Norm Kloosterman Sums over $\mathbb{Z}[i]$

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The classical Kloosterman sums first appeared in the paper of Kloosterman[1] in connection with the representation of positive integers quadratic forms. These sums and their generalizations find the various applications in additive number theory.

In our talk we study the n-dimensional norm Kloosterman sums over the ring of Gaussian integers that have no analogue in the ring $\mathbb{Z}$.

For the Gaussian integers $\alpha_0, \alpha_1, \ldots, \alpha_n$ and positive integer h we define n-dimensional Kloosterman sum

$$\widetilde{K}(\alpha_0, \alpha_1, \ldots, \alpha_n; q, h) := \sum_{S(C)} e^{2\pi i \Re(\alpha_0 x_0 + \ldots \alpha_n x_n)/q},$$

where the notation $S(C)$ means that summation passes under condition

$$C : \{x_j \in \mathbb{Z}[i]/q\mathbb{Z}[i],\ j = 0, 1, \ldots, n;\ N(x_0, x_1, \ldots, x_n) \equiv h \pmod q\}.$$

(here $N(x)$ denotes the norm of $x \in \mathbb{Z}[i]$, i.e. $N(x) = (\Re(x))^2 + (\Im(x))^2$).

We will obtain the non-trivial estimates for $\widetilde{K}(\alpha_0, \alpha_1, \ldots, \alpha_n)$. In particular, we have

Theorem 1. *Let h is a norm residue modulo p and $(h,p)=1$ and let $\alpha_0 \in \mathbb{Z}[i]$, $\alpha_0 \not\equiv 0 \pmod{p}$. Then*

$$|\widetilde{K}(\alpha_0,\alpha_1,\ldots,\alpha_n;p^m,h)| \le 2(4n-1)p^{2n(m-m_0)} I(\alpha_1,\ldots,\alpha_n;p^m),$$

where $I(\alpha_1,\ldots,\alpha_n;p^m)$ is the number of solutions of the system of congruences over unknowns $u_j, v_j \in \mathbb{Z}_{p^{m-m_0}}$

$$\begin{cases} a_j v_j + b_j u_j \equiv 0 \pmod{p^{m-m_0}}, \\ N(\alpha_0)u_j + 2ka_j \prod\limits_{j=1}^{n} \left(u_j^2 + v_j^2\right)^2 \equiv 0 \pmod{p^{m-m_0}}, \\ j = 1,\ldots,n. \end{cases}$$

(here $m_0 = [m+1/2]$).

Let χ be a Dirichlet character modulo q_1, $q_1|q$. We study the twisted norm Kloosterman sum

$$\widetilde{K}_\chi(\alpha,\beta;q,h) = \sum_{\substack{x,y \in (\mathbb{Z}[i]/q\mathbb{Z}[i]) \\ N(xy)\equiv h \pmod{q}}} \overline{\chi} e^{2\pi i \Re(\alpha x + \beta y)/q}.$$

For sum $\widetilde{K}_\chi(\alpha,\beta;q,h)$ we also obtain "the root" estimate. These results generalize the results from [2],[3].

1. Kloosterman H.D., On the representation of numbers in the form $ax^2+by^2+cz^2+dt^2$, Acta Math., 49, 407-464.
2. Varbanets S.P., The norm Kloosterman sums over $\mathbb{Z}[i]$, Anal. Probab. Methods Number Theory, A. Laurinčikas and E. Manstavičius(Eds.), (2007), 225-239.
3. Savastru O. Varbanets S., Norm Kloosterman sums over $\mathbb{Z}[i]$, Algebra and Discrete Mathematics, 11(2), (2011), 82-91.