УДК 511

I. N. Velichko
Odessa National University

ARITHMETICAL FUNCTIONS IN MATRIX RING OF ORDER 4 AND HIGHER

Величко И. М. Арифметичнi функцiї над кiльцем матриць розмiрностi 4 i вище. Побудован ряд Дiрiхле для функцiї дiльникiв у кiльцi матриць порядку 4. Отриманi оцiнки розподiлу деяких арифметичних функцiй.
Ключовi слова: функцiя дiльникiв матриць, арифметичнi функцiї у кiльцi матриць, оцiнки функцiї дiльникiв матриць.

Величко И. Н. Арифметические функции над кольцом матриц размерности 4 и выше. Построен производящий ряд Дирихле для функции делителей в кольце целочисленных матриц порядка 4. Получены оценки распределения некоторых арифметических функций.
Ключевые слова: функция делителей матриц, арифметические функции в кольце матриц, оценки функции делителей матриц.

Velichko I. N. Arithmetical functions in matrix ring of order 4 and higher. We constructed the generating Dirichlet series for divisor function in matrix ring of order 4. Estimates for distributions of some arithmetical functions were obtained.
Key words: divisor function of integer matrices, arithmetical functions in matrix ring, estimates of matrix divisor function.

Introduction.

Let $M_k(\mathbb{Z})$ denotes the ring of integer matrices of order k, $GL_k(\mathbb{Z})$ is the unite group of $M_k(\mathbb{Z})$. We denote the number of different (to association) representations of matrix $C \in M_k(\mathbb{Z})$ in the form $C = A_1A_2,\ A_1, A_2 \in M_k(\mathbb{Z})$ as $\tau_k(C)$.

G. Bhowmik and H. Menzer [1], H.-Q. Liu [2], H. Menzer [3] studied the distribution of function $t_2(n)$, where

$$t_2(n) = \sum_{\substack{C\in M_2(\mathbb{Z})\\ |\det C|=n}} \tau_2(C).$$

In particular, the asymptotic formula

$$T_2(x) := \sum_{n\le x} t_2(n) = K_1 x\log^2 x + K_2 x\log x + K_3 x + \Delta_2(x)$$

was proved. The best estimate of $\Delta_2(x)$ that is presently known was obtained by G. Bhowmik and J. Wu [4]:

$$\Delta_2(x) \ll x^{5/8}\log^4 x.$$

A. Ivič [5] gave bounds for second moment of remainder term $\Delta_2(x)$:

$$\int_1^x \Delta_2^2(x)dx = O(x^2(\log x)^{31/3}),$$

© I. N. Velichko, 2008

$$\int_1^x \Delta_2^2(x)dx = \Omega(x^2 \log^2 x).$$

Study of the distribution of function $t_k(n)$ for $k \geq 3$ has some difficulties. N.Fugelo, I.Velichko [6] constructed the generating Dirichlet series for $t_3(n)$:

$$F(s) = \sum_{n=1}^{\infty} \frac{t_3(n)}{n^s}, \qquad (Re\, s > 1).$$

Besides, one proved the asymptotic formula

$$T_3(x) := \sum_{n \leq x} t_3(n) = xP_4(\log x) + O(x^{5/6}),$$

where $P_4(u)$ is the polynomial of fourth degree.

In this paper we study the distribution of function $t_4(n)$ and prove the following statements

Theorem 1. *For $x \to \infty$ the estimate*

$$T_4(x) := \sum_{n \leq x} t_4(n) = C_0 x^{5/4} + O(x^{9/8} \log^2 x)$$

holds.

Theorem 2. *For any natural number $k > 1$ we have*

$$T_k^*(x) := \sum_{n \leq x}{}' t_k(n) = xP_2(\log x) + O(x^{1/2} \log^5 x),$$

where the sign $'$ in sum $\sum'$ indicates that the summation runs over all square-free numbers, $P_2(u)$ is the polynomial of second degree.

Notations: Let $gcd(a,b)$ denotes the greatest common divisor of $a, b \in \mathbb{Z}$, $\mu(n)$ is the Möbius function. Vinogradov's symbol "$\ll$" is used in a sense of Landau symbol "O", in other words $f(x) \ll g(x)$ is equivalent to $f(x) = O(g(x))$. For complex number $s = \sigma + it$ we denote zeta-function as $\zeta(s)$. $res_a\, F(s)$ is the residue of function $F(s)$ at the point $s = a$.

Notation and auxiliary lemmas

We say, that matrices A_1, A_2 are

(1) associated on the left (accordingly, on the right), if $A_1 = A_2 V$ (accordingly, $A_1 = V A_2$) holds for some matrix $V \in GL_k(\mathbb{Z})$;

(2) associated, if $A_1 = U A_2 V$ holds for some matrix $U, V \in GL_k(\mathbb{Z})$.

Two representations $C = A_1 B_1 = A_2 B_2$ are considered equivalent, if matrices A_1 and A_2 are associated on the left (or B_1 and B_2 are associated on the right). $\tau_k(C)$ denotes the number of nonequivalent representations C, $C \in M_k(\mathbb{Z})$ in the form $C = AB$, $A, B \in M_k(\mathbb{Z})$.

Let $S(C)$ be the Smith normal form of matrix $C \in M_k(\mathbb{Z})$, $|C| \neq 0$. Then there are matrices $U, V \in GL_k(\mathbb{Z})$ such that $C = US(C)V$. Besides, if $C = AB$ and $C = US(C)V$, then $S(C) = U^{-1}ABV^{-1}$ and, $U^{-1}A|S(C)$. So, we have identical correspondence between left divisors of the matrices C and $S(C)$, hence, $\tau_k(C) = = \tau_k(S(C))$.

It is well known, that there are a unique lower triangular matrix $\hat{A} = \{a_{ij}\}$ where $0 \leq a_{ij} < a_{jj}$, $i < j$, which is associated with $A \in M_k(\mathbb{Z})$. Hence, $\tau_k(C)$ is equal to the number of lower triangular matrices $\hat{A} = \{a_{ij}\} \in M_k(\mathbb{Z})$, $0 \leq a_{ij} < a_{jj}$, $i < j$, for which $\hat{A}^{-1}S(C) \in M_k(\mathbb{Z})$ holds.

Let consider the case $k = 4$. Let

$$\hat{A} = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix},$$

where $a_{11},\ a_{22},\ a_{33},\ a_{44} \in \mathbb{N}$, $0 \leq a_{21} < a_{22}$, $0 \leq a_{31}, a_{32} < a_{33}$, $0 \leq a_{41}, a_{42}, a_{43} < < a_{44}$;

$$C = \begin{pmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & c_{44} \end{pmatrix},$$

where $c_{11}|c_{22},\ c_{22}|c_{33},\ c_{33}|c_{44}$.

Then we have, that the relation $\hat{A}^{-1}S(C) \in M_4(\mathbb{Z})$ holds if and only if the system of the congruences

$$\begin{cases} c_{11} \equiv 0 \ mod\ a_{11},\ c_{22} \equiv 0 \ mod\ a_{22}, \\ c_{33} \equiv 0 \ mod\ a_{33},\ c_{44} \equiv 0 \ mod\ a_{44}, \\ a_{21}\dfrac{c_{11}}{a_{11}} \equiv 0 \ mod\ a_{22}, \\ a_{32}\dfrac{c_{22}}{a_{22}} \equiv 0 \ mod\ a_{33}, \\ a_{43}\dfrac{c_{33}}{a_{33}} \equiv 0 \ mod\ a_{44}, \\ \dfrac{a_{21}a_{32}\ c_{11}}{a_{11}a_{22}} - a_{31}\dfrac{c_{11}}{a_{11}} \equiv 0 \ mod\ a_{33}, \\ \dfrac{a_{32}a_{43}\ c_{22}}{a_{22}a_{33}} - a_{42}\dfrac{c_{22}}{a_{22}} \equiv 0 \ mod\ a_{44}, \\ -\dfrac{a_{43}(a_{21}\ \ a_{32} - a_{31}a_{22})c_{11}}{a_{11}a_{22}a_{33}} + \dfrac{a_{21}a_{42}c_{11}}{a_{11}a_{22}} - a_{41}\dfrac{c_{11}}{a_{11}} \equiv 0 \ mod\ a_{44}. \end{cases} \tag{1}$$

has solutions.

Hence, $\tau_4(C)$ is equal to the number of different solutions of system (5) with variables $a_{11},\ a_{21},\ a_{22},\ a_{31},\ a_{32},\ a_{33},\ a_{44},\ a_{41},\ a_{42},\ a_{43}$.

Using the fact, that $\sharp\{x \mid ax \equiv b \ mod\ q,\ 0 \leq x < q\} = \frac{1}{q}\sum\limits_{x=0}^{q-1}\sum\limits_{h=0}^{q-1} e^{2\pi i\frac{(ax-b)h}{q}}$, it is easy to show, that $\tau_4(C_1C_2) = \tau_4(C_1)\tau_4(C_2)$, $(|C_1|, |C_2|) = 1$. In other words $\tau_4(C)$ is multiplicative.

Let consider $t_4(n) := \sum\limits_{|C|=n} \tau_4(C)$. Note, that $t_4(n)$ is multiplicative too.

Now we need the following statements:

Lemma 1. *For any prime number p and integers i, j, k, c_1, c_2, c_3 : $0 \le i < c_1$, $0 \le j < c_2$, $0 \le k < c_3$, $c_1 \le c_2 \le c_3$ the system of congruences*

$$\begin{cases} x_1 p^{c_1 - i} \equiv 0 \ mod \ p^j, \\ x_2 p^{c_2 - j} \equiv 0 \ mod \ p^k, \\ x_1 x_2 p^{c_1 - i - j} - x_3 p^{c_1 - i} \equiv 0 \ mod \ p^k. \end{cases} \tag{2}$$

has $p^{\min(c_1 - i + k,\ \min(j,\, c_1 - i) + \min(k,\, c_2 - j) + \min(k,\, c_1 - i))}$ solutions.

Proof. We remark that the relation $gcd(p^a, p^b) = p^{\min(a,b)}$ holds. Then, from the first and second congruences we have $x_1 = t_1 p^{j - \min(j, c_1 - i)}$, $x_2 = t_2 p^{k - \min(k, c_2 - j)}$, $t_1 = 0, ..., p^{\min(j, c_1 - i)} - 1$, $t_2 = 0, ..., p^{\min(k, c_2 - j)} - 1$.

Therefore the congruence $x_1 x_2 p^{c_1 - i - j} - x_3 p^{c_1 - i} \equiv 0 \ mod \ p^k$ has a solution if and only if first summand of this congruence is divisible by $p^{\min(k, c_1 - i)}$, and in that case it has precisely $p^{\min(k, c_1 - i)}$ solutions. But this divisibility is equivalent to realization of the inequality

$$ord_p(t_1 t_2) + c_1 - i + k - \min(j, c_1 - i) - \min(k, c_2 - j) - \min(k, c_1 - i) \ge 0. \tag{3}$$

Now if $\eta := \min(j,\, c_1 - i) + \min(k,\, c_2 - j) + \min(k,\, c_1 - i) - c_1 + i - k \ge 0$ holds, we infer that (3) can be realize only if $ord_p(t_1 t_2) \ge \eta$. And if $\eta < 0$, then (3) holds for all t_1, t_2. So, the number of solutions of the system (2) is equal to

$$\begin{aligned} & p^{\min(j,\, c_1 - i) + \min(k,\, c_2 - j) + \min(k,\, c_1 - i) - \max(0, \eta)} = \\ & = p^{\min(c_1 - i + k,\ \min(j,\, c_1 - i) + \min(k,\, c_2 - j) + \min(k,\, c_1 - i))}. \end{aligned} \tag{4}$$

The proof of Lemma 1 is complete. ■

Since $t_4(n)$ is multiplicative, it is enough to calculate $t_4(n)$ for integer degrees of prime p. So we can suppose, that $c_{11} = p^{c_1}, c_{22} = p^{c_2}, c_{33} = p^{c_3}, c_{44} = p^{c_4}$, $0 \le c_1 \le c_2 \le c_3 \le c_4$, $c_1 + c_2 + c_3 + c_4 = n$. Since the congruence $ax \equiv b \ mod \ c$ has no more than $gcd(a, c)$ solutions $mod \ c$ (exactly $gcd(a, c)$, if $gcd(a, c) | b$), then the system

$$\begin{cases} a_{21} \dfrac{c_{11}}{a_{11}} \equiv 0 \ mod \ a_{22}, \\ a_{32} \dfrac{c_{22}}{a_{22}} \equiv 0 \ mod \ a_{33}, \\ a_{43} \dfrac{c_{33}}{a_{33}} \equiv 0 \ mod \ a_{44}, \\ \dfrac{a_{21} a_{32}\ c_{11}}{a_{11} a_{22}} - a_{31} \dfrac{c_{11}}{a_{11}} \equiv 0 \ mod \ a_{33}, \\ \dfrac{a_{32} a_{43}\ c_{22}}{a_{22} a_{33}} - a_{42} \dfrac{c_{22}}{a_{22}} \equiv 0 \ mod \ a_{44}, \\ -\dfrac{a_{43}(a_{21}\ \ a_{32} - a_{31} a_{22}) c_{11}}{a_{11} a_{22} a_{33}} + \dfrac{a_{21} a_{42} c_{11}}{a_{11} a_{22}} - a_{41} \dfrac{c_{11}}{a_{11}} \equiv 0 \ mod \ a_{44}. \end{cases} \tag{5}$$

has no more than

$$\left(\frac{c_{11}}{a_{11}}, a_{22}\right) \cdot \left(\frac{c_{22}}{a_{22}}, a_{33}\right) \cdot \left(\frac{c_{33}}{a_{33}}, a_{44}\right) \cdot \left(\frac{c_{11}}{a_{11}}, a_{33}\right) \cdot \left(\frac{c_{22}}{a_{22}}, a_{44}\right) \cdot \left(\frac{c_{11}}{a_{11}}, a_{44}\right)$$

solutions. So, setting $a_{11} = p^i, a_{22} = p^j, a_{33} = p^k, a_{44} = p^k,\ 0 \le i \le c_1, 0 \le j \le \le c_2, 0 \le k \le c_3, 0 \le l \le c_4$, we obtain

$$\tau_4(p^n) \le \sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} \sum_{i=0}^{c_1} \sum_{j=0}^{c_2} \sum_{k=0}^{c_3} \sum_{l=0}^{c_4} p^{\gamma(c_1,c_2,c_3,c_4,i,j,k,l)},$$

where $\gamma(c_1, c_2, c_3, c_4, i, j, k, l) = \min(j,\, c_1 - i) + \min(k,\, c_2 - j) + \min(l,\, c_3 - k) + \min(k,\, c_1 - i) + \min(l,\, c_2 - j) + \min(l,\, c_1 - i)$.

Moreover, using Lemma 1, it can be shown, that the system composed from the first, second, fourth congruences of (5) coincides with the system of congruences (2), and, hence, has

$$p^{\min(c_1-i+k,\ \min(j,\, c_1-i)+\min(k,\, c_2-j)+\min(k,\, c_1-i))}$$

solutions. Similarly, the system composed from second, third, fifth congruences of (5) has

$$p^{\min(c_2-j+l,\ \min(k,\, c_2-j)+\min(l,\, c_3-k)+\min(l,\, c_2-j))}$$

solutions. Now if we denote

$$\lambda_1(i,j,k,l) := \min(c_1 - i + k,\ \min(j,\, c_1 - i) + \min(k,\, c_2 - j) + \min(k,\, c_1 - i)) + \min(l,\, c_3 - k) + \min(l,\, c_2 - j) + \min(l,\, c_1 - i),$$

$$\lambda_2(i,j,k,l) := \min(c_2 - j + l,\ \min(k,\, c_2 - j) + \min(l,\, c_3 - k) + \min(l,\, c_2 - j)) + \min(j,\, c_1 - i) + \min(k,\, c_1 - i) + \min(l,\, c_1 - i),$$

$$\lambda_3(i,j,k,l) := \min(j,\, c_1 - i) + \min(k,\, c_2 - j) + \min(k,\, c_1 - i) + \min(l,\, c_3 - k) + \min(l,\, c_2 - j) + \min(l,\, c_1 - i).$$

Then we infer

$$\tau_4(p^n) \le \sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} \sum_{i=0}^{c_1} \sum_{j=0}^{c_2} \sum_{k=0}^{c_3} \sum_{l=0}^{c_4} p^{\lambda_1(i,j,k,l)} \le$$

$$\le \sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} \sum_{i=0}^{c_1} \sum_{j=0}^{c_2} \sum_{k=0}^{c_3} \sum_{l=0}^{c_4} p^{\lambda_3(i,j,k,l)},$$

$$\tau_4(p^n) \le \sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} \sum_{i=0}^{c_1} \sum_{j=0}^{c_2} \sum_{k=0}^{c_3} \sum_{l=0}^{c_4} p^{\lambda_2(i,j,k,l)} \le$$

$$\le \sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} \sum_{i=0}^{c_1} \sum_{j=0}^{c_2} \sum_{k=0}^{c_3} \sum_{l=0}^{c_4} p^{\lambda_3(i,j,k,l)}.$$

Furthermore, we remark that

$$\begin{aligned}
&\lambda_3(i,j,k,l) \le (c_1-i+k)+(c_3-k)+(c_2-j)+(c_1-i) = \\
&= 2c_1+c_2+c_3-2i-j; \\
&\lambda_3(i,j,k,l) \le (c_1-i+k)+(c_3-k)+l+(c_1-i) = 2c_1+c_3-2i+l; \\
&\lambda_3(i,j,k,l) \le j+k+(c_3-k)+k+(c_2-j)+(c_1-i) = \\
&= c_1+c_2+c_3-i+k; \\
&\lambda_3(i,j,k,l) \le (c_2-j+l)+j+k+c_1-i = c_1+c_2-i+k+l; \\
&\lambda_3(i,j,k,l) \le (c_1-i+k)+(c_3-k)+c_2-j+l = c_1+c_2+c_3-i-j+l.
\end{aligned} \tag{6}$$

We define $\lambda(i,j,k,l) := \min(\lambda_1(i,j,k,l), \lambda_2(i,j,k,l), \lambda_3(i,j,k,l))$. Then for all prime number $p > 15$ the inequality

$$\begin{aligned}
\tau_4(p^n) \le \sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} \sum_{i=0}^{c_1}\sum_{j=0}^{c_2}\sum_{k=0}^{c_3}\sum_{l=0}^{c_4} p^{\lambda(i,j,k,l)} \le \\
\le \sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} \Bigg(\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=c_2-j}^{c_3}\sum_{l=c_2-j}^{c_4} p^{\lambda(i,j,k,l)} + \\
+\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=c_2-j}^{c_3}\sum_{l=0}^{c_2-j-1} p^{\lambda(i,j,k,l)} + \sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=0}^{c_2-j-1}\sum_{l=c_2-j}^{c_4} p^{\lambda(i,j,k,l)} + \\
+\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=0}^{c_2-j-1}\sum_{l=0}^{c_2-j-1} p^{\lambda(i,j,k,l)} + \sum_{i=0}^{c_1}\sum_{j=c_2-c_1+1}^{c_2}\sum_{k=c_2-j}^{c_3}\sum_{l=c_2-j}^{c_4} p^{\lambda(i,j,k,l)} + \\
+\sum_{i=0}^{c_1}\sum_{j=c_2-c_1+1}^{c_2}\sum_{k=c_2-j}^{c_3}\sum_{l=0}^{c_2-j-1} p^{\lambda(i,j,k,l)} + \sum_{i=0}^{c_1}\sum_{j=c_2-c_1+1}^{c_2}\sum_{k=0}^{c_2-j-1}\sum_{l=c_2-j}^{c_4} p^{\lambda(i,j,k,l)} + \\
+\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=0}^{c_2-j-1}\sum_{l=0}^{c_2-j-1} p^{\lambda(i,j,k,l)} \Bigg)
\end{aligned} \tag{7}$$

holds.

Using (6), we can estimate the sums from (7) in the following way

$$\begin{aligned}
\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=c_2-j}^{c_3}\sum_{l=0}^{c_2-j-1} p^{\lambda(i,j,k,l)} \le \sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=c_2-j}^{c_3}\sum_{l=0}^{c_2-j-1} p^{2c_1+c_3-2i+l} \ll \\
\ll \sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1} (c_3-c_1+1+i)p^{2c_1+c_3-2i+c_2-j-1} \ll (c_3-c_1+1)p^{2c_1+c_3+c_2-1};
\end{aligned}$$

$$\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=0}^{c_2-j-1}\sum_{l=c_2-j}^{c_4} p^{\lambda(i,j,k,l)} \le \sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=0}^{c_1-i-1}\sum_{l=c_2-j}^{c_4} p^{c_1+c_2+c_3-i+k} +$$

$$+\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=c_1-i}^{c_2-j-1}\sum_{l=c_2-j}^{c_4} p^{2c_1+c_2+c_3-2i-j} \ll (c_2-c_1+1)(c_4-c_1+1)p^{2c_1+c_2+c_3-1} +$$

$$+(c_2-c_1)(c_4-c_2+1)p^{2c_1+c_2+c_3};$$

$$\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=0}^{c_2-j-1}\sum_{l=0}^{c_2-j-1} p^{\lambda(i,j,k,l)} \le \sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=0}^{c_1-i-1}\sum_{l=0}^{c_2-j-1} p^{c_1+c_2-i+k+l} +$$

$$+\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=c_1-i}^{c_2-j-1}\sum_{l=0}^{c_2-j-1} p^{2c_1+c_3-2i+l} \ll p^{2c_1+2c_2-2} + (c_2-c_1)p^{2c_1+c_2+c_3-1};$$

$$\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=0}^{c_2-j-1}\sum_{l=0}^{c_2-j-1} p^{\lambda(i,j,k,l)} \le \sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=0}^{c_2-j-1}\sum_{l=0}^{c_2-j-1} p^{c_1+c_2-i+k+l} \ll$$

$$\ll \sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1} p^{c_1+3c_2-i-2j-2} \ll p^{3c_1+c_2-4}.$$

$$\sum_{i=0}^{c_1}\sum_{j=c_2-c_1+1}^{c_2}\sum_{k=c_2-j}^{c_3}\sum_{l=c_2-j}^{c_4} p^{\lambda(i,j,k,l)} \le \sum_{i=0}^{c_1}\sum_{j=c_2-c_1+1}^{c_2}\sum_{k=c_2-j}^{c_3}\sum_{l=c_2-j}^{c_4} p^{2c_1+c_2+c_3-2i-j} \ll$$

$$\ll \sum_{i=0}^{c_1}\sum_{j=c_2-c_1+1}^{c_2} (c_3-c_2+j+1)(c_4-c_2+j+1)p^{2c_1+c_2+c_3-2i-j} \ll$$

$$\ll (c_3-c_1)(c_4-c_1)p^{3c_1+c_3-1};$$

$$\sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=c_2-j}^{c_3}\sum_{l=0}^{c_2-j-1} p^{\lambda(i,j,k,l)} \le \sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1}\sum_{k=c_2-j}^{c_3}\sum_{l=0}^{c_2-j-1} p^{2c_1+c_2+c_3-2i-j} \ll$$

$$\ll \sum_{i=0}^{c_1}\sum_{j=0}^{c_2-c_1} (c_3-c_2+j+1)(c_4-c_2+j+1)p^{2c_1+c_2+c_3-2i-j} \ll$$

$$\ll (c_3-c_2+1)(c_4-c_2+1)p^{2c_1+c_2+c_3};$$

$$\sum_{i=0}^{c_1}\sum_{j=c_2-c_1+1}^{c_2}\sum_{k=0}^{c_2-j-1}\sum_{l=c_2-j}^{c_4} p^{\lambda(i,j,k,l)} \le \sum_{i=0}^{c_1}\sum_{j=c_2-c_1+1}^{c_2}\sum_{k=0}^{c_2-j-1}\sum_{l=c_2-j}^{c_4} p^{c_1+c_2+c_3-i+k} \ll$$

$$\ll \sum_{i=0}^{c_1} \sum_{j=c_2-c_1+1}^{c_2} (c_4-c_2+j+1)p^{c_1+2c_2+c_3-i-j-1} \ll (c_4-c_2+1)p^{2c_1+c_2+c_3-2};$$

$$\sum_{i=0}^{c_1} \sum_{j=c_2-c_1+1}^{c_2} \sum_{k=c_2-j}^{c_3} \sum_{l=0}^{c_2-j-1} p^{\lambda(i,j,k,l)} \le \sum_{i=0}^{c_1} \sum_{j=c_2-c_1+1}^{c_2} \sum_{k=c_2-j}^{c_3} \sum_{l=0}^{c_2-j-1} p^{c_1+c_2+c_3-i-j+l} \ll$$

$$\ll \sum_{i=0}^{c_1} \sum_{j=c_2-c_1+1}^{c_2} (c_3-c_2+j+1)p^{c_1+2c_2+c_3-i-2j-1} \ll (c_3-c_2+j+1)p^{3c_1+c_3-3};$$

Finally we obtain

$$\tau_4(p^n) \ll \sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} ((c_3-c_2+1)(c_4-c_2+1)+(c_2-c_1)(c_4-c_2+1))p^{2c_1+c_2+c_3} =$$

$$= \sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} (c_3-c_1+1)(c_4-c_2+1)p^{2c_1+c_2+c_3} \le \tag{8}$$

$$\le \sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} (c_4-c_1+1)^2 p^{2c_1+c_2+c_3}.$$

Now we use the following statement:

Lemma 2. *Let* $c_1, c_2, c_3, c_4 \in \mathbb{Z}$, $0 \le c_1 \le c_2 \le c_3 \le c_4$, $p > 15$ *be a prime number, then we have*

$$\sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0\le c_1\le c_2\le c_3\le c_4}} (c_4-c_1+1)^2 p^{2c_1+c_2+c_3} = (n-n_0+1)^2 p^{n_0}(1+O(1/p)), \tag{9}$$

where $n_0 = \begin{cases} n, & \text{if } n = 4k \\ n-1, & \text{otherwise} \end{cases}$.

Proof. Note, that $2c_1+c_2+c_3 = c_1+n-c_4$. Thus the summand with the greatest degree of p is a term with maximum of c_1-c_4. Since $c_1 \le c_4$, then $c_1-c_4 \le \le 0$. The relation $c_1-c_4=0$ holds only if $c_1=c_2=c_3=c_4=:k$, in that case $n = 2c_1+c_2+c_3 = 4k$. If $n \not\equiv 0 \mod 4$, then $c_1-c_4 \le -1$. And equality takes place: for $c_1=c_2=c_3=k, c_4=k+1$ (if $n=4k+1$), for $c_1=c_2=k, c_3=c_4=k+1$ (if $n=4k+2$), for $c_1=k, c_2=c_3=c_4=k+1$ (if $n=4k+3$).

Since $0 \le c_1 \le c_2 \le c_3 \le c_4$, then the values $a_2 := c_2-c_1$, $a_3 := c_3-c_2$, $a_4 := c_4-c_3$ are nonnegative, and the set of (c_1,c_2,c_3,c_4), where $0 \le c_1 \le c_2 \le c_3 \le \le c_4$, $c_1+c_2+c_3+c_4=n$, one-at-one corresponds to the set of (c_1,a_2,a_3,a_4), where $0 \le c_1,a_2,a_3,a_4$, $4c_1+3a_2+2a_3+a_4=n$.

Let Θ_t denotes the set of (c_1,c_2,c_3,c_4), for which $0 \le c_1 \le c_2 \le c_3 \le c_4$, $c_1+ +c_2+c_3+c_4=n$ and $2c_1+c_2+c_3=t$. Besides, Ω_t denotes the set of (c_1,a_2,a_3,a_4), for which $c_1,a_2,a_3,a_4 \ge 0$, $4c_1+3a_2+2a_3+a_4=n$ and $4c_1+2a_2+a_3=t$. Let's show that $\frac{|\Theta_{t-1}|}{|\Theta_t|} \leq 15$. But, there are one-at-one correspondence between Θ_t and Ω_t, so, $\frac{|\Theta_{t-1}|}{|\Theta_t|} = \frac{|\Omega_{t-1}|}{|\Omega_t|}$.

Note, that the Ω_{t-1} can be presented as sum of sets $\Omega_1 = \{(c_1, a_2, a_3, a_4) \in \in \Omega_{t-1} | \, a_2 \geq 2\}$, $\Omega_2 = \{(c_1, a_2, a_3, a_4) \in \Omega_{t-1} | \, a_2 < 2, \ a_3 \geq 3\}$, $\Omega_3 = \{(c_1, a_2, a_3, a_4) \in \in \Omega_{t-1} | \, a_2 < 2, \ a_3 < 3, \ a_4 \geq 2\}$ and $\Omega_4 = \Omega_{t-1}/(\Omega_1 \cup \Omega_2 \cup \Omega_3)$.

Let's introduce the following mappings:

1) $\varphi_1 : \ \Omega_1 \to \Omega_t, \ \varphi_1(c_1, a_2, a_3, a_4) = (c_1 + 1, a_2 - 2, a_3 + 1, a_4) \in \Omega_t$;

2) $\varphi_2 : \ \Omega_2 \to \Omega_t, \ \varphi_2(c_1, a_2, a_3, a_4) = (c_1, a_2 + 2, a_3 - 3, a_4) \in \Omega_t$;

3) $\varphi_3 : \ \Omega_3 \to \Omega_t, \ \varphi_3(c_1, a_2, a_3, a_4) = (c_1, a_2, a_3 + 1, a_4 - 2) \in \Omega_t$.

Since the mappings $\varphi_1, \ \varphi_2, \ \varphi_3$ are injective, then $|\varphi_1(\Omega_1)| = |\Omega_1|$, $|\varphi_2(\Omega_2)| = |\Omega_2|, \ |\varphi_3(\Omega_3)| = |\Omega_3|$. Without loss of generality, we can suppose, that

$$\max(|\varphi_1(\Omega_1)|, \ |\varphi_2(\Omega_2)|, \ |\varphi_3(\Omega_3)|) = |\varphi_1(\Omega_1)|.$$

Let's consider the elements of Ω_4 in more detail. If $(c_1, a_2, a_3, a_4) \in \Omega_4$, then for c_1, a_2, a_3, a_4 the following conditions

$$4c_1 + 3a_2 + 2a_3 + a_4 = n, a_2 < 2, a_3 < 3, a_4 < 2.$$

hold. Hence, $|\Omega_4| \leq 12$.

Finally we have

$$\frac{|\Theta_{t-1}|}{|\Theta_t|} = \frac{|\Omega_{t-1}|}{|\Omega_t|} = \frac{|\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4|}{|\Omega_t|} \leq$$

$$\leq \frac{|\Omega_1| + |\Omega_2| + |\Omega_3| + |\Omega_4|}{|\varphi_1(\Omega_1)|} \leq \frac{3|\Omega_1| + 12}{|\Omega_1|} \leq 15.$$

Hence, for the sum (9) we have inequality

$$\sum_{\substack{c_1+c_2+c_3+c_4=n \\ 0 \leq c_1 \leq c_2 \leq c_3 \leq c_4}} (c_4 - c_1 + 1)^2 p^{2c_1+c_2+c_3} \leq \sum_{t=0}^{n_0} (n - t + 1)^2 15^{n_0 - t} p^t =$$

$$= p^{n_0} \sum_{t=0}^{n_0} (n - t + 1)^2 (\frac{15}{p})^{n_0 - t} = (n - n_0 + 1)^2 p^{n_0} (1 + O(1/p)),$$

which required. ■

Thus, we proved the relation

$$t_4(p^n) \ll (n - n_0 + 1)^2 p^{n_0} (1 + O(1/p)),$$

where $p > 15, \ n_0 = \begin{cases} n, \text{ if } \ n = 4k \\ n - 1, \text{ otherwise} \end{cases}$.

Note, that using (5), it can be shown, that matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a_{31} & a_{32} & p^k & 0 \\ a_{41} & a_{42} & 0 & p^k \end{pmatrix}, \ 0 \leq a_{31}, a_{32}, a_{41}, a_{42} < p^k,$$

is a divisor of the matrix

$$\begin{pmatrix} p^k & 0 & 0 & 0 \\ 0 & p^k & 0 & 0 \\ 0 & 0 & p^k & 0 \\ 0 & 0 & 0 & p^k \end{pmatrix}.$$

Therefore $t_4(p^{4k}) > p^{4k}$, and for $n = 4k$ the following equality

$$t_4(p^n) = p^n(1 + O(1/p))$$

holds. So, we proved the following statement

Lemma 3. *Let $p > 15$ be a prime number, $n \in \mathbb{N}$, then the estimate*

$$t_4(p^n) \ll (n - n_0 + 1)^2 p^{n_0}(1 + O(1/p)), \tag{10}$$

holds, where $n_0 = \begin{cases} n, & \text{if } n = 4k \\ n-1, & \text{otherwise} \end{cases}$*, moreover, for $n = 4k$ we have*

$$t_4(p^n) = p^n(1 + O(1/p)), \tag{11}$$

(The constant in symbol "O" is absolute.)

Besides, we will use some well-known statements about Rieman ζ-function:

Lemma 4. *For $1/2 \le \sigma \le 1$, $|t| > 1$ the following estimates*

$$|\zeta(\sigma + it)| \ll t^{\frac{1}{3}(1-\sigma)}(\log(|t| + 3))^{2\sigma-1},$$

$$\int_1^T |\zeta(\frac{1}{2} + it)|^2 dt \ll T \log T$$

hold.

Proof of the Theorem 1

Let's consider a summatory function $T_4(x) := \sum\limits_{n \le x} t_4(n)$. Using lemma 3, we can obtain the generating Dirichlet series for $t_4(n)$, $Re\, s \ge 2$:

$$F(s) = \sum_{n=1}^{\infty} \frac{t_4(n)}{n^s} = \prod_p (1 + \frac{t_4(p)}{p^s} + \frac{t_4(p^2)}{p^{2s}} + ...) =$$

$$= \prod_p (1 + \frac{2}{p^s} + \frac{p+6}{p^{2s}} + \frac{2p^2 + 4p + 12}{p^{3s}} + \frac{p^4 + 3p^3 + 9p^2 + 12p + 25}{p^{4s}} +$$

$$+ \frac{2p^4 + 8p^3 + 20p^2 + 20p + 36}{p^{5s}} + ...) = \prod_p (1 - \frac{1}{p^{4s-4}})^{-1} \cdot (1 + \frac{2}{p^s} + \frac{1}{p^{2s-1}} + ...) \cdot (1 - \frac{1}{p^{4s-4}}) =$$

$$= \zeta(4s - 4) G_1(s),$$

where the Dirichlet series for $G_1(s)$ is absolutely convergent for $Re\, s > 1$. By using the Perron's summation formula, we have:

$$T_4(x) = \sum_{n\le x} t_4(n) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \zeta(4s-4)G_1(s)\frac{x^s}{s}ds + O\Big(\frac{x^b}{T(b-1)}\Big) + O\Big(\frac{xA(2x)\log x}{T}\Big).$$

Taking into account lemma 3, we can think that $t_4(n) < Cn,\ C > 1$, and suppose $A(x) := Cx,\ b := 2$.

Then, shifting the line of integration on the line $Re\, s = 9/8$ and taking into account that the function $F(s)$ has pole of 1-st order at the point $s = \frac{5}{4}$, we obtain

$$\int_{b-iT}^{b+iT} \zeta(4s-4)G_1(s)\frac{x^s}{s}ds = res_{5/4}\zeta(4s-4)G_1(s)\frac{x^s}{s} + \int_{9/8+iT}^{b+iT} \zeta(4s-4)G_1(s)\frac{x^s}{s}ds +$$

$$+\int_{9/8-iT}^{9/8+iT} \zeta(4s-4)G_1(s)\frac{x^s}{s}ds - \int_{9/8-iT}^{b-iT} \zeta(4s-4)G_1(s)\frac{x^s}{s}ds;$$

$$res_{5/4}\zeta(4s-4)G_1(s)\frac{x^s}{s} = C_1x^{5/4},\ \text{where } C_1 = \frac{4}{5}G_1(\frac{5}{4}).$$

We can estimate integrals from upper equality in the following way:

$$\left|\int_{9/8+iT}^{b+iT} \zeta(4s-4)G_1(s)\frac{x^s}{s}ds\right| \ll \int_{9/8}^{b} |\zeta(4\sigma-4+4iT)|\frac{x^\sigma}{|\sigma+iT|}d\sigma \ll$$

$$\ll \frac{1}{T}\int_{9/8}^{b} |\zeta(4\sigma-4+4iT)|x^\sigma d\sigma \ll \frac{1}{T}\int_{1/2}^{4b-4} |\zeta(u+4iT)|x^{1+\frac{u}{4}}du \ll$$

$$\ll \frac{1}{T}\int_{1/2}^{4b-4} T^{\frac{1}{3}(1-u)}\log T x^{1+\frac{u}{4}}du \ll \frac{x\log T}{T^{2/3}}\int_{1/2}^{4b-4}\left(\frac{x^{1/4}}{T^{1/3}}\right)^u du \ll \frac{x^{5/4+\epsilon}}{T^{2/3}};$$

$$\left|\int_{9/8-iT}^{9/8+iT} \zeta(4s-4)G_1(s)\frac{x^s}{s}ds\right| \ll \int_{-T}^{T} |\zeta(\frac{1}{2}+4it)|\frac{x^{9/8}}{|\frac{9}{8}+it|}dt \ll x^{9/8}\int_{1}^{T} |\zeta(\frac{1}{2}+4it)|\frac{dt}{t} \ll$$

$$\ll x^{9/8}\log T \max_{1\le Q\le T/2}\frac{1}{Q}\int_{Q}^{2Q} |\zeta(\frac{1}{2}+4it)|dt \ll$$

$$\ll x^{9/8}\log T \max_{1\le Q\le T/2}\frac{1}{Q}\left(\int_{1}^{2Q} |\zeta(\frac{1}{2}+4it)|^2dt\right)^{1/2}\left(\int_{1}^{2Q} dt\right)^{1/2} \ll$$

$$\ll x^{9/8}\log T \max_{1\le Q\le T/2}\frac{1}{Q}2Q\log^{1/2}2Q \ll x^{9/8}\log^2 T.$$

Now setting $T = x^2$, we have

$$T_4(x) = \sum_{n\le x} t_4(n) = C_1 x^{5/4} + O(x^{9/8}\log^2 x), \tag{12}$$

The proof of Theorem 1 is complete. ■

Proof of the Theorem 2

By the analogy with $t_4(n)$, we can define $t_k(n) := \sum_{|C|=n} \tau_k(C)$, where $C \in M_k(\mathbb{Z})$. Let's find an estimation of the sum

$$T_k^*(x) := \sum_{n\le x}{}' t_k(n),$$

where the sign $'$ in sum $\sum'$ indicates that the summation runs over all square-free numbers.

Note, that

$$T_k^*(x) = \sum_{n\le x}{}' t_k(n) = \sum_{n\le x} \mu^2(n) t_k(n). \tag{13}$$

It is easy to obtain the generating Dirichlet series for $Re\, s > 1$

$$F(s) = \sum_{n=1}^{\infty}\frac{\mu^2(n)t_k(n)}{n^s} = \prod_p \left(1 + \frac{\mu^2(p)t_k(p)}{p^s} + \frac{\mu^2(p^2)t_k(p^2)}{p^{2s}} + \dots\right) = \prod_p \left(1 + \frac{2}{p^s}\right) =$$

$$= \zeta^2(s)\prod_p\left(1+\frac{2}{p^s}\right)\left(1-\frac{2}{p^s}+\frac{1}{p^{2s}}\right) = \zeta^2(s)\prod_p\left(1-\frac{3}{p^{2s}}+\frac{2}{p^{3s}}\right)\frac{(1+\frac{1}{p^{2s}})^3}{(1+\frac{1}{p^{2s}})^3} =$$

$$= \frac{\zeta^2(s)\zeta^3(2s)}{\zeta^3(4s)}\prod_p\left(1+\frac{2}{p^{3s}}-\frac{6}{p^{4s}}+\frac{6}{p^{5s}}-\frac{8}{p^{6s}}+\frac{6}{p^{7s}}-\frac{3}{p^{8s}}+\frac{2}{p^{9s}}\right) = \frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s),$$

where the Dirichlet series for $G_2(s)$ is absolutely convergent for $Re\, s > 1/4$.

By using the Perron's summation formula, we have:

$$T_k^*(x) = \frac{1}{2\pi i}\int_{b-iT}^{b+iT}\frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s)\frac{x^s}{s}ds + O\left(\frac{x^b}{T(b-1)}\right) + O\left(\frac{xA(2x)\log x}{T}\right).$$

We can suppose, that for our case $A(x) = x^{\varepsilon_0}$, $\varepsilon_0 > 0$, $b = 1 + \frac{1}{\log x}$. Then, shifting the line of integration on the line $Re\, s = 1/2$ and going around the point $s = 1/2$ on the left, it can be obtained the following relation

$$\int_{b-iT}^{b+iT}\frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s)\frac{x^s}{s}ds = res_1\frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s) +$$

$$+\int_{1/2-iT}^{b-iT}\frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s)\frac{x^s}{s}ds+\int_{1/2-iT}^{1/2+iT}\frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s)\frac{x^s}{s}ds-$$

$$-\int_{1/2+iT}^{b+iT}\frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s)\frac{x^s}{s}ds;$$

$$res_1\frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s)\frac{x^s}{s}=xP_2(\log x);$$

$$res_{1/2}\frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s)\frac{x^s}{s}=x^{1/2}P_3(\log x)=O(x^{1/2}\log^5 x).$$

We can estimate integrals from upper equality in the following way:

$$\left|\int_{1/2+iT}^{b+iT}\frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s)\frac{x^s}{s}ds\right|\ll\int_{1/2}^{b}|\zeta^2(\sigma+iT)|\frac{x^\sigma}{|\sigma+iT|}d\sigma\ll$$

$$\ll\frac{1}{T}\int_{1/2}^{b}|\zeta(\sigma+iT)|^2x^\sigma d\sigma\ll\frac{\log T}{T^{1/3}}\int_{1/2}^{b}\left(\frac{x}{T^{2/3}}\right)^\sigma d\sigma\ll\frac{x^b\log T}{T};$$

$$\left|\int_{1/2-iT}^{1/2+iT}\frac{\zeta^2(s)\zeta^3(2s)\zeta^2(3s)}{\zeta^3(4s)}G_2(s)\frac{x^s}{s}ds\right|\ll\int_{1}^{T}|\zeta(\frac{1}{2}+it)|^2|\zeta(1+it)|^3\frac{x^{1/2}}{|1/2+it|}dt+$$

$$+\int_{-1}^{1}\frac{x^{7/16}}{|7/16+it|}dt+\int_{7/16}^{1/2}\frac{x^\sigma}{|\sigma+i|}d\sigma\ll x^{1/2}\log^3 T\int_{1}^{T}|\zeta(\frac{1}{2}+it)|^2\frac{dt}{t}\ll x^{1/2}\log^5 T.$$

Now setting $T=x$, we have

$$T_k^*(x)=\sum_{n\le x}{}'t_k(n)=\sum_{n\le x}\mu^2(n)t_k(n)=xP_2(\log x)+O(x^{1/2}\log^5 x).\qquad(14)$$

The proof of Theorem 2 is complete. ■

Conclusion.

We obtained estimates of functions $T_4(x)$ and $T_k^*(x)$ with the help of Perron's summation formula. Using the estimates in the asymmetric divisor problem, the orders of the remainder terms for $T_k(x),\ k=3,4$ probably can be improved.

1. **Bhowmik G.** On the number of subgroups of finite Abelian groups [text] / G. Bhowmik, H. Menzer // Abh. Math. Sem. Univ. Hamburg. – 1997. – № 67. – P. 117–121.
2. **Liu H.-Q.** Divisor problems of 4 and 3 dimensions [text] / H.-Q. Liu // Acta Arith. – 1995. – № 73. – P. 249–269.
3. **Menzer H.** On the number of subgroups of finite Abelian groups [text] / H. Menzer // Proc. Conf. Analytic and Elementary Number Theory, July 18-20 1996. – 1996. – P. 181–188.
4. **Bhowmik G.** On the asymptotic behaviour of number of subgroups of finite abelian groups [text] / G. Bhowmik, J. Wu // Archiv der Mathematik. – 1997. – № 69. – P. 95–104.
5. **Ivič A.** On the number of subgroups of finite abelian groups [text] / A. Ivič // Theorie Nombres de Bordeaux. – 1997. – № 9. – P. 371–381.
6. **Fugelo N.** Average orders of divisor function of integer matrices in $M_3(\mathbb{Z})$ [text] / N. Fugelo, I. Velichko // Annales Univ. Sci. Budapest. – 2008. – № 28. – P. 261–270.
7. **Ivič A.** The Riemann zeta-function [text] / A. Ivič. – New York: John Wiley & Sons, 1985. – 562 p.