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NEW APPROACH OF ANALYTICAL INVERSION OF LAPLACE TRANSFORM FOR SOME CASES

Laplace transform is a useful tool for solving of dynamic elasticity problems. However, the problem of analytical inversion of Laplace transform has not yet completely solved. Therefore, it is relevant to consider the new methods that allow to derive the analytical form of the original function by the known transform.

In this paper, the new method of analytical inversion of Laplace transform for the transforms of the certain form containing exponents in the denominator that linearly depend on Laplace transform parameter is proposed. The cases of correlation between the exponential indices are considered. The theorem is proved according to which the transform is expanded into the Taylor series, and the original function is derived by term-by-term application of the inverse Laplace transform. The correctness of the term-by-term application of the inverse Laplace transform is proved. The results derived by the use of the new method are verified by comparing them with the previously known formulas. The originals of Laplace transforms that were not found in the literature are derived.
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1. Introduction

Dynamic problems for elastic bodies can be solved with the help of Laplace transform. But the analytical inversion of Laplace transform in many cases is a complex problem. So, instead of Laplace transform, steady-state oscillations are sometimes considered. But, of course, they cannot describe an arbitrary dependence of the time variable.

Some asymptotic schemes are usually used to determine the function's behavior at the points $t = 0$ and $t \to \infty$ [1], [2]. Numerical methods for inverting Laplace transform are usually applied, but their correctness should be confirmed by at least some asymptotic methods, because the Laplace transform inversion problem is not correct [3]. Some numerical inversion methods of Laplace transform dealing with Laguerre polynomials are used in [4]. These methods are inverted numerically. The Laplace transform inversion problem for some functions can be reduced to the problem of solving the Volterra integral equation of the first or second kind [6], which are usually solved numerically. The relations dispensing contour integration were derived by the change of variables in [5].

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In some cases, the original function can be found as the series of residuals of the transform function [1], [6]. But in many cases the analytical finding of all poles of the transform function is impossible.

The approach by which the transform function is expanded into series was proposed in [7]. According to it, the transform function can be expanded not only into power series, but also into series of exponential functions and even into series of arbitrary functions if they satisfy the conditions indicated there. But there were no examples of dealing with generalized functions.

Thus, the problem of analytical inversion of Laplace transform has not yet been completely solved, but its application is extremely important in solving dynamic problems.

2. Main Results

The following Laplace transform is considered

$$\frac{1}{c_0 + \sum\limits_{i=1}^{N} c_i e^{-sA_i}} \tag{1}$$

Here $A_i > 0, i = \overline{1,N}$, $c_i, i = \overline{1,N}, c_0 \neq 0$ are real constants or functions, which do not depend on parameter of Laplace transform s, $N \geq 1$ is natural digit.

Let's consider the case when $A_i = n_i A_m, n_i \in \mathbb{N}, i = \overline{1,N}$ for some fixed number $1 \leq m \leq N$. Then the transform (1) can be rewritten in the following form

$$\frac{1}{c_0 + \sum\limits_{i=1}^{N} c_i e^{-sn_iA_m}} \tag{2}$$

Denote the single-valued function of the complex variable s e^{-sA_m} as z. Since $\Re s > 0$, then $|e^{-sA_m}| = |z| < 1$. The expression (2) can be rewritten as

$$f(z) = \frac{1}{c_0 + \sum\limits_{k=1}^{N} c_k z^{n_k}} \tag{3}$$

It is obvious that the function (3) has $\max\limits_{1\leq k\leq N} n_k = \eta$ singular points $z_i = \alpha_i, i = \overline{1,\eta}$. So, the points $s_i = -\frac{1}{A_m}\ln \alpha_i, i = \overline{1,\eta}$ are singular points for the function (2). Since γ in the formula of the inverse Laplace transform $\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} f(s)e^{st}dt$ is the abscissa in the semi-plane of the Laplace integral's absolute convergence [7], so $\Re s > \nu > 0$, where $\nu = \max\limits_{1\le i\le \nu}\left\{-\frac{1}{A_m}\ln\alpha_i\right\}$. Thus, when $\Re s > \nu > 0$ it is fulfilled that $|e^{-sA_m}| = |z| < \vartheta < 1$, where $\vartheta = e^{-\nu A_m \min\limits_{1\le i\le N} n_i}$. So, the function (3) in the domain $|z| < \vartheta < 1$ does not have any singular points.

First the following lemma should be proved.

Lemma. *The function (3) satisfies Cauchy-Riemann conditions in the domain $|z| < \vartheta < 1$ where it has no singular points.*

Proof. Cauchy-Riemann conditions for the function $f(z) = u(x, y) + iv(x, y)$ have the following form [8]:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{4}$$

First let's present the function (3) in the form $f(z) = u(x, y) + iv(x, y)$:

$$f(z) = \frac{1}{c_0+\sum\limits_{k=1}^{N} c_k z^{n_k}} = \frac{1}{c_0+\sum\limits_{k=1}^{N} c_k (x+iy)^{n_k}} = \frac{1}{c_0+\sum\limits_{k=1}^{N} c_k \sum\limits_{l=0}^{n_k} C_{n_k}^{l} x^{n_k-l}(iy)^l} =$$
$$= \frac{1}{c_0+\sum\limits_{k=1}^{N} c_k \sum\limits_{l=0}^{[n_k/2]} C_{n_k}^{2l} x^{n_k-2l}(-1)^l y^{2l} + i\sum\limits_{k=1}^{N} c_k \sum\limits_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1} x^{n_k-2l-1}(-1)^l y^{2l+1}} =$$
$$= \frac{1}{Re+iIm} = \frac{Re-iIm}{Re^2+Im^2}$$

Here

$$Re(x, y) = c_0 + \sum_{k=1}^{N} c_k \sum_{l=0}^{[n_k/2]} C_{n_k}^{2l} x^{n_k-2l}(-1)^l y^{2l},$$
$$Im(x, y) = \sum_{k=1}^{N} c_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1} x^{n_k-2l-1}(-1)^l y^{2l+1},$$

$[n_k/2]$, $[(n_k-1)/2]$ are integer parts of division.

Then $f(z) = u(x, y) + iv(x, y)$ where $u(x, y) = \frac{Re}{Re^2+Im^2}, v(x, y) = -\frac{Im}{Re^2+Im^2}$.

Calculate the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$.

$$\frac{\partial u}{\partial x} = \frac{-Re'_x Re^2+Re'_x Im^2-2Im'_x ReIm}{(Re^2+Im^2)^2};$$
$$\frac{\partial v}{\partial y} = \frac{-Im'_y Re^2+Im'_y Im^2+2Re'_y ReIm}{(Re^2+Im^2)^2};$$
$$\frac{\partial u}{\partial y} = \frac{-Re'_y Re^2+Re'_y Im^2-2Im'_y ReIm}{(Re^2+Im^2)^2};$$
$$\frac{\partial v}{\partial x} = \frac{-Im'_x Re^2+Im'_x Im^2+2Re'_x ReIm}{(Re^2+Im^2)^2}.$$

Note that Cauchy-Riemann conditions (4) for the function (3) are fulfilled when

$$Re'_x = Im'_y, Re'_y = -Im'_x \tag{5}$$

Calculate $Re'_x, Im'_y, Re'_y, Im'_x$.

$$Re'_x = \frac{\partial Re}{\partial x} = \sum_{k=1}^{N} c_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l}(n_k - 2l)x^{n_k-2l-1}(-1)^l y^{2l};$$

$$Re'_y = \frac{\partial Re}{\partial y} = \sum_{k=1}^{N} c_k \sum_{l=0}^{[n_k/2]} C_{n_k}^{2l} x^{n_k-2l}(-1)^l (2l) y^{2l-1};$$

$$Im'_x = \frac{\partial Im}{\partial x} = \sum_{k=1}^{N} c_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1}(n_k - 2l - 1)x^{n_k-2l-2}(-1)^l y^{2l+1};$$

$$Im'_y = \frac{\partial Im}{\partial y} = \sum_{k=1}^{N} c_k \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1} x^{n_k-2l-1}(-1)^l (2l+1) y^{2l}.$$

To check (5) the following differences are calculated:

$$Re'_x - Im'_y = \sum_{k=1}^{N} c_k \left(\sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l}(n_k - 2l)x^{n_k-2l-1}(-1)^l y^{2l} - \right.$$
$$\left. - \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1} x^{n_k-2l-1}(-1)^l (2l+1) y^{2l} \right) =$$
$$= \sum_{k=1}^{N} c_k \left(\sum_{l=0}^{[(n_k-1)/2]} x^{n_k-2l-1}(-1)^l y^{2l} \left(\frac{n_k!}{(2l)!(n_k-2l)!}(n_k - 2l) - \right.\right.$$
$$\left.\left. - \frac{n_k!}{(2l+1)!(n_k-2l-1)!}(2l+1) \right) \right) = 0;$$

$$Re'_y + Im'_x = \sum_{k=1}^{N} c_k \left(\sum_{l=0}^{[n_k/2]} C_{n_k}^{2l} x^{n_k-2l}(-1)^l (2l) y^{2l-1} + \right.$$
$$\left. + \sum_{l=0}^{[(n_k-1)/2]} C_{n_k}^{2l+1}(n_k - 2l - 1)x^{n_k-2l-2}(-1)^l y^{2l+1} \right) =$$
$$= \sum_{k=1}^{N} c_k \left(\sum_{l=0}^{[n_k/2]} \frac{n_k!}{(2l)!(n_k-2l)!}(2l) x^{n_k-2l}(-1)^l y^{2l-1} - \right.$$
$$\left. - \sum_{l=0}^{[n_k/2]} \frac{n_k!}{(2l-1)!(n_k-2l+1)!}(n_k - 2l + 1)x^{n_k-2l}(-1)^l y^{2l-1} \right) = 0.$$

It is derived that conditions (5) and, correspondingly, Cauchy-Riemann conditions (4) for the function $f(z)$ of the form (3) are fulfilled for all $|z| < \vartheta < 1$.

Theorem 1. $L^{-1}\left[\frac{1}{c_0 + \sum_{i=1}^{N} c_i e^{-s n_i A_m}}\right] = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\delta(t - kA_m)$, *where the function* $f(z)$ *has the form (3).*

Proof.

By the proved lemma the function (3) satisfies Cauchy-Riemann conditions and, therefore, it is holomorphic and regular [8] for all $|z| < \vartheta < 1$.

According to the theorems [8] the regular function in the circle $K : |z - a| < R$ can be presented by Taylor series $f(z) = \sum\limits_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$, which is convergent everywhere in the circle K.

The circle for the function (3) has the form $K : |z| < \vartheta$. Inside this circle the function is regular. So, the following equality holds

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k \tag{6}$$

Power series inside the circle of convergence can be term-by-term integrated and differentiated any number of times, moreover the radius of convergence of the derived series is equal to the radius of convergence of the original series [9].

Thus, the series (6) has the radius of convergence $R = \vartheta$, within which this series can be term-by-term integrated. That is the following is true:

$$L^{-1}\left[\frac{1}{c_0 + \sum\limits_{i=1}^{N} c_i e^{-s n_i A_m}}\right] = L^{-1}\left[f(z)\right] = L^{-1}\left[\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k\right] =$$
$$= \frac{1}{2\pi i}\int\limits_{\gamma - i\infty}^{\gamma + i\infty} \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} e^{-skA_m} e^{st} dt = \frac{1}{2\pi i}\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \int\limits_{\gamma - i\infty}^{\gamma + i\infty} e^{-skA_m} e^{st} dt =$$
$$= \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} L^{-1}\left[e^{-skA_m}\right] = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \delta(t - kA_m)$$

Let's prove that the derived series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \delta(t - kA_m) \tag{7}$$

converges in the sense that all series

$$\left(\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \delta(t - kA_m), \varphi(t)\right) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \varphi(kA_m) \tag{8}$$

absolutely converge for all functions $\varphi(t) \in S \cup K^0$, where S is the main space containing all infinitely differentiable functions which when $|x| \to \infty$ tends to zero with all their derivatives of any order faster than any power of $1/|x|$ [10], K^0 is the main space containing all continuous functions that are zero outside some bounded domain [10]. Obviously, if the absolute convergence of series (8) is proved for all functions from the main spaces S and K^0, then it will also take place for the functions from the main spaces $K^m, m > 0$ and K, since $K \subset K^m \subseteq K^0, K \subset S$ [10].

To prove the convergence of the series

$$\sum_{k=0}^{\infty} \frac{\left|f^{(k)}(0)\right|}{k!} \left|\varphi(kA_m)\right| \tag{9}$$

which is real-valued, let's use the following theorem, accordingly to which if, at least starting from some place (say, for $n > N$), the following inequality holds $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$, then the convergence of the series $\sum\limits_{n=1}^{\infty} b_n$ with positive terms implies the convergence of the series $\sum\limits_{n=1}^{\infty} a_n$ with positive terms [11].

The comparison will be made with the series

$$\sum_{k=0}^{\infty} \frac{\left|f^{(k)}(0)\right|}{k!} \left|z_0^k\right| \tag{10}$$

By Abel's theorem [9] if the power series $\sum\limits_{n=0}^{\infty} c_n z^n$ converges at the point $z_* \neq 0$, then it absolutely converges in the circle $K_0 : |z| < |z_*|$, and in any smaller circle $K_1 : |z| \leq R_1 < |z_*|$ this series converges uniformly. In this case the point $0 < |z_0| < \vartheta - \varepsilon_0 < \vartheta$ is chosen for some small fixed $\varepsilon_0 > 0$. Then, by Abel's theorem, using the convergence of the series (6), it is derived that the series (10) converges (converges absolutely).

Let's prove that

$$\frac{\frac{\left|f^{(k+1)}(0)\right|}{(k+1)!} \left|\varphi((k+1)A_m)\right|}{\frac{\left|f^{(k)}(0)\right|}{k!} \left|\varphi(kA_m)\right|} \leq \frac{\frac{\left|f^{(k+1)}(0)\right|}{(k+1)!} \left|z_0^{k+1}\right|}{\frac{\left|f^{(k)}(0)\right|}{k!} \left|z_0^k\right|} \tag{11}$$

for the functions $\varphi(t) \in S$ (it is fair for them that $|\varphi(kA_m)| \neq 0$ for all k).

Obviously, the inequality (11) will take place if the inequality $\frac{|\varphi((k+1)A_m)|}{|\varphi(kA_m)|} \leq |z_0|$ holds or, equivalently, the following inequality holds

$$\frac{|\varphi((k+1)A_m)|}{|z_0|} \leq |\varphi(kA_m)| \tag{12}$$

By definition of the main space S [10] $\lim\limits_{|x|\to\infty} x^q \varphi(x) = 0$ for all $q = 0, 1, 2, ...$ For definiteness, the value $q = 1$ is chosen. According to the definition of the limit of the sequence [12] the following holds: for each $\varepsilon > 0$, no matter how small it may be, there exists a number N such that for all $n > N$: $|n\varphi(n)| < \varepsilon$. Accordingly, the following is true for $k > N - 1$ $(A_m > 0)$

$$|(k+1)A_m \varphi((k+1)A_m)| < \varepsilon \tag{13}$$

When $0 < |z_0| < \vartheta - \varepsilon_0 < \vartheta$, obviously, there will be such number N_0 that for all $k > N_0 - 1$ the following inequality holds

$$\frac{1}{|z_0|} < (k+1)A_m \tag{14}$$

Let's choose such a small digit $\varepsilon_* > 0$ for which

$$|\varphi(kA_m)| \geq \varepsilon_* \tag{15}$$

Obviously, for the function $|\varphi(kA_m)|$ that does not turn to 0, such a digit $\varepsilon_* > 0$ can always be chosen. Let's fix it. For this $\varepsilon_* > 0$ there is some number N_* that for all $k > N_* - 1$ (13) will be true. Let's choose $k > \max\{N_*, N_0\} - 1$ and combine the inequalities (13)-(15):

$$\frac{|\varphi((k+1)A_m)|}{|z_0|} < |(k+1)A_m\varphi((k+1)A_m)| < \varepsilon_* \leq |\varphi(kA_m)|,$$

that is the inequality (12) and therefore (11) takes places. Then, by the theorem, the series (9) is convergent for all functions $\varphi(t) \in S$.

Note that for the functions $\varphi(t) \in K^0$, since they are equal to zero outside some bounded domain, there exists a number N such that $|\varphi(kA_m)| = 0$ for $k > N$. In this case, the convergence of the series (9) can be proved by another theorem, according to which if, at least starting from some place (say, for $n > N$), the inequality $a_n \leq b_n$ holds, then the convergence of the series $\sum\limits_{n=1}^{\infty} b_n$ with positive terms implies the convergence of the series $\sum\limits_{n=1}^{\infty} a_n$ with positive terms [11]. Then for $k > N$ the following correspondence takes place $0 = \frac{|f^{(k)}(0)|}{k!}|\varphi(kA_m)| \leq \frac{|f^{(k)}(0)|}{k!}|z_0^k|$. Hence the series (9) is convergent for all functions $\varphi(t) \in K^0$. Thus, it is proved that the series (8) converges absolutely for all functions $\varphi(t) \in S \cup K^0$, and the series (7) converges in the sense indicated earlier.

The proved convergence of the series (7) implies the correctness of the term-by-term application of the series (7) to any function from the main spaces $K^m, m \geq 0, K, S$.

Now let's prove that the resulting series (7) is the original for the Laplace transform (2). For this, the Laplace transform is applied to the series (7)

$$L\left[\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}\delta(t - kA_m)\right] = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} L\left[\delta(t - kA_m)\right] = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} e^{-skA_m}$$

Let's prove that the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} e^{-skA_m} \tag{16}$$

converges to the known transform (2).

The series (16), taking into account the change of variables $z = e^{-sA_m}$, can be written as (6), that is, it is an expansion of the function $f(z)$ (3) in Taylor series. According to the theorems [8] and the proved regularity of the function $f(z)$, it is derived that the series (16) converges to the function $f(z)$ (3) with the radius of convergence $R = \vartheta$, which corresponds to the entire range of the variable $|z| < \vartheta$.

The statement of the theorem is proved.

The approbation of the proposed method is done on the known transform. The result of applying of the proposed method to the known transform gave the same result to the previously known result [13]. The detailed verification is given in Appendix A.

Let's consider some examples of application of the proved theorem. Consider the following functions $\frac{1}{(1-e^{-sA})^\alpha}$ and $\frac{1}{(1+e^{-sA})^\alpha}$ when $A > 0$, α is a natural digit.

The Taylor series can be easily constructed for the functions $f(z) = \frac{1}{(1-z)^\alpha}$ and $g(z) = \frac{1}{(1+z)^\alpha}$:

$$f(z) = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+k-1)}{k!} z^k$$
$$g(z) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k \alpha(\alpha+1)...(\alpha+k-1)}{k!} z^k$$

According to theorem 1

$$\begin{aligned}
L^{-1}\left[\frac{1}{(1-e^{-sA})^\alpha}\right] &= \left[z = e^{-sA}\right] = L^{-1}\left[f(z)\right] \\
&= L^{-1}\left[1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+k-1)}{k!} z^k\right] = \\
&= \delta(t) + \sum_{k=1}^{\infty} \frac{\alpha(\alpha+1)...(\alpha+k-1)}{k!} \delta(t-kA) \\
L^{-1}\left[\frac{1}{(1+e^{-sA})^\alpha}\right] &= \left[z = e^{-sA}\right] = L^{-1}\left[g(z)\right] \\
&= L^{-1}\left[1 + \sum_{k=1}^{\infty} \frac{(-1)^k \alpha(\alpha+1)...(\alpha+k-1)}{k!} z^k\right] = \\
&= \delta(t) + \sum_{k=1}^{\infty} \frac{(-1)^k \alpha(\alpha+1)...(\alpha+k-1)}{k!} \delta(t-kA)
\end{aligned}$$

Finally the following formulas are derived

$$L^{-1}\left[\frac{1}{(1-e^{-sA})^{\alpha}}\right]=\delta(t)+\sum_{k=1}^{\infty}\frac{\alpha(\alpha+1)\ldots(\alpha+k-1)}{k!}\delta(t-kA)$$
$$L^{-1}\left[\frac{1}{(1+e^{-sA})^{\alpha}}\right]=\delta(t)+\sum_{k=1}^{\infty}\frac{(-1)^{k}\alpha(\alpha+1)\ldots(\alpha+k-1)}{k!}\delta(t-kA) \tag{17}$$

Let's consider the transform that corresponds to the general function of the form (1)

$$x^{L}(s)=\frac{f^{L}(s)}{c_{0}+K^{L}(s)} \tag{18}$$

The expression (18) can be rewritten in the following form

$$c_{0}x^{L}(s)+x^{L}(s)K^{L}(s)=f^{L}(s) \tag{19}$$

By the convolution theorem of originals the Volterra integral equation of the second kind [6] is derived from (19)

$$c_{0}x(t)+\int_{0}^{t}x(\tau)K(t-\tau)d\tau=f(t) \tag{20}$$

For the function of the form (1) $f(t)=\delta(t), K(t)=\sum_{i=1}^{N}c_{i}\delta(t-A_{i})$. Since these functions are equal to zero when $t<0$, the equation (20) can be written using convolution as follows [10]

$$\left[c_{0}\delta(t)+\sum_{i=1}^{N}c_{i}\delta(t-A_{i})\right]*x(t)=\delta(t) \tag{21}$$

That is, finding the original $x(t)$ is reduced to the solving of the convolution equation (21). The solution of the convolution equation of the form $a(x)*y(x)=b(x)$ is uniquely determined by the formula $y(x)=a^{-1}(x)*b(x)$ in the case when the inverse generalized function $a^{-1}(x)$ exists [10]. By the definition, if the generalized function $f(x)$ has its inverse function $f^{-1}(x)$, then [10]

$$f^{-1}(x)*f(x)=f(x)*f^{-1}(x)=\delta(x) \tag{22}$$

From the above the following consequence can be formulated

Consequence. $\left[c_{0}\delta(t)+\sum_{i=1}^{N}c_{i}\delta(t-n_{i}A_{m})\right]^{-1}=\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\delta(t-kA_{m})$, *where the function* $f(z)$ *has the form (3).*

The verification of the formulas (17) by the fulfilment of the equality (22) for them is done in Appendix B.

The more general cases when $A_i = n_i A_d + m_i A_q, n_i, m_i \in \mathbb{N}, i = \overline{1,N}$ for some fixed numbers $1 \le d, q \le N, d \ne q$ or even when $A_i = \sum\limits_{j=1}^{m} n_{ij} A_{q_j}, n_{ij} \in \mathbb{N}, i = \overline{1,N}, j = \overline{1,m}$ for some fixed numbers $1 \le q_j \le N, j = \overline{1,m}, q_j \ne q_k, j \ne k, j,k = \overline{1,m}$ can be also considered. For these cases the transform (1) can be rewritten in the forms $\frac{1}{c_0 + \sum\limits_{i=1}^{N} c_i e^{-s(n_i A_d + m_i A_q)}}$ or $\frac{1}{c_0 + \sum\limits_{i=1}^{N} c_i e^{-s\sum\limits_{j=1}^{m} n_{ij} A_{q_j}}}$ respectively. So, the following theorems take place.

Theorem 2. $L^{-1}\left[\frac{1}{c_0 + \sum\limits_{i=1}^{N} c_i e^{-s(n_i A_d + m_i A_q)}}\right] = \sum\limits_{k=0}^{\infty}\sum\limits_{l=0}^{\infty} \frac{1}{k!l!} \frac{\partial^{k+l} f(0,0)}{\partial z^k \partial \zeta^l} \delta(t - kA_d - lA_q)$, *where* $f(z,\zeta) = \frac{1}{c_0 + \sum\limits_{k=1}^{N} c_k z^{n_k} \zeta^{m_k}}$. *Here the single-valued functions of the complex variable s* e^{-sA_d} *and* e^{-sA_q} *are denoted as* z *and* ζ *respectively.*

Theorem 3. $L^{-1}\left[\frac{1}{c_0 + \sum\limits_{i=1}^{N} c_i e^{-s\sum\limits_{j=1}^{m} n_{ij} A_{q_j}}}\right] =$

$$\sum_{k_1,\ldots,k_m=0}^{\infty} \frac{1}{k_1!\ldots k_m!} \frac{\partial^{k_1+\ldots+k_m} f(0,\ldots,0)}{\partial z_1^{k_1}\ldots\partial z_m^{k_m}} \delta(t - k_1 A_{q_1} - \ldots - k_m A_{q_m}),$$

where

$$f(z_1,\ldots,z_m) = \frac{1}{c_0 + \sum\limits_{k=1}^{N} c_k \prod\limits_{j=1}^{m} z_j^{n_{k_j}}}.$$

Here the single-valued functions of the complex variable s $e^{-sA_{q_j}}$ *are denoted as* $z_j, j = \overline{1,m}$.

The proof of these theorems is beyond the scope of this article.

Appendix A. Method validation on known originals

Consider the functions $\frac{1}{1-e^{-sA}}$ and $\frac{1}{1+e^{-sA}}$ when $A > 0$. From [13] it is known that

$$L^{-1}\left[\frac{1}{1-e^{-sA}}\right] = \sum_{n=0}^{\infty}\delta(t-nA), L^{-1}\left[\frac{1}{1+e^{-sA}}\right] = \sum_{n=0}^{\infty}(-1)^n\delta(t-nA) \quad (A.1)$$

Let's show that the results derived from theorem 1 are consistent with the known results (A.1).

According to theorem 1

$$\begin{gathered}
L^{-1}\left[\tfrac{1}{1-e^{-sA}}\right]=\left[z=e^{-sA}\right]=L^{-1}\left[\tfrac{1}{1-z}\right]=L^{-1}\left[f(z)\right]=\\
=L^{-1}\left[\sum_{k=0}^{\infty}\tfrac{f^{(k)}(0)}{k!}z^k\right]=L^{-1}\left[\sum_{k=0}^{\infty}z^k\right]=\sum_{k=0}^{\infty}\delta(t-kA),\\
L^{-1}\left[\tfrac{1}{1+e^{-sA}}\right]=\left[z=e^{-sA}\right]=L^{-1}\left[\tfrac{1}{1+z}\right]=L^{-1}\left[f(z)\right]=\\
=L^{-1}\left[\sum_{k=0}^{\infty}\tfrac{f^{(k)}(0)}{k!}z^k\right]=L^{-1}\left[\sum_{k=0}^{\infty}(-1)^k z^k\right]=\sum_{k=0}^{\infty}(-1)^k\delta(t-kA)
\end{gathered}\qquad (A.2)$$

So, the known results (A.1) are equal to the results derived from theorem 1 (A.2).

Appendix B. Verification of derived formulas using convolution

The fulfillment of the formula (22) for the functions (17) can be verified for any fixed α. Let's prove this for $\alpha=2$.

According to (17)

$$\begin{gathered}
L^{-1}\left[\tfrac{1}{(1-e^{-sA})^2}\right]=\delta(t)+\sum_{k=1}^{\infty}(k+1)\delta(t-kA),\\
L^{-1}\left[\tfrac{1}{(1+e^{-sA})^2}\right]=\delta(t)+\sum_{k=1}^{\infty}(-1)^k(k+1)\delta(t-kA)
\end{gathered}\qquad (B.1)$$

Consider the following convolution

$$\begin{gathered}
\left(\left[\delta(t)-2\delta(t-A)+\delta(t-2A)\right]*\left[\delta(t)+\sum_{k=1}^{\infty}(k+1)\delta(t-kA)\right],\varphi(t)\right)=\\
=\iint\limits_{\mathbb{R}^2}\left[\delta(\xi)-2\delta(\xi-A)+\delta(\xi-2A)\right]\\
\times\left[\delta(x-\xi)+\sum_{k=1}^{\infty}(k+1)\delta(x-\xi-kA)\right]\varphi(x)dxd\xi=\\
=\int\limits_{\mathbb{R}}\left[\delta(\xi)-2\delta(\xi-A)+\delta(\xi-2A)\right]\left[\varphi(\xi)+\sum_{k=1}^{\infty}(k+1)\varphi(\xi+kA)\right]d\xi=\\
=\varphi(0)-2\varphi(A)+\varphi(2A)+\sum_{k=1}^{\infty}(k+1)\varphi(kA)-\\
-2\sum_{k=1}^{\infty}(k+1)\varphi((k+1)A)+\sum_{k=1}^{\infty}(k+1)\varphi((k+2)A)=\\
=\varphi(0)-2\varphi(A)+\varphi(2A)+\sum_{k=1}^{\infty}(k+1)\varphi(kA)-\\
-2\sum_{k=2}^{\infty}k\varphi(kA)+\sum_{k=3}^{\infty}(k-1)\varphi(kA)=\\
=\varphi(0)+\varphi(2A)+\sum_{k=2}^{\infty}(1-k)\varphi(kA)+\sum_{k=3}^{\infty}(k-1)\varphi(kA)=\varphi(0)=(\delta(t),\varphi(t))
\end{gathered}$$

So, it is proved that

$$\left[\delta(t)-2\delta(t-A)+\delta(t-2A)\right]*\left[\delta(t)+\sum_{k=1}^{\infty}(k+1)\delta(t-kA)\right]=\delta(t).$$

The inequality $\left[\delta(t) + \sum\limits_{k=1}^{\infty}(k+1)\delta(t-kA)\right] * [\delta(t) - 2\delta(t-A) + \delta(t-2A)] =$ $\delta(t)$ is proved similarly. So, the correctness of the first formula in (B.1) is shown.

Consider the following convolution

$$\left([\delta(t) + 2\delta(t-A) + \delta(t-2A)] * \left[\delta(t) + \sum_{k=1}^{\infty}(-1)^k(k+1)\delta(t-kA)\right], \varphi(t)\right) =$$
$$= \iint\limits_{\mathbb{R}^2} [\delta(\xi) + 2\delta(\xi - A) + \delta(\xi - 2A)] \times$$
$$\times \left[\delta(x-\xi) + \sum_{k=1}^{\infty}(-1)^k(k+1)\delta(x-\xi-kA)\right]\varphi(x)dxd\xi =$$
$$= \varphi(0) + 2\varphi(A) + \varphi(2A) + \sum_{k=1}^{\infty}(-1)^k(k+1)\varphi(kA) +$$
$$+2\sum_{k=1}^{\infty}(-1)^k(k+1)\varphi((k+1)A) +$$
$$+\sum_{k=1}^{\infty}(-1)^k(k+1)\varphi((k+2)A)$$
$$= \varphi(0) + 2\varphi(A) + \varphi(2A) + \sum_{k=1}^{\infty}(-1)^k(k+1)\varphi(kA) -$$
$$-2\sum_{k=2}^{\infty}(-1)^k k\varphi(kA) + \sum_{k=3}^{\infty}(-1)^k(k-1)\varphi(kA) = \varphi(0) = (\delta(t), \varphi(t))$$

So, it is proved that

$$[\delta(t) + 2\delta(t-A) + \delta(t-2A)] * \left[\delta(t) + \sum_{k=1}^{\infty}(-1)^k(k+1)\delta(t-kA)\right] = \delta(t).$$

The inequality

$$\left[\delta(t) + \sum_{k=1}^{\infty}(-1)^k(k+1)\delta(t-kA)\right] * [\delta(t) + 2\delta(t-A) + \delta(t-2A)] = \delta(t)$$

is proved similarly. So, the correctness of the second formula in (B.1) is shown.

3. Conclusion

1. The new method for the analytical inversion of the Laplace transform for the functions of the certain structure is proposed. The proof of this method is carried out.

2. The results derived by the new method of analytical inversion of Laplace transform are compared with the formulas for the original functions known in literature.

3. Due to the use of the proposed method, the originals of new transforms that are important for use in mechanics are derived.

Журавльова З. Ю.

Новий підхід до аналітичного обернення перетворення Лапласа для деяких випадків

Резюме

Перетворення Лапласу є корисним інструментов для розв'язання динамічних задач теорії пружності. Тим не менш, проблема аналітичного обернення перетворення Лапласу до сих пір повністю не розв'язана. Тому актуальним є розгляд нових методів, за допомогою яких можна отримати аналітичне подання оригіналу за відомою трансформантою.

У даній роботі запропоновано новий метод аналітичного обернення перетворення Лапласу для трансформант певного вигляду, що містять у знаменнику експоненти, які лінійно залежать від параметра перетворення Лапласу. Розглянуто випадки співвідношень між показниками експоненти. Доведено теорему, згідно з якою трансформанта розвивається у ряд Тейлора, і оригінал отримується шляхом почленного застосування оберненого перетворення Лапласу. Коректність почленного застосування оберненого перетворення Лапласу доведена. Проведена перевірка результатів, що отримані з використанням нового методу, з відомими раніше формулами. Отримані оригінали від трансформант Лапласу, які раніше не зустрічались у літературі.
Ключові слова: перетворення Лапласу, аналітичне обернення, розвинення в ряди Тейлора, узагальнені функції, згортка.

Журавлёва З. Ю.

Новый подход к аналитическому обращению преобразования Лапласа для некоторых случаев

Резюме

Преобразование Лапласа является полезным инструментом для решения динамических задач теории упругости. Тем ни менее, проблема аналитического обращения преобразования Лапласа до сих пор полностью не решена. Поэтому актуальным является рассмотрение новых методов, с помощью которых можно получить аналитическое представление оригинала по известной трансформанте.

В данной работе предложен новый метод аналитического обращения преобразования Лапласа для трансформант определённого вида, содержащих в знаменателе экспоненты, линейно зависящие от параметра преобразования Лапласа. Рассмотрены случаи соотношений между показателями экспоненты. Доказана теорема, согласно которой трансформанта раскладывается в ряд Тейлора, и оригинал получается путём почленного применения обратного преобразования Лапласа. Корректность почленного применения обратного преобразования Лапласа доказана. Проведена проверка результатов, полученных с использованием нового метода, с известными ранее формулами. Получены оригиналы от трансформант Лапласа, ранее не встречавшиеся в литературе.
Ключевые слова: преобразование Лапласа, аналитическое обращение, разложение в ряды Тейлора, обобщённые функции, свёртка.

References

1. Schiff, J. L. (1999). *The Laplace transform. Theory and Applications.* New York: Springer-Verlag, 245 p.
2. Tuan, V. K. and Duc, D. T. (2000). Convergence rate of Post-Widder approximate inversion of the Laplace transform. *Vietnam J. Math.*, Vol. 28(1), P. 93–96.
3. Krylov, V. I. and Skoblya, N. S. (1977). *A Handbook of Methods of Approximate Fourier Transformation and Inversion of the Laplace Transform.* Moscow: Mir, 224 p.
4. Boumenir, A. and Al-Shuaibi, A. (2000). On the numerical inversion of the Laplace transform by the use of optimized Legendre polynomials. *Approx. Theory Appl.*, Vol. 16(4), P. 17–32.
5. Berberan-Santos, M. N. (2005). Analytical inversion of the Laplace transform without contour integration: application to luminescence decay laws and other relaxation functions. *Journal of Mathematical Chemistry*, Vol. 38(2), P. 165–173.
6. Guz, A. N. and Kubenko, V. D. (1982). *Teoriya nestazionarnoy aerogidrouprugosti obolochek [Theory of nonstationary aerohidroelasticity of shells], Vol. 5.* Kyiv: Naukova dumka, 400 p.
7. Doetsch, G. (1974). *Introduction to the Theory and Application of the Laplace Transformation.* New York: Springer-Verlag, 326 p.
8. Sidorov, Yu. V., Fedoryuk, M. V. and Shabunin, M. I. (1989). *Lekzii po teorii funkziy kompleksnogo peremennogo [Lectures on functions of complex variable theory].* Moscow: Nauka, 477 p.
9. Sveshnikov, A. G. and Tikhonov, A. N. (2005). *Teoriya funkziy kompleksnoy peremennoy [Theory of complex variable functions].* Moscow: Fizmatlit, 336 p.
10. Kecs, W. and Teodorescu, P. P. (1978). *Vvedenie v teoriyu obobshchennih funkzii s prilogeniyami v tekhnike [Introduction in theory of generalized functions with applications in technique].* Moscow: Mir, 520 p.
11. Fikhtengolz, G. M. (2001). *Kurs differentsialnogo i integralnogo ischisleniya [A Course of Differential and Integral Calculus], Vol. II.* Moscow: Fizmatlit, 864 p.
12. Fikhtengolz, G. M. (2001). *Kurs differentsialnogo i integralnogo ischisleniya [A Course of Differential and Integral Calculus], Vol. I.* Moscow: Fizmatlit, 680 p.
13. Abramowitz, M. and Stegun, I. (1979). *Spravochnik po spezialnim funkziyam s formulami, grafikami i matematicheskimi tablizami [Handbook of mathematical functions with formulas, graphs and mathematical tables].* Moscow: Nauka, 834 p.