+q^2)\frac{\kappa-1}{\kappa+1} & 0 \\ 0 & \beta^2\frac{\kappa+1}{\kappa-1}+q^2 \end{pmatrix}$, $\vec{f}_s(x) = \begin{pmatrix} \frac{3-\kappa}{\kappa+1}\chi_s'(x) \\ -\beta\frac{\kappa+1}{\kappa-1}\chi_s(x) \end{pmatrix}$, $\chi(x) = v|_{y=0}, \chi'(x) = v'|_{y=0}$, $\vec{y}_{s\beta}(x) = \begin{pmatrix} u_{s\beta}(x) \\ v_{s\beta}(x) \end{pmatrix}$.

The solution of inhomogeneous equation in the vector boundary problem (6) is constructed as the superposition

$$\vec{y}_{s\beta}(x) = \vec{y}^0_{s\beta}(x) + \vec{y}^1_{s\beta}(x) \tag{7}$$

here $\vec{y}^0_{s\beta}(x)$ is the general solution of the vector homogeneous equation (6) and $\vec{y}^1_{s\beta}(x)$ is the partial solution of the vector inhomogeneous equation.

The general solution is constructed with the help of matrix differential calculation. Accordingly to it the corresponding matrix equation is considered $L_2 Y_{s\beta}(x) = 0$. The matrix $Y_{s\beta}(x)$ is chosen in the form $Y_{s\beta}(x) = e^{\xi x} I$ and substituted into the matrix equation. As the result, the equality $L_2 e^{\xi x} I = M(\xi) e^{\xi x}$ is derived, where $M(\xi) = I\xi^2 + 2\beta Q\xi - P_{s\beta}$. The solution of the matrix homogeneous equation is constructed as the following $Y(x) = \frac{1}{2\pi i}\oint_C e^{\xi x} M^{-1}(\xi) d\xi$. The determinant of the matrix $M(\xi)$ has four different roots $\xi_{1,2} = \pm\sqrt{\beta^2+q^2}, \xi_{3,4} = \pm\sqrt{\beta^2+q^2\frac{\kappa-1}{\kappa+1}}$, and the system of fundamental matrix solutions has the following form

$$Y_{1,2}(x) = e^{\pm x\sqrt{\beta^2+q^2}} \begin{pmatrix} \mp\frac{\beta^2(\kappa+1)}{2q^2\sqrt{\beta^2+q^2}(\kappa-1)} & -\frac{\beta}{2q^2} \\ \frac{\beta(\kappa+1)}{2q^2(\kappa-1)} & \pm\frac{\sqrt{\beta^2+q^2}}{2q^2} \end{pmatrix},$$

$$Y_{3,4}(x) = e^{\pm x\sqrt{\beta^2+q^2\frac{\kappa-1}{\kappa+1}}} \begin{pmatrix} \pm\frac{\beta^2(\kappa+1)+q^2(\kappa-1)}{2q^2\sqrt{\beta^2+q^2\frac{\kappa-1}{\kappa+1}}(\kappa-1)} & \frac{\beta}{2q^2} \\ -\frac{\beta(\kappa+1)}{2q^2(\kappa-1)} & \mp\frac{\beta^2}{2q^2\sqrt{\beta^2+q^2\frac{\kappa-1}{\kappa+1}}} \end{pmatrix}.$$

The partial solution is derived with the help of Green's matrix function $G_s(x,\xi)$. The Green's matrix function is constructed with the help of matrix sin-, cos- integral Fourier's transformation method with the kernel $H(x,\alpha_n) = \begin{pmatrix} \sin\alpha_n x & 0 \\ 0 & \cos\alpha_n x \end{pmatrix}, \alpha_n = \frac{n\pi}{a}, n = 0,1,2...$. Green's matrix function is derived in the bilinear expansion form [7]

$$G_s(x,\xi) = \frac{2}{a}\sum_{n=0}^{\infty}{}' H(x,\alpha_n)\Omega^{-1}_{s\beta}(\alpha_n) H(\xi,\alpha_n),$$

here $\Omega_{s\beta}(\alpha_n) = -I\alpha_n^2 - 2\beta\alpha_n\tilde{Q} - P_{s\beta}$, $\tilde{Q} = \begin{pmatrix} 0 & \frac{1}{\kappa+1} \\ \frac{1}{\kappa-1} & 0 \end{pmatrix}$, stroke means that the zeroth member is multiplied by ½.

So, the formula (7) can be rewritten [7] as

$$\vec{y}_\beta(x) = \left(Y_1(x)+Y_3(x)\right)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \left(Y_2(x)+Y_4(x)\right)\begin{pmatrix} c_3 \\ c_4 \end{pmatrix} + \int_0^a G_s(x,\xi)\,\vec{f}_s(\xi)\,d\xi \qquad (8)$$

where constants $c_i,\ i=\overline{1,4}$ are found from the boundary conditions on the semi-infinite sides.

The inverse Fourier transformation is applied to (8) and the solution of the stated problem is constructed in Laplace's transform domain in an analytical form. This representation contains unknown function $\chi(x)$. It can be obtained from the first condition in (1). The substitution of the expressions for the displacements into the condition $\sigma_y\big|_{y=0} = p(x,t)$ reduces the solving of the problem to the solving of the singular integral equation regarding to the unknown function $\chi(x)$.

The detalization of the problem was done for the subcase of the steady-state load applied to a short edge of a semi-strip. The unknown function is expanded into series by the Chebyshev polynomials of the second kind and the singular integral equation is solved by the orthogonalization scheme.

Conclusions

1. The proposed approach allows to construct the analytical solution of the problem in the Laplace's transform domain. However, it is necessary to inverse the mutual Fourier-Laplace's transform to derive the final solution.

2. The derived solution in the Laplace's transform domain allows to consider the problem for the steady-state oscillations and to investigate the semi-strip's stress state in regard of the oscillation frequency.

References

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