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**ON A ONE CLASS OF THE SOLUTIONS OF THE NONLINEAR
FIRST-ORDER DIFFERENTIAL EQUATION
WITH OSCILLATING COEFFICIENTS**

Щоголев С. А. Про один клас розв’язків нелінійного диференціального рівняння першого порядку з коливними коефіцієнтами. Для нелінійного диференціального рівняння першого порядку, коефіцієнти якого зображувані у вигляді абсолютно та рівномірно збіжних рядів Фур’є з повільно змінними коефіцієнтами та частотою, отримано умови існування частинного розв’язку аналогічної структури в резонансному випадку.

Ключові слова: диференціальний, повільно змінний, ряди Фур’є.

Щёголев С. А. Об одном классе решений нелинейного дифференциального уравнения первого порядка с осциллирующими коэффициентами. Для нелинейного дифференциального уравнения первого порядка, коэффициенты которого представимы в виде абсолютно и равномерно сходящихся рядов Фурье с медленно меняющимися коэффициентами и частотой, получены условия существования частного решения аналогичной структуры в резонансном случае.

Ключевые слова: дифференциальный, медленно меняющийся, ряды Фурье.

Shchogolev S. On a one class of the solutions of the nonlinear first-order differential equation with oscillating coefficients. For the nonlinear first-order differential equation, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency, the conditions of existence of the particular solution of analogous structure are obtained at resonance case.

Key words: differential, slowly-varying, Fourier series.

INTRODUCTION. This paper is a continuation of research initiated in paper [1]. Here we using the definitions and designations from [1]. In this paper are considered the next system of the differential equations:

$$\frac{dx_j}{dt} = \sum_{k=1}^2 a_{jk}(t, \varepsilon)x_k + f_j(t, \varepsilon, \theta(t, \varepsilon)) + \mu X_j(t, \varepsilon, \theta(t, \varepsilon), x_1, x_2), \quad j = 1, 2, \quad (1)$$

where $t, \varepsilon \in G(\varepsilon_0) = \{t, \varepsilon : 0 < \varepsilon < \varepsilon_0, -L\varepsilon^{-1} \leq t \leq L\varepsilon^{-1}, 0 < L < +\infty\}$, $\text{colon}(x_1, x_2) \in D \subset \mathbf{R}^2$, $a_{jk} \in S(m, \varepsilon_0)$, $f_j \in F(m, l, \varepsilon_0, \theta)$, $X_1, X_2 \in F(m, l, \varepsilon_0, \theta)$ with respect t, ε, θ and analytic with respect $x_1, x_2 \in D$; $\mu \in (0, \mu_0) \subset \mathbf{R}^+$. Functions a_{jk} , f_j , X_j ($j, k = 1, 2$) are real, and eigenvalues of matrix $(a_{jk}(t, \varepsilon))$ have a form $\pm i\omega(t, \varepsilon)$, where $\omega \in \mathbf{R}^+$.

In paper [1] the conditions of existence of the particular solutions belongs to class $F(m^*, l^*, \varepsilon^*, \theta)$ ($m^* \leq m, l^* \leq l, \varepsilon^* \leq \varepsilon_0$) are obtained (the definitions of classes $S(m, \varepsilon_0)$, $F(m, l, \varepsilon_0, \theta)$ given in [1]). It was assumed that the conditions:

$$\inf_{G(\varepsilon_0)} |a_{12}(t, \varepsilon)| > 0,$$

$$\inf_{G(\varepsilon_0)} |k\omega(t, \varepsilon) - n\varphi(t, \varepsilon)| \geq \gamma > 0, \quad k = 1, 2; \quad n \in \mathbf{Z},$$

$\varphi(t, \varepsilon) = d\theta/dt$, means considered noresonance case. The purpose of this paper is to obtain analogous results in resonance case, means when eigenvalues of matrix $(a_{jk}(t, \varepsilon))$ have a form $\pm ir\varphi(t, \varepsilon)$, $r \in \mathbf{N}$. In order to simplify the presentation instead of system (1) we consider the first-order differential equation of special kind. The results for this equation can be easily extended to a system (1) and to the same systems of the more general kind [2].

MAIN RESULTS

1. Statement of the Problem. We consider the next first-order differential equation:

$$\frac{dx}{dt} = f(t, \varepsilon, \theta(t, \varepsilon)) + \mu X(t, \varepsilon, \theta(t, \varepsilon), x), \quad (2)$$

where $t, \varepsilon \in G(\varepsilon_0)$, $|x| \leq d < +\infty$, $f \in F(m, l, \varepsilon_0, \theta)$, $X \in F(m, l, \varepsilon_0, \theta)$ with respect t, ε, θ and analytic with respect x , at $|x| \leq d$.

We study the problem of existence of the particular solutions of the classes $F(m^*, l^*, \varepsilon^*, \theta)$ ($m^* \leq m, l^* \leq l, \varepsilon^* \leq \varepsilon_0$) of the equation (2).

2. Auxiliary results.

Lemma 1. *Suppose we are given the following linear first-order differential equation*

$$\frac{dx}{dt} = \lambda(t, \varepsilon)x + u(t, \varepsilon, \theta(t, \varepsilon)), \quad (3)$$

where $\lambda \in S(m, \varepsilon_0)$, $u \in F(m, l, \varepsilon_0, \theta)$. Let condition:

$$\inf_{G(\varepsilon_0)} |\operatorname{Re}\lambda(t, \varepsilon)| = \gamma > 0. \quad (4)$$

Then the equation (3) has a particular solution $x(t, \varepsilon, \theta) \in F(m, l, \varepsilon_0, \theta)$ for any function $u \in F(m, l, \varepsilon_0, \theta)$, and exists $K_0 \in (0, +\infty)$ such that

$$\|x\|_{F(m, l, \varepsilon_0, \theta)} \leq \frac{K_0}{\gamma} \|u\|_{F(m, l, \varepsilon_0, \theta)}. \quad (5)$$

Proof. We represent the function u in the form of Fourier-series:

$$u(t, \varepsilon, \theta) = \sum_{n=-\infty}^{\infty} u_n(t, \varepsilon) \exp(in\theta).$$

The desired solution will be sought in the form of a Fourier series:

$$x(t, \varepsilon, \theta) = \sum_{n=-\infty}^{\infty} x_n(t, \varepsilon) \exp(in\theta). \quad (6)$$

Then for coefficients $x_n(t, \varepsilon)$ we obtain the following differential equations:

$$\frac{dx_n}{dt} = \sigma_n(t, \varepsilon)x_n + u_n(t, \varepsilon), \quad n \in \mathbf{Z}, \quad (7)$$

where $\sigma_n(t, \varepsilon) = \lambda(t, \varepsilon) - in\varphi(t, \varepsilon)$.

We consider the following solution of equation (7):

$$x_n(t, \varepsilon) = \int_{\pm \frac{L}{\varepsilon}}^t u_n(\tau, \varepsilon) \exp \left(\int_{\tau}^t \sigma_n(s, \varepsilon) ds \right) d\tau, \quad (8)$$

where the sign in lower limit of integration coincides with the sign of $\operatorname{Re}\lambda(t, \varepsilon)$.

We consider the case $m = 0$ and $\operatorname{Re}\lambda(t, \varepsilon) \leq -\gamma < 0$. We have:

$$\begin{aligned} x_n(t, \varepsilon) &= \int_{-\frac{L}{\varepsilon}}^t u_n(\tau, \varepsilon) \exp \left(\int_{\tau}^t \sigma_n(s, \varepsilon) ds \right) d\tau, \\ \sup_{G(\varepsilon_0)} |x_n(t, \varepsilon)| &\leq \sup_{G(\varepsilon_0)} |u_n(t, \varepsilon)| \int_{-\frac{L}{\varepsilon}}^t \exp \left(\int_{\tau}^t \operatorname{Re}\lambda(s, \varepsilon) ds \right) d\tau \leq \\ &\leq \sup_{G(\varepsilon_0)} |u_n(t, \varepsilon)| \int_{-\frac{L}{\varepsilon}}^t \exp(-\gamma(t - \tau)) d\tau = \\ &= \frac{1}{\gamma} \sup_{G(\varepsilon_0)} |u_n(t, \varepsilon)| \left(1 - \exp \left(-\gamma \left(t + \frac{L}{\varepsilon} \right) \right) \right) < \frac{1}{\gamma} \sup_{G(\varepsilon_0)} |u_n(t, \varepsilon)|. \end{aligned} \quad (9)$$

It is easy to show that a similar estimate holds in the case $\operatorname{Re}\lambda(t, \varepsilon) \geq \gamma > 0$. Thus in case $m = 0$ Lemma are proved. For the case $m \geq 1$ using arguments similar to those given in [3], and using estimation (9), we obtain the Lemma.

We suppose, that

$$\int_0^{2\pi} f(t, \varepsilon, \theta) d\theta = 0 \quad \forall (t, \varepsilon) \in G(\varepsilon_0). \quad (10)$$

We consider the function:

$$\xi_0(t, \varepsilon, \theta) = M_0(t, \varepsilon) + \tilde{\xi}(t, \varepsilon, \theta),$$

where

$$\tilde{\xi}(t, \varepsilon, \theta) = L[f(t, \varepsilon, \theta)] = \sum_{\substack{\nu=-\infty \\ (\nu \neq 0)}}^{\infty} \frac{\Gamma_{\nu}[f]}{in\nu} \exp(in\theta),$$

and function $M_0(t, \varepsilon)$ are defined as the root of equation:

$$P(t, \varepsilon, M) = \int_0^{2\pi} X(t, \varepsilon, \theta, M + \tilde{\xi}(t, \varepsilon, \theta)) d\theta = 0. \quad (11)$$

Lemma 2. *Let the equation (2) such that:*

- 1) *the function $f(t, \varepsilon, \theta)$ satisfy condition (10);*

2) the equation (11) has a root $M_0(t, \varepsilon)$ such that

$$\inf_{G(\varepsilon_0)} \left| \operatorname{Re} \frac{\partial P(t, \varepsilon, M_0)}{\partial M} \right| = \gamma_0 > 0. \quad (12)$$

Then exists $\mu_1 \in (0, \mu_0)$ such that $\forall \mu \in (0, \mu_1)$ exists the non-degenerate transformation of kind

$$x = \psi_1(t, \varepsilon, \theta, \mu) + \psi_2(t, \varepsilon, \theta, \mu)y,$$

where $\psi_1, \psi_2 \in F(m, l, \varepsilon_0, \theta)$, reducing the equation (2) to kind:

$$\begin{aligned} \frac{dy}{dt} = & \mu \lambda_0(t, \varepsilon)y + \mu^2 r(t, \varepsilon, \theta, \mu)y + \mu \varepsilon v(t, \varepsilon, \theta, \mu)y + \\ & + \varepsilon c(t, \varepsilon, \theta, \mu) + \mu^2 d(t, \varepsilon, \theta, \mu) + \mu Y(t, \varepsilon, \theta, y, \mu), \end{aligned} \quad (13)$$

where $\lambda_0 \in S(m, \varepsilon_0)$, $r, d \in F(m, l, \varepsilon_0, \theta)$, $v \in F(m-1, l, \varepsilon_0, \theta)$, function Y belong to class $F(m, l, \varepsilon_0, \theta)$ with respect t, ε, θ and contain terms not lower than second order with respect y .

Proof. We make in the equation (2) the substitution:

$$x = \xi_0(t, \varepsilon, \theta) + z, \quad (14)$$

where z – the new unknown function, for which we obtain the equation:

$$\frac{dz}{dt} = \varepsilon g(t, \varepsilon, \theta) + \mu h(t, \varepsilon, \theta) + \mu p(t, \varepsilon, \theta)z + \mu Z(t, \varepsilon, \theta, z, \mu), \quad (15)$$

where

$$\begin{aligned} g = & -\frac{1}{\varepsilon} \frac{\partial \xi_0}{\partial t} \in F(m-1, l, \varepsilon_0, \theta), \quad h = X(t, \varepsilon, \theta, M_0 + \tilde{\xi}) \in F(m, l, \varepsilon_0, \theta), \\ p = & \frac{\partial X(t, \varepsilon, \theta, M_0 + \tilde{\xi})}{\partial x} \in F(m, l, \varepsilon_0, \theta), \quad Z = \frac{1}{2} \frac{\partial^2 X(t, \varepsilon, \theta, \xi_0 + \nu z)}{\partial x^2} z^2 \quad (0 < \nu < 1). \end{aligned}$$

By condition (11) we have:

$$\Gamma_0[h(t, \varepsilon, \theta)] \equiv 0.$$

We make in equation (15) the substitution:

$$z = \mu z_0(t, \varepsilon, \theta) + \tilde{z}, \quad (16)$$

where $z_0 = L[h(t, \varepsilon, \theta)] \in F(m, l, \varepsilon_0, \theta)$, and \tilde{z} – new unknown function. We obtain:

$$\begin{aligned} \frac{d\tilde{z}}{dt} = & \varepsilon c_1(t, \varepsilon, \theta, \mu) + \mu^2 d_1(t, \varepsilon, \theta, \mu) + \mu p(t, \varepsilon, \theta)\tilde{z} + \\ & + \mu^2 q(t, \varepsilon, \theta, \mu)\tilde{z} + \mu \tilde{Z}(t, \varepsilon, \theta, \tilde{z}, \mu), \end{aligned} \quad (17)$$

where

$$c_1 = -\frac{\mu}{\varepsilon} \frac{\partial z_0}{\partial t} + g \in F(m-1, l, \varepsilon_0, \theta), \quad d_1 = p z_0 + \frac{1}{\mu} Z(t, \varepsilon, \theta, \mu z_0, \mu) \in F(m, l, \varepsilon, \theta),$$

$$q = \frac{1}{\mu} \frac{\partial Z(t, \varepsilon, \theta, \mu z_0, \mu)}{\partial z} \in F(m, l, \varepsilon_0, \theta), \quad \tilde{Z} = \frac{1}{2} \frac{\partial^2 Z(t, \varepsilon, \theta, \mu z_0 + \nu_1 \tilde{z}, \mu)}{\partial z^2} \tilde{z}^2 \quad (0 < \nu_1 < 1).$$

We make in equation (17) the transformation:

$$\tilde{z} = (1 + \mu \tilde{\psi}(t, \varepsilon, \theta))y, \quad (18)$$

where $\tilde{\psi} = L[p(t, \varepsilon, \theta)]$. For sufficiently small μ this transformation is non-degenerate, and as result of its application we obtain the equation (13), in which:

$$\lambda_0(t, \varepsilon) = \Gamma_0[p(t, \varepsilon, \theta)], \quad (19)$$

$$c = (1 + \mu \tilde{\psi})^{-1} c_1, \quad d = (1 + \mu \tilde{\psi})^{-1} d_1, \quad r = (1 + \mu \tilde{\psi})^{-1} (p \tilde{\psi} - q(1 + \mu \tilde{\psi}) - \tilde{\psi} \lambda_0),$$

$$v = -\frac{1}{\varepsilon} (1 + \mu \tilde{\psi})^{-1} \frac{\partial \tilde{\psi}}{\partial t}, \quad Y = (1 + \mu \tilde{\psi})^{-1} \tilde{Z}(t, \varepsilon, \theta, (1 + \tilde{\psi})y, \mu).$$

Lemma 2 are proved.

3. Principal results.

Theorem. *Let the equation (2) satisfy conditions of Lemma 2. Then exists $\mu_2 \in (0, \mu_0)$, $\varepsilon_1(\mu) \in (0, \varepsilon_0)$ such that $\forall \mu \in (0, \mu_2)$, $\varepsilon \in (0, \varepsilon_1(\mu))$ the equation (2) has a particular solution $x(t, \varepsilon, \theta, \mu) \in F(m-1, l, \varepsilon_1(\mu), \theta)$.*

Proof. Based on Lemma 2 for sufficiently small μ we reduce the equation (2) to the equation (13). In equation (13) we make the substitution:

$$y = \frac{\varepsilon + \mu^2}{\mu} \tilde{y}, \quad (20)$$

where \tilde{y} – new unknown function. Since the function Y contain the terms njt lower the second order with respect y , we obtain:

$$\begin{aligned} \frac{d\tilde{y}}{dt} &= \mu \lambda_0(t, \varepsilon) \tilde{y} + \mu^2 r(t, \varepsilon, \theta, \mu) \tilde{y} + \mu \varepsilon v(t, \varepsilon, \theta, \mu) \tilde{y} + \\ &+ \frac{\varepsilon \mu}{\varepsilon + \mu^2} c(t, \varepsilon, \theta, \mu) + \frac{\mu^3}{\varepsilon + \mu^2} d(t, \varepsilon, \theta, \mu) + (\varepsilon + \mu^2) \tilde{Y}(t, \varepsilon, \theta, \tilde{y}, \mu). \end{aligned} \quad (21)$$

Consider corresponding to equation (21) the linear nonhomogeneous equation:

$$\frac{d\tilde{y}_0}{dt} = \mu \lambda_0(t, \varepsilon) \tilde{y}_0 + \frac{\varepsilon \mu}{\varepsilon + \mu^2} c(t, \varepsilon, \theta, \mu) + \frac{\mu^3}{\varepsilon + \mu^2} d(t, \varepsilon, \theta, \mu). \quad (22)$$

Based on (19) and condition (12) we have:

$$\inf_{G(\varepsilon_0)} |\operatorname{Re} \lambda_0(t, \varepsilon)| > 0.$$

Then based on Lemma 1 the equation (22) has a particular solution $\tilde{y}_0(t, \varepsilon, \theta, \mu) \in F(m-1, l, \varepsilon_0, \theta)$, and exists $K_1 \in (0, +\infty)$ such that:

$$\|\tilde{y}_0\|_{F(m-1, l, \varepsilon_0, \theta)} \leq K_1 \left(\frac{\varepsilon}{\varepsilon + \mu^2} \|c\|_{F(m-1, l, \varepsilon_0, \theta)} + \frac{\mu^2}{\varepsilon + \mu^2} \|d\|_{F(m-1, l, \varepsilon_0, \theta)} \right).$$

We construct the process of successive approximations, defining as initial approximation \tilde{y}_0 , and subsequent approximations defining as solutions from class $F(m-1, l, \varepsilon_0, \theta)$ of the equations:

$$\begin{aligned} \frac{d\tilde{y}_{s+1}}{dt} = & \mu\lambda_0(t, \varepsilon)\tilde{y}_{s+1} + \mu^2 r(t, \varepsilon, \theta, \mu)\tilde{y}_s + \mu\varepsilon v(t, \varepsilon, \theta, \mu)\tilde{y}_s + \frac{\varepsilon\mu}{\varepsilon + \mu^2} c(t, \varepsilon, \theta, \mu) + \\ & + \frac{\mu^3}{\varepsilon + \mu^2} d(t, \varepsilon, \theta, \mu) + (\varepsilon + \mu^2) \tilde{Y}(t, \varepsilon, \theta, \tilde{y}_s, \mu), \quad s = 0, 1, 2, \dots \end{aligned} \quad (23)$$

Using techniques contraction mapping principle [4] it is easy to show that exists $\mu_2 \in (0, \mu_0)$ and $\varepsilon_2(\mu) = K_2\mu$, where K_2 – sufficiently small constant, such that $\forall \mu \in (0, \mu_2), \forall \varepsilon \in (0, \varepsilon_2(\mu))$ the process (23) converges to the solution $\tilde{y}(t, \varepsilon, \theta, \mu)$ of the equation (21), From its based on (21) and Lemma 2 we obtain the theorem.

CONCLUSION. Thus, for the equation (2) with the oscillating coefficients the sufficient conditions of the existence of the solution which represented by a Fourier-series with slowly varying coefficients and frequency are obtained in a one critical case.

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