МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ

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МЕХАНІКА.

Частина 1. КІНЕМАТИКА

Методичні вказівки з механіки
англійською мовою
для студентів фізичного факультету

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Детально висвітлюються методи розв'язування ряду типових задач з кінематики. Основна увага приділяється питанням, що часто виникають при переході від шкільної програми до систематичного вивчення механіки в рамках загального курсу фізики у вищих навчальних закладах. Вказівки покликані полегшити цей перехід та розвинути потрібні навички. Вони будуть корисними і для студентів четвертого курсу, що слухають курс з методики викладання фізики та готуються до державного іспиту з фізики.

Виклад ведеться (американською) англійською мовою. Це надає студентам-фізикам можливість ознайомитися зі стандартною фізико-математичною термінологією та типовими лексичними зворотами, необхідними для спілкування англійською мовою у професійній сфері.

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ЗМІСТ

1. FRAMES OF REFERENCE IN KINEMATICS

Any change in position of an object relative to other objects is called mechanical motion. The object itself is said to be moving. Remaining in place is a case of motion, too: we say that the object is at rest.

Mechanical motion is the subject of study of mechanics. Mechanics is customarily divided into kinematics, which describes how objects move, and dynamics, which investigates what causes objects to move. The concept of frame of reference is an integral part of both.

Frames of reference first come into play in kinematics. Dynamics then proves a special role of *inertial frames of reference* in stating laws of physics, and *special theory of relativity* reveals the relation between the concept of frame of reference and modern ideas of space and time.

The present text introduces several frames of reference that are significant for kinematics. As we mentioned before, kinematics describes the motion without explicit reference to its causes. The description is carried out in terms of position, velocity, and acceleration. In calculating values of these quantities, *it is necessary to take into account two basic facts*:

(A) *Mechanical motion is always relative:* even the same object seems to move differently with respect to different objects. For instance, a passenger sitting in a train is moving at a certain velocity relative to the ground, but he is at rest with respect to the train. In other words, two observers, one standing on the ground and the other seated in the train, are watching him moving with different velocities. Moreover, if the train is accelerating, the observers also have different points of view on the passenger's acceleration: with respect to the ground, he is accelerating along with the train, but relative to the train, he has no acceleration. Lastly, the passenger is constantly changing his location on the ground (Brooklyn, Manhattan, the Bronx, etc.), but remains in place relative to the train.

(B) *Mechanical motion occurs in time*: an object changes its position relative to other objects as time passes.

So, the motion of an object looks different to different observers. To avoid misinterpretation of his results, each observer needs to tell others relative to what and how his measurements of distances and time intervals are made. Frames of reference serve this purpose.

A particular frame of reference is defined by pointing out its following elements:

1) **an observer, who is usually associated with some object relative to which the motion is being studied;**
2) **a coordinate system attached to the observer, to measure distances and directional angles;**
3) **a clock held by the observer, to measure time intervals.**

A coordinate system and a clock, both moving with the train, is one frame of reference; a coordinate system and a clock, both stationary on the ground, is another.

Remember: it makes no sense to analyze a mechanical motion until a frame of reference is specified. Both the coordinate system and the clock are 'tied' up to the observer. The observer uses his coordinate system to establish where the motion occurs and his clock to determine when.

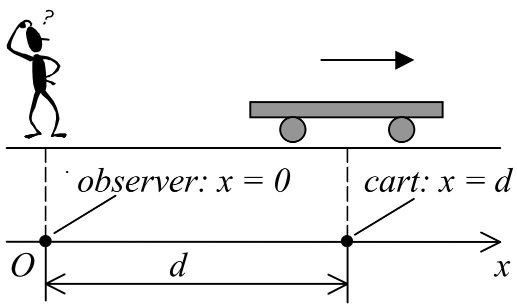
In every day life, we usually deal with objects whose speeds v are very small in comparison with the speed $c = 3.00\times10^{+8}$ m/s of light in vacuum: $v << c$. In this case, called Newtonian mechanics, time intervals measured between two events remain the same with respect to different frames of reference. We say that time is *absolute* and that it passes identically in different frames of reference. Time is considered as a convenient mathematical parameter, rather than a physical quantity. This parameter is used just to arrange events in an orderly fashion, one after another.

Therefore, in Newtonian mechanics there is no need to focus attention on giving the precise definition of time for a particular frame of reference. The problem of reference frame reduces to a choice of (1) *an observer and* (2) *an observer-based coordinate system.*

A coordinate system is usually drawn as a *set of coordinate axes* that intersect at one common point, the *origin*. The number of the axes to be used depends on the dimension of motion, the least number of independent quantities which uniquely determine any position of the object under study. With respect to the axes, the object is located by its *coordinates*. These are either distances measured along the axes or angles measured from the axes.

In general physics courses we usually deal with two kinds of coordinate systems, the *Cartesian coordinate system* and the *polar coordinates*, a simplified version of the *cylindrical coordinate system*.

Example 1.1. A cart moving along a straight level track.

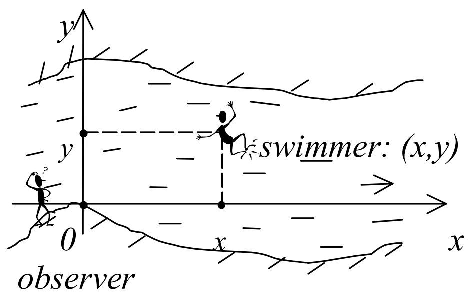


It is convenient to study the motion of the cart with respect to a ground-based observer who is standing next to the railroad track. Since the observer is associated with the track, we can say that the motion is treated relative to the track.

Any motion of the cart is one-dimensional. To locate the cart, it is enough to fix an arbitrary point O on the track and then measure the distance d between O and the cart, d being assigned a certain sign.

Suppose d is positive when measured to the right from O, and negative when to the left. If the straight level track is now taken to be the x-axis, with the point O as the origin, then different positions of the cart are uniquely determined by a single quantity x defined as the distance d taken with the proper sign. The quantity x represents the coordinate of the moving cart.

Example 1.2. A swimmer crossing a river.

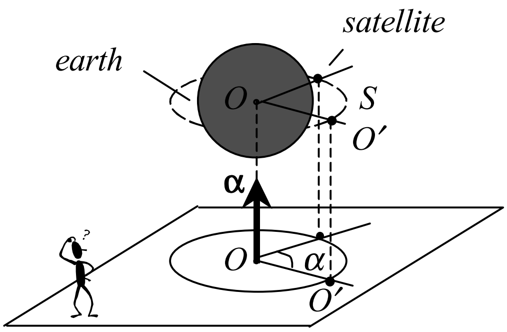


The motion is convenient to analyze relative to an observer standing on the ground. For the sake of brevity, we say that the motion is studied relative to the ground.

The motion is two-dimensional, for two coordinates, x and y, are required to determine locations of the swimmer on the water. One can be used to describe the motion of the swimmer across the river. The other shows the position of the swimmer along the bank of the river. The axes are perpendicular to each other. The starting point serves as the origin O.

This system is a two-dimensional variant of the Cartesian coordinate system.

Example 1.3. A satellite in a plane circular orbit around the earth.



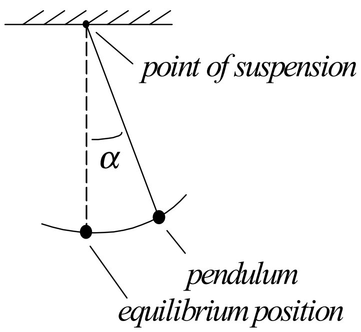
The motion can be treated with respect to an earth-based observer or, simply, relative to the earth. Even though the motion occurs in a plane, it is one-dimensional. Any point O', arbitrarily fixed on the circular orbit, can serve as the origin. The distance S, measured around the orbit clock- or counterclockwise, can be taken as the coordinate of the satellite. By convention, it is assigned the positive sign when measured in the counterclockwise sense; otherwise S is negative.

The S-axis is a curved line. Since description of mechanical motion in curvilinear coordinates is complicated, it is more convenient to consider the center O of the earth as the origin of another coordinate system, and use the line OO' as the line of reference (axis). Then the satellite can be located by the angle α measured from OO' to the line joining the point O and the satellite. The angle α is also assigned a sign, according to its sense.

Note that this angle can be treated as a vector $\boldsymbol{\alpha}$ perpendicular to the plane of motion and with its tail at the center O of the orbit.

The above coordinate system is called the polar coordinates. It represents a simplified version of the cylindrical coordinates.

Example 1.4. A point mass suspended by a weightless, inextendable string in a uniform gravitational field (a *simple pendulum*).



Since the motion occurs along the arc of a circle, we can use the angle α measured from the vertical which passes through the point of suspension and the equilibrium position. The choice of the sign of α is the same as that in *Example 1.3*.

The motion is one-dimensional.

2. MATH PREREQUISITES AND GENERAL APPROACH TO KINEMATICS PROBLEMS

Proficiency in math is a prerequisite for *Mechanics*, the first physics course you are taking. Many students find this course hard and time consuming. Even those who have no difficulties with the physics do have them with the math involved.

As any other physics course, Mechanics requires certain skills in various areas of math: arithmetic, algebra, trigonometry, geometry, and others. You learned most of those during your years in high school and college. Some new topics are introduced in the beginning of the course (Chapters 1, 2, etc.). You should invest a lot of time and effort in studying those, but be aware that additional review may be necessary. And, of course, you should know how to use your calculator.

The next example is given in order to point out to several math concepts that are crucial for Mechanics. You should be familiar with those. The example covers essential material on kinematics, usually from Chapters 1 – 3 or 1 – 4 of standard textbooks. The names of the concept are **boldfaced,** to let you know what to look up even without solving the problem.

The solution of the example also provides you with a general approach to most problems of kinematics. Try to follow it to develop your problem-solving skills.

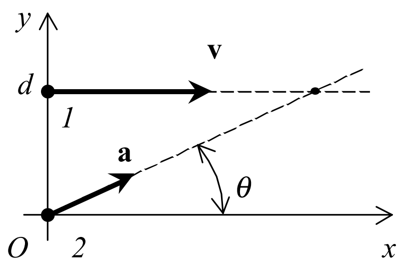
You are expected to solve this kind of problems in about 15 – 20 minutes.

Example 2.1. A particle *1* moves along the line $y = d = 30\,\text{m}$ at a constant 3-m/s velocity, **v**, directed parallel to the positive *x*-axis. At the instant when the particle passes the *y*-axis, another particle *2* starts from the origin *O* with zero speed and a constant 0.4-m/s^2 acceleration, **a**. Find the angle θ between the vector **a** and the positive *x*-axis which results in a collision between the particles.

Reasoning and solution. **The general procedure for solving most kinematics problems includes several steps to follow:**

1) Sketch a picture of the situation being studied.

2) Choose a convenient frame of reference.



Steps *1* and *2* are usually taken simultaneously. The motion occurs in a plane and is therefore two-dimensional. We use a reference frame that is stationary with respect to the ground and includes: a **plane Cartesian coordinate system**, with origin *O* and two coordinate axes, *x* and *y,* to locate the objects; and a clock, to measure time.

3) Specify general equations of motion.

The objects are moving with *constant accelerations*. In this case, the general equations of a two-dimensional motion, written in the Cartesian system, are of the form ($i = 1,2$):

$$
\begin{aligned}
x_i(t) &= x_{i0} + v_{i0x}t + \frac{1}{2}a_{ix}t^2, \qquad v_{ix}(t) = v_{i0x} + a_{ix}t \\
y_i(t) &= y_{i0} + v_{i0y}t + \frac{1}{2}a_{iy}t^2, \qquad v_{iy}(t) = v_{i0y} + a_{iy}t
\end{aligned}
\tag{1}
$$

These are formulated in terms of the **x- and y-coordinates** of the objects' locations in the plane, and the **x- and y-scalar components** of their velocities and accelerations. Namely, the symbols in Equations (1) stand for the following quantities (the subscript i is unnecessary and should be omitted if the motion of a single object is under study):

$x_i(t)$ – *x*-coordinate (position) of object i at time *t*;

x_{i0} – *x*-coordinate (position) of object i at initial time $t = 0$: $x_{i0} = x_i(0)$;

$v_{ix}(t)$ – x-component of the velocity of object i at t;

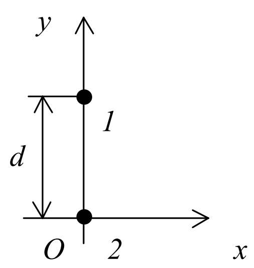
v_{i0x} – x-component of the velocity of object i at $t = 0$: $v_{i0x} = v_{ix}(0)$;

a_{ix} – x-component of the acceleration of object i (a_{ix} is constant throughout the motion).

The quantities $y_i(t)$, y_{i0}, $v_{iy}(t)$, v_{i0y}, and a_{iy} are related to the y-axis and have similar meaning.

The current time t is measured from $t = 0$; if the motion starts at a nonzero time t_0 instead of $t = 0$, t in Equations (1) must be replaced by the difference $t - t_0$.

4) Analyze the objects' initial location, initial velocity, and acceleration to obtain the equations describing the motion under given initial conditions.



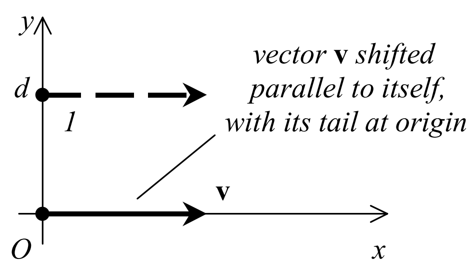
At this step, we simplify the above general equations by taking advantage of the known initial conditions (the objects' initial locations and velocities) and information about the objects' accelerations.

At $t = 0$, object *1* is on the y-axis, at the distance d away from the origin O, so:

$$x_{10} = 0, \qquad y_{10} = d$$

Object *2* is initially located at the origin, therefore,

$$x_{20} = 0, \qquad y_{20} = 0$$



At $t = 0$, object *1* is moving parallel to the positive x-axis, the magnitude of its velocity (its speed) being v. Then:

$$v_{10x} = +v, \qquad v_{10y} = 0$$

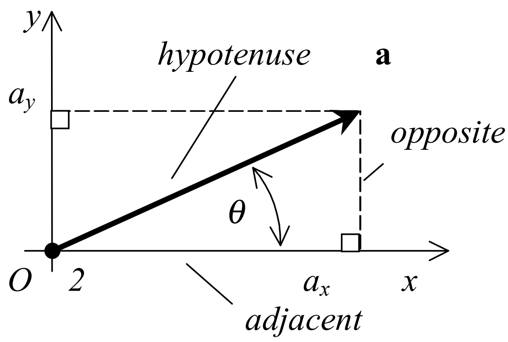
Note that you can always shift a vector parallel to itself to place its tail at the origin. This does not alter the vector, but makes it easier to find its components.

The initial speed of object *2* is zero; therefore, its velocity is **zero vector**, with components

$$v_{20x} = 0, \qquad v_{20y} = 0$$

During the motion, object *1* has a constant velocity and, therefore, zero acceleration, so

$$a_{1x} = 0, \qquad a_{1y} = 0$$



adjacent = hypotenuse times cosine
opposite = hypotenuse times sine

The acceleration of object *2* makes an angle θ above the positive *x*-axis. The **basic trigonometric functions** defined in terms of the **hypotenuse**, the **adjacent side**, and the **opposite side** of a **right triangle** lead to

$$a_{2x} = a\cos\theta, \qquad a_{2y} = a\sin\theta$$

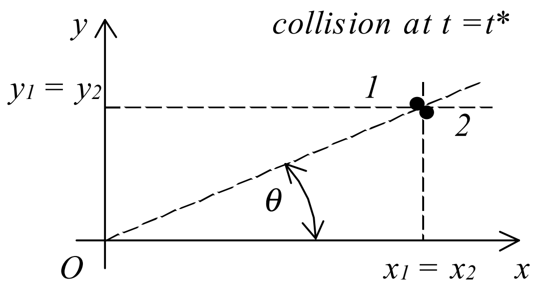
Now that all the given data has been used, Equations (1) take the form:

Object 1:
$$x_1(t) = vt, \qquad v_{1x}(t) = v$$
$$y_1(t) = d, \qquad v_{1y}(t) = 0 \tag{2}$$

Object 2:
$$x_2(t) = \frac{1}{2}a\cos\theta\, t^2, \qquad v_{2x}(t) = a\cos\theta\, t$$
$$y_2(t) = \frac{1}{2}a\sin\theta\, t^2, \qquad v_{2y}(t) = a\sin\theta\, t \tag{3}$$

5) Analyze the event of interest to you (which occurs at a particular point on the objects' paths at a certain moment) and set up a closed algebraic system for quantities describing the event.

The event of interest to us is the collision of particles.



Let the particles collide at a certain time $t = t^*$. Then their locations coincide, which means that their *x*- and *y*-coordinates are equal in pairs:

$$x_1 = x_2, \qquad y_1 = y_2$$

Together with Formulas (3) and (4), these relations lead to a **system of algebraic equations**:

$$vt^* = \frac{1}{2}a\cos\theta\, t^{*2}, \qquad d = \frac{1}{2}a\sin\theta\, t^{*2} \tag{4}$$

The physical part of the problem is almost exhausted by Equations (4). There is no need to use the equations for velocities, and we leave them out.

6) Solve the algebraic system for the desired quantities.

Simple **algebraic manipulations** help us solve the first equation in (4) for t^*, the time when the collision occurs:

$$t^* = 2v/a\cos\theta$$

Substitution of t^* in the other equation yields an equation for the angle θ:

$$d = 2v^2\sin\theta/a\cos^2\theta$$

Further manipulations, involving the **trigonometric identity** $\sin^2\theta + \cos^2\theta = 1$, lead to a **quadratic equation** in the variable $z = \sin\theta$:

$$z^2 + 2bz - 1 = 0$$

where $b = v^2/ad$.

The **quadratic equation formulas** yield two roots:

$$z_1 = -b + \sqrt{b^2+1}, \qquad z_2 = -b - \sqrt{b^2+1}$$

Since $b = 3\times3/(0.4\times30) = 3/4$, **arithmetic calculations**, including **operations with fractions** and **radicals**, result in

$$z_1 = \frac{1}{2}, \qquad z_2 = -2$$

We discard the root $z_2 = \sin\theta = -2$, for the number -2 does not belong to the closed interval [-1; 1] representing the **range** of the sine function. Thus, the angle θ obeys the equation

$$\sin\theta = \frac{1}{2}$$

Its solution is given by the inverse sine, one of the **inverse trigonometric functions**:

$$\theta = \sin^{-1}\frac{1}{2} = 30^\circ$$

3. FREE FALL AND ACCELERATION DUE TO GRAVITY

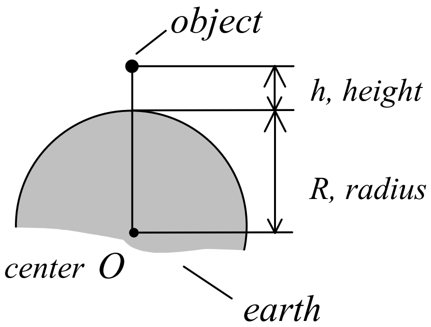
We consider two different objects, a planet and a body moving above its surface. Under 'planet' we understand any object whose size and mass are vastly greater than those of the body. Some examples are the earth, the sun, the moon, etc. as compared with a stone, a car, a plane, etc. We also assume that the only force acting on the moving body is the gravitational force exerted by the planet, all other forces, including air resistance, being neglected.

The motion of a body near a planet is called free fall if the planet's gravity is the only force acting on the body. The acceleration of a freely falling body is called the acceleration due to gravity, $\mathbf{g}$.

According to Newton's law of universal gravitation, the *gravity force between two objects is always attractive. It acts along the line joining the centers of gravity of the objects, and has a magnitude F directly proportional to the product of the objects' masses m and M and inversely proportional to the square of the distance r between their centers of gravity:*

$$F = G\frac{mM}{r^2} \tag{5}$$

The coefficient G is called the *universal gravitational constant*. Its modern experimental value is 6.673×10^{-11} N·m²/kg².



Suppose the planet can be treated as a uniform sphere of radius R, whereas a freely falling body as a point at a height h above the planet's surface. Then $\mathbf{g}$ points towards the center of the planet; or, relative to a person on the planet's surface, downward. Moreover, since r is the distance between the sphere's center and the point, $r = R + h$, the magnitude g of the acceleration due to gravity is

$$g = \frac{F}{m} = G\frac{M}{(R+h)^2} \tag{6}$$

It depends upon the planet's mass and size and upon the altitude above the planet's surface. *The mass of a freely falling body is insignificant.*

In the limiting case when the object is located near the planet's surface so that its altitude h can be ignored in comparison with R ($h << R$),

$$g = G\frac{M}{R^2} \tag{7}$$

At a given height, all objects fall freely with the same acceleration, regardless of their size or mass.

Objects falling in the air are not in free fall: they experience two forces, the gravity and the air resistance. The air resistance depends on the object's geometrical shape and size. As a result, different objects falling in the air have different accelerations.

Exercise 3.1. Calculate the magnitude of the acceleration due to gravity on the surfaces of (a) the earth (M = 5.98×10^{24} kg, R = 6.38×10^{6} m); (b) the sun (M = 1.99×10^{30} kg, R = 6.95×10^{8} m); (c) the moon (M = 7.36×10^{22} kg, R = 1.74×10^{6} m).

Answers: Formula (6) gives: (a) 9.80 m/s^2; (b) 2.75x10^2 m/s^2; (c) 1.62 m/s^2.

Exercise 3.2. Estimate the acceleration due to gravity at (a) 10 km (representative for motion of airplanes), (b) 200 km (orbital motion of artificial satellites), and (c) 1000 km above the ground. The parameters of the earth are given in *Exercise 3.1*.

Answers: (a) g = 9.77 m/s^2, (b) g = 9.22 m/s^2, and (c) g = 7.33 m/s^2.

Formulas (6) and (7) were derived for a uniform sphere. Actually, g varies slightly according to various locations on the planet and its mass distribution. On the earth's equator, for instance, the measured value of g is 9.780 m/s^2, whereas at the city of New York it equals 9.803 m/s^2. Usually, we can ignore these variations to accept g on the ground to be constant and equal to 9.80 m/s^2. The dependence g upon

h is essential for problems dealing with orbital (h is less than or comparable with R) and planetary motion ($h >> R$).

The complete description of the motion of a freely falling body is a rather complicated problem involving integral calculus. Within a general physics course, *we usually restrict the discussion to the case of a constant-in-magnitude acceleration due to gravity and then classify different problems according to the initial conditions – the object's initial position and velocity – which determine the path followed by the object.* As a result, we deal with three kinds of problems on free fall:

1) **Freely falling objects**: both the initial velocity and the acceleration due to gravity are directed vertically, resulting in a *one-dimensional motion along a vertical line perpendicular to the level ground*;

2) **Projectile motion**: the initial velocity and the acceleration due to gravity are neither parallel nor perpendicular to each other, resulting in a *two-dimensional motion along a plane parabola*;

3) **Orbital motion**: the velocity and the acceleration due to gravity are always perpendicular to each other, resulting in a *one-dimensional motion around a plane circle*.

4. FREELY FALLING OBJECTS

In solving one-dimensional problems of this kind, the above step-by-step general procedure still applies. Here are some relevant comments.

Step 2. We choose the vertical to be the *y*-axis of the reference frame. The positive *y*-axis can be directed either upward or downward. The choice of the origin O is arbitrary as well. Nevertheless, the reference frame with the *y*-axis pointed upward and the origin located on the ground is most widely used.

Step 3. Since a freely-falling object is moving with a constant acceleration, the general equations of motion are of the form (1):

$$y(t) = y_0 + v_{0y}t + \frac{1}{2}a_y t^2, \qquad v_y(t) = v_{0y} + a_y t \tag{8}$$

where the symbols represent the following kinematic variables of the object:

$y(t)$ – y-coordinate (position) at time t;

y_0 – y-coordinate (position) at initial time $t = 0$: $y_0 = y(0)$;

$v_y(t)$ – y-component of the velocity at t;

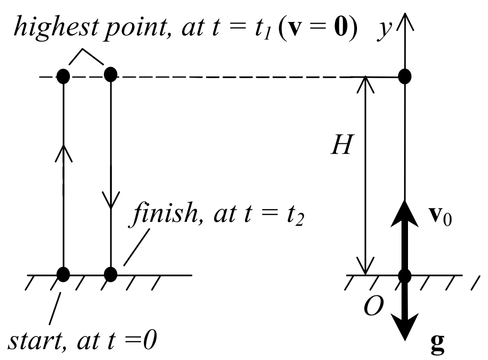
v_{0y} – y-component of the velocity at $t = 0$: $v_{0y} = v_y(0)$;

a_y – y-component of the acceleration.

Should the initial time t_0 be nonzero, t is replaced by the difference $t - t_0$.

Step 6. For a single event, the number of unknown independent quantities to be found does not exceed two. Make sure all available information has been used or interpreted correctly. The algebraic system being sought for consists at most of two equations, which are either linear or quadratic.

Example 4.1. A ball is thrown upward with an initial speed of v_0 = 20.0 m/s. In the absence of air resistance, (a) how high does the ball go, (b) what is the total time the ball is in the air before returning to the ground, and (c) how fast is the ball traveling just before striking the ground?



Solution. In the absence of air resistance, the ball is a freely falling object. The motion occurs along the vertical. Suppose that the upward direction is chosen as the positive direction of the y-axis, and the origin O is placed at the point where the ball was initially.

The general equations of the ball's motion are given by (8). In order to rewrite them for our case, we use the given initial conditions and information about the object's acceleration:

at $t = 0$, the ball is located at the origin, so $y_0 = 0$;

at $t = 0$, its initial velocity is directed along the positive y-axis, so $v_{0y} = +v_0$;

during the motion, the ball's acceleration points along the negative y-axis, so

$$a_y = -g$$

(the negative sign in front of g is due to the fact that the vector **g** points in the negative direction of the y-axis, *not because* **g** *is directed downward*; if the positive y-axis were directed downward, we would have $a_y = +g$).

The general equations (8) change to

$$y(t) = v_0 t - \frac{1}{2} g t^2, \qquad v_y(t) = v_0 - gt \tag{9}$$

Equations (9) are the equations of the ball's motion under the given initial conditions. They describe the behavior of the ball's coordinate (position) y and velocity component v_y with time t.

According to *Step 5*, events (a), (b), and (c) are now to be considered. They occur at different points on the ball's path, and we need learn how and in what terms to analyze them. This step is a frequent source of many troubles.

(a) The ball attains a maximum height and stops there before returning down. At that point, its instantaneous velocity becomes equal to zero. To reflect this fact mathematically, we introduce the following quantities: H – the maximum height, t_1 – the point in time when the ball is at H. Then we state:

$$y = H \quad \text{and} \quad v_y = 0 \quad \text{when } t = t_1$$

Equivalently,

$$y(t_1) = H, \qquad v_y(t_1) = 0$$

Substituting these data into Equation (9) gives a closed algebraic system with the desired height H present:

$$H = v_0 t_1 - \frac{1}{2} g {t_1}^2, \qquad 0 = v_0 - g t_1 \tag{10}$$

From the second equation, $t_1 = v_0/g = 2.04\,\text{s}$. Substitution of t_1 into the first one yields the answer:

$$H = \frac{{v_0}^2}{2g} = 20.4\,\text{m}$$

(b) When the ball hits the ground, its vertical coordinate becomes equal to zero. Let t_2 be the point in time as this event occurs. We state:

$$y = 0 \quad \text{when} \quad t = t_2$$

That is,

$$y(t_2) = 0$$

Then it follows from Equations (9) that

$$0 = v_0 t_2 - \frac{1}{2} g t_2{}^2, \qquad v_y = v_0 - g t_2 \tag{11}$$

In fact, we need the first equation only. It is readily factored:

$$t_2 \left(v_0 - \frac{1}{2} g t_2 \right) = 0$$

There are two solutions: $t_2 = 0$ and $t_2 = 2v_0/g = 4.04\,\text{s}$. The former corresponds to the initial time, when the ball was thrown and was at $y = 0$ for the first time. The latter corresponds to the final point when the ball just returns to the ground. The time interval the ball spends in the air equals 4.08 s. It is twice as much as t_1. *It takes the same time for the ball to reach the highest point and to return from there to the ground.*

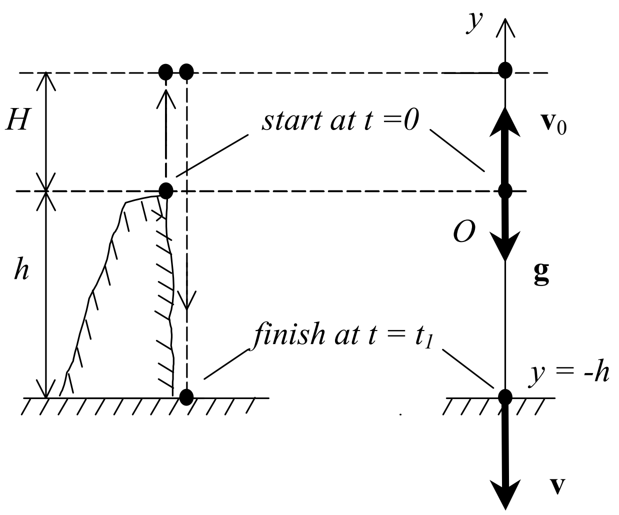
(c) We need to calculate the ball's velocity just before striking the ground. By repeating reasoning similar to that in (b), we obtain System (11). The desired velocity is given by the second equation with $t_2 = 2v_0/g$:

$$v_y = -v_0$$

The negative sign reveals that the velocity of the ball just before striking the ground is directed along the negative y-axis, downward in the case. But *its magnitude at that time is equal to the initial speed of the ball.*

Note that the velocity and the acceleration of the ball are not necessarily in the same direction. The velocity changes both its magnitude and direction. The acceleration remains constant at all times, including the highest point of the ball's path.

Example 4.2. A stone is thrown vertically upward with a speed of v_0 = 10 m/s from the edge of a cliff h = 50 m high. What are (a) the time taken by the stone to reach the bottom of the cliff, (b) its speed just before hitting the ground, (c) its total displacement, and (d) its total distance traveled?



Solution. We treat the stone as a freely falling object. Let the y-axis be directed upward, and the origin O be placed on the edge of the cliff. The general equations of the stone's motion are given by (8). The initial position, the y-component of the initial velocity, and that of the acceleration of the stone are:

$$y_0 = 0, \quad v_{0y} = +v_0, \quad a_y = -g$$

The equations describing the stone's motion under the above conditions take the form (9).

(a) In the chosen reference frame, the bottom of the cliff has a coordinate of $y = -h$. Let t_1 be the time when the stone strikes it. We state:

$$y(t_1) = -h$$

Equations (9), written for $t = t_1$, give:

$$-h = v_0 t_1 - \frac{1}{2} g t_1{}^2, \qquad v_y = v_0 - g t_1 \tag{12}$$

Of (12), the first equation is quadratic in t_1. We rewrite it in the standard form

$$a t_1{}^2 + b t_1 + c = 0$$

where the coefficients are: $a = g/2$, $b = -v_0$, $c = -h$. The discriminant

$$D = b^2 - 4ac = v_0{}^2 + 2gh$$

is positive, therefore, there are two real solutions. They are given by the quadratic formulas:

$$t_1 = \frac{-b + \sqrt{D}}{2a} = \frac{v_0 + \sqrt{v_0{}^2 + 2gh}}{g} = 4.4\,\text{s}, \quad t_1 = \frac{-b - \sqrt{D}}{2a} = \frac{v_0 - \sqrt{v_0{}^2 + 2gh}}{g} = -2.3\,\text{s}$$

The latter solution is negative and must be discarded as physically meaningless. So, the answer is t_1 = 4.4 s.

(b) The y-component of the velocity just before striking the bottom is given by the second equation in (12) as $t = t_1$:

$$v_y = v_0 - gt_1 = \sqrt{v_0^2 + 2gh} = -33\,\text{m/s}$$

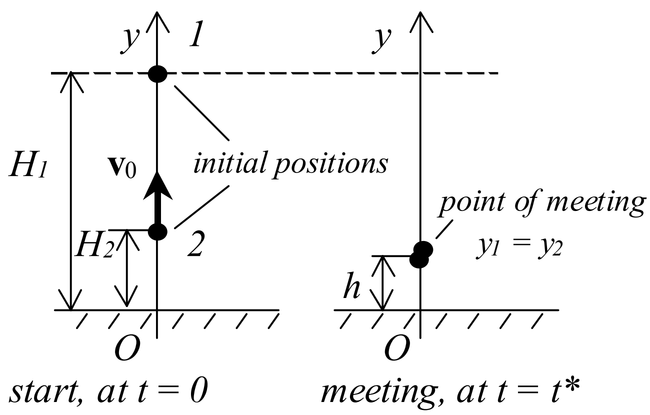
At $t = t_1$, the instantaneous velocity is directed along the negative y-axis, downward, its magnitude being equal to 33 m/s.

(c) The total displacement $S_y = y - y_0 = -h = -50\,\text{m}$ is directed downward and has a magnitude of 50 m.

(d) The distance traveled $d = d_1 + d_2$ where $d_1 = H$ is the distance traveled by the stone on its way up, from the launch point to the maximum height, and $d_2 = H + h$ represents the distance traveled by the stone down, from the highest point to the bottom. We can use the known result (*Example 4.1a*): $H = v_0^2/g$, then

$$d = h + \frac{v_0^2}{g} = 60\,\text{m}$$

Example 4.3. A stone 1 is released from a height of $H_1 = 47\,\text{m}$ above the ground. Simultaneously, a stone 2 is thrown upward from a height of $H_2 = 2.0\,\text{m}$ with a speed of $v_0 = 15.0\,\text{m/s}$. (a) How long does it take for the stones to meet each other and (b) how high above the ground do they meet?



Solution. Let us direct the y-axis upward and place the origin on the ground. The stones are falling freely. To describe their motion, we use the general equations (8), written for each of the stones separately, and follow up with the analysis of the initial conditions and the objects' accelerations.

Stone 1:

the initial conditions: $y_{10} = H_1$, $v_{10y} = 0$;

the y-component of the acceleration: $a_{1y} = -g$;

the equations describing the motion of stone 1 under the above conditions:

$$y_1(t) = H_1 - \frac{1}{2}gt^2, \qquad v_{1y}(t) = -gt \tag{13}$$

Stone 2:

the initial conditions: $y_{20} = H_2$, $v_{20y} = +v_0$;

the y-component of the acceleration: $a_{2y} = -g$;

the equations describing the motion of stone 2 under the these conditions:

$$y_2(t) = H_2 + v_0 t - \frac{1}{2}gt^2, \qquad v_{2y}(t) = v_0 - gt \tag{14}$$

When the stones meet at a certain time $t*$, their positions coincide:

$$y_1 = y_2 \quad \text{when} \quad t = t*$$

That is,

$$y_1(t*) = y_2(t*)$$

Together with the first equations in (13) and (14), this yields:

$$H_1 - \frac{1}{2}gt*^2 = H_2 + v_0 t* - \frac{1}{2}gt*^2$$

The identical terms on both sides cancel out, and we obtain the desired time:

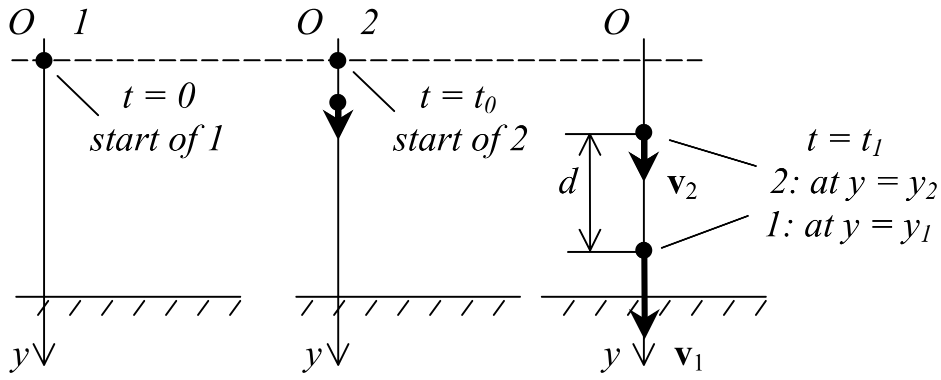
$$t* = \frac{H_1 - H_2}{v_0} = 3.0\,\text{s}$$

The stones meet in 3.0 s at a surprisingly low height:

$$h = y_1(t*) = y_2(t*) = 2.8\,\text{m}$$

Make sure that it takes 1.53 s for stone 2 to reach its maximum height of $13.5\,\text{m}$. Stone 1 overtakes stone 2 on its way down.

Example 4.4. A stone is dropped from the roof of a high building. A second stone is dropped $t_0 = 1\,\text{s}$ later. How far apart are the stones when the second one reaches a speed of $v_2 = 12.5\,\text{m/s}$?



Solution. For convenience, we direct the y-axis downward and place the origin at the point where the stones are released. Then we write Equations (8) for each stone separately and analyze the initial conditions and the objects' accelerations. We should take into account that the stones start at different times. Let the time when the first stone starts be equal to zero; then the second stone starts at $t_0 = 1\,\text{s}$.

Stone 1:

the initial conditions ($t = 0$): $y_{10} = 0, \quad v_{10y} = 0$;

the y-component of the acceleration: $a_{1y} = +g$;

the equations describing the motion of the first stone at times $t > 0$:

$$y_1(t) = \frac{1}{2}gt^2, \qquad v_{1y}(t) = gt \tag{15}$$

Stone 2:

the general equations:

$$y_2 = y_{20} + v_{20y}(t - t_0) + \frac{1}{2}a_{2y}(t - t_0)^2, \quad v_{2y} = v_{20y} + a_{2y}(t - t_0)$$

the initial conditions ($t = t_0$): $y_{20} = 0, \quad v_{20y} = 0$;

the y-component of the acceleration: $a_{2y} = +g$;

the equations describing the motion of the second stone at times $t > t_0$:

$$y_2(t) = \frac{1}{2}g(t - t_0)^2, \qquad v_{2y}(t) = g(t - t_0) \tag{16}$$

We are interested in determining the stones' locations y_1 and y_2 at the instant t_1 when $v_{2y} = v_2$. The second equation in (16) gives for $t = t_1$:

$$v_2 = g(t_1 - t_0)$$

whence $t_1 = t_0 + v_2/g = 2.58\,\mathrm{s}$. At this instant, the stones are located at

$$y_1 = y_1(t_1) = \frac{1}{2}gt_1^{\,2} = 32.62\,\mathrm{m}, \qquad y_2 = y_2(t_1) = \frac{1}{2}g(t_1 - t_0)^2 = 12.23\,\mathrm{m}$$

They are separated by the distance

$$d = y_1 - y_2 = 20.4\,\mathrm{m}$$

5. PROJECTILE MOTION

A projectile is any body that is projected with an initial velocity near the surface of a planet. Some common examples are: a thrown ball, a speeding bullet, and an athlete doing a jump.

Two basic suggestions about a projectile in motion are usually made: (1) it experiences no air resistance and (2) the curvature and rotation of the planet are ignored.

Projectile motion is the motion of a projectile under conditions (1) and (2), when the planet's gravity is the only force acting on the body.

Projectile motion is a particular example of *free fall*, with the acceleration due to the planet's gravity. Generally, the initial velocity $\mathbf{v}_0$ and the acceleration $\mathbf{g}$ of a projectile are not parallel to each other. As a result, *the projectile motion is two-dimensional and occurs in a vertical plane perpendicular to the level ground. This plane is generated by the vectors* $\mathbf{v}_0$ *and* $\mathbf{g}$ *and remains fixed throughout the motion.*

The path (trajectory) followed by a projectile is a parabola. The instantaneous velocity of a projectile at any point is tangent to the path of the projectile at that point. It changes both its magnitude and direction.

In order to describe a projectile motion, we apply the general procedure, with the following adjustments:

Step 2. We usually use a reference frame that is stationary with respect to the ground. Its x-axis is parallel to the horizontal, and its y-axis points along the vertical. The positive y-axis can be directed either upward or downward, but frequently upward. The origin O is usually placed at the initial position of a projectile, or on the ground.

Note that reference frames with their axes making angles with the horizontal and the vertical are also used.

Step 3. A projectile is moving with constant acceleration. Its general equations of motion are of the form (1). The motion can be treated as a superposition of two one-dimensional motions, one occurring along the x-axis and the other along the y-axis.

In the most widely used reference frame with the x-axis directed horizontally,

$$a_x = 0$$

Then the projectile motion is a combination of a horizontal motion with constant velocity and a vertical motion with constant acceleration.

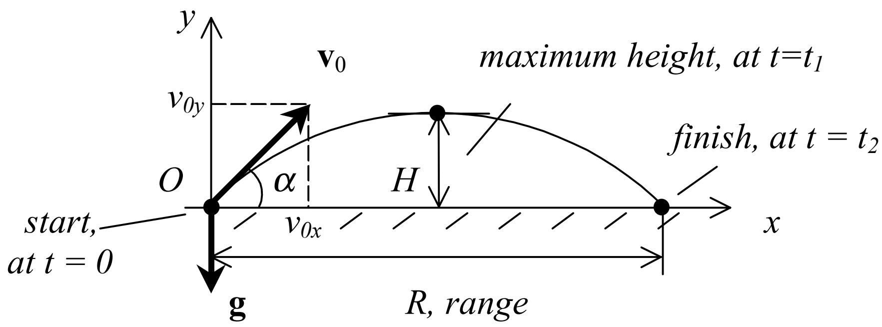
If the origin is placed at the launch point of a projectile, then $x_0 = 0$, $y_0 = 0$.

Step 6. For a single event, the number of unknown independent quantities to be found does not exceed four. Make sure all available information has been used or interpreted correctly. The algebraic system being sought for consists at most of four equations, two for positions and two for velocity components, but usually we deal with a less number of equations, which are either linear or quadratic.

Example 5.1. A ball is thrown with a speed of $v_0 = 20\,\text{m/s}$ at an angle $\alpha = 60^\circ$ above the ground. In the absence of air resistance, determine (a) the highest barrier the ball can clear, (b) the total time of travel before the ball hits the ground, (c) the horizontal range of the ball, and (d) the velocity of the ball just before hitting the ground. If the angle α is allowed to vary, what are (e) the maximum height the ball can reach, and (f) its maximum horizontal range?

Solution. In the absence of air resistance, the ball is in projectile motion which occurs in the vertical plane generated by the vectors $\mathbf{v}_0$ and **g**.

Let the x-axis be directed along the ground, the upward direction be chosen as the positive y-axis, and the origin O be placed at the launch point of the ball.



The general equations of the ball's motion are given by (1). To rewrite them for our case, we use the initial conditions – the object's initial position and velocity – and information about the object's acceleration:

at $t = 0$, the ball is located at the origin, so $x_0 = 0$, $y_0 = 0$;

at $t = 0$, the ball's velocity makes the angle α above the positive x-axis, so

$$v_{0x} = +v_0 \cos\alpha, \qquad v_{0y} = +v_0 \sin\alpha$$

during the motion, the ball's acceleration points along the negative y-axis, so

$$a_x = 0,\ a_y = -g$$

(the negative sign before g is due to the fact that the vector **g** is directed along the negative y-axis, *not because* **g** *points downward*. If the positive y-axis were directed down, we would have $a_y = +g$).

Equations (1) change to:

x-axis:
$$x(t) = v_0 \cos\alpha\, t \tag{17}$$

$$v_x(t) = v_0 \cos\alpha \tag{18}$$

y-axis:
$$y(t) = v_0 \sin\alpha\, t - \frac{1}{2} g t^2 \tag{19}$$

$$v_y(t) = v_0 \sin\alpha - gt \tag{20}$$

Equations (17) – (20) describe the behavior of the ball's coordinates (*x*- and *y*-positions) and velocity components (v_x and v_y) as time *t* passes. They enable us to analyze any single event that happens to the ball moving under the given initial conditions. Each event is described in terms of the ball's positions and velocity components taken at the corresponding time. We are supposed to identify their values in each particular case.

(a) As a matter of fact, we have to determine the *maximum height* the ball attains before returning down. At that point, the ball stops going up, and the vertical component of its instantaneous velocity vanishes. We can reflect this fact by introducing the following quantities: H – the maximum height, t_1 – the time when the ball is at H. Then we state:

$$y(t_1) = H, \qquad v_y(t_1) = 0$$

Note that the ball keeps moving horizontally at that point. For now, however, we have no need to use any information about its horizontal motion.

Equations (19) and (20), written for $t = t_1$, give a closed algebraic system where the desired height *H* is involved:

$$H = v_0 \sin\alpha\, t_1 - \frac{1}{2} g {t_1}^2, \qquad 0 = v_0 \sin\alpha - g t_1$$

From the second equation, $t_1 = v_0 \sin\alpha / g = 1.77\,\mathrm{s}$. Substitution of t_1 into the first equation yields:

$$H = \frac{{v_0}^2 \sin^2\alpha}{2g} = 15.3\,\mathrm{m}$$

(b) When the ball hits the ground, its vertical coordinate becomes equal to zero. Let t_2 be the moment as this event happens. Then

$$y(t_2) = 0$$

The *x*-position of the ball at $t = t_2$ is unknown as yet. From among equations (17) – (20), written for $t = t_2$, we need only this one:

$$0 = v_0 \sin\alpha\, t_2 - \frac{1}{2} g {t_2}^2$$

After being factored,

$$t_2\left(v_0 \sin\alpha - \frac{1}{2} g t_2\right) = 0$$

it yields two solutions: $t_2 = 0$ and $t_2 = 2v_0 \sin\alpha / g = 3.53\,\text{s}$. The former corresponds to the initial time, when the ball was thrown and was first at $y = 0$. The latter corresponds to the final point when the ball just returns to the ground. The total time of the ball's travel in the air equals 3.53 s. It is twice as much as t_1: *it takes the same time for the ball to reach the highest point and to return from there to the ground.*

Our numerical calculations seem to contradict this statement: $2t_1$ = 3.54 s, but not 3.53 s. In fact, there is no contradiction here. These numbers are approximate. They have been rounded off to the desired three significant figures.

(c) We are interested in how far away from the launch point the ball lands. This distance, called the *horizontal range R*, is given by the *x*-coordinate of the ball at $t = t_2$. We say

$$x(t_2) = R, \qquad y(t_2) = 0$$

and rewrite equations (17) and (19) for $t = t_2$:

$$R = v_0 \cos\alpha \; t_2, \qquad 0 = v_0 \sin\alpha \, t_2 - \frac{1}{2} g t_2^{\,2}$$

As in (b), the second equation leads to $t_2 = 2v_0 \sin\alpha / g$. Then, with the use of the first equation and the double-angle formula $\sin 2\alpha = 2\cos\alpha \sin\alpha$, we get:

$$R = \frac{v_0^{\,2} \sin 2\alpha}{g} = 25.3\,\text{m}$$

(d) The reasoning identical to that in (b) leads to the above t_2. The components of the desired velocity are given by Equations (18), (20) with $t = t_2$:

$$v_x = v_0 \cos\alpha, \qquad v_y = v_0 \sin\alpha - g t_2 = -v_0 \sin\alpha$$

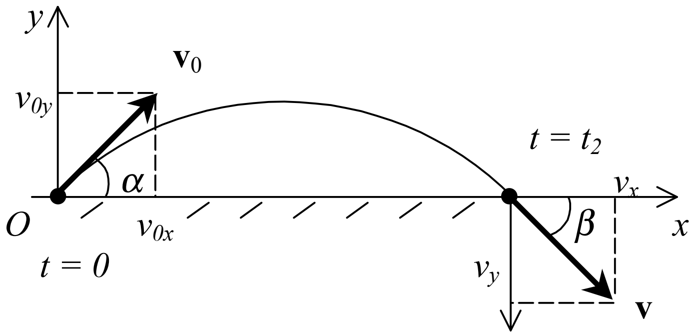
Just before hitting the ground, the instantaneous velocity of the ball has the magnitude

$$v = \sqrt{v_x^{\,2} + v_y^{\,2}} = \sqrt{(v_0 \cos\alpha)^2 + ((-v_0 \cos\alpha))^2} = v_0$$

and makes the angle

$$\beta = \tan^{-1}\frac{v_y}{v_x} = \tan^{-1}\frac{-v_0\sin\alpha}{v_0\cos\alpha} = -\tan^{-1}(\tan\alpha) = -\alpha$$

below the horizontal. The ball returns to the same horizontal level at the same speed as it was projected. At this instant, the angle between the instantaneous velocity of the ball and the horizontal is equal to the initial angle α.



In general, *any change in velocity of a projectile is caused by the acceleration due to gravity, which affects only the vertical component of the velocity. The horizontal velocity does not change at all, neither in magnitude nor in direction.* (Do not confuse the horizontal component and the x-component of the velocity: unless the x-axis is horizontal, they are different.)

(e), (f) According to (a) and (c), the height H and the horizontal range R of a projectile are given by

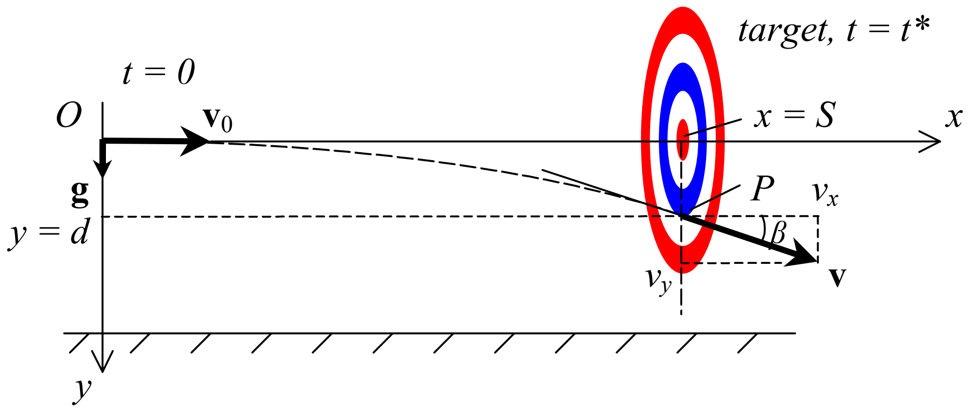
$$H = \frac{v_0^2\sin^2\alpha}{2g}, \qquad R = \frac{v_0^2\sin 2\alpha}{g} \tag{21}$$

They attain their maximum values $H_{\max}$ and $R_{\max}$ as the maximum values (unity) of $\sin\alpha$ and $\sin 2\alpha$ occur; that is, when α = 90° (the projectile is launched straight up), and when α = 45° (the projectile is launched at an angle of 45° above the horizontal):

$$H_{\max} = \frac{v_0^2}{2g} = 20.4\,\text{m}, \qquad R_{\max} = \frac{v_0^2}{g} = 40.8\,\text{m}$$

Example 5.2. An arrow is fired horizontally at the center of a vertical target which is S = 20.0 m away. The speed of the arrow is v_0 = 50.0 m/s. (a) How far away from the center and (b) at what angle with respect to the horizontal is the arrow expected to hit the target? (c) Would the arrow hit the center if, simultaneously with the shot, the target were released to fall straight down?

Solution. We neglect the air resistance and treat the arrow as a projectile.



Suppose that the x-axis is parallel to the ground, the y-axis is directed downward, and the origin O is placed at the point where the arrow's tip was located initially. The general equations of the arrow's (strictly speaking, of the arrow's tip's) motion are given by (1). In order to develop them for our case, we analyze the arrow's initial position, initial velocity, and acceleration:

at $t = 0$, the arrow is located at the origin, so $x_0 = 0$, $y_0 = 0$;

at $t = 0$, the arrow's velocity is parallel to the positive x-axis, so

$$v_{0x} = +v_0,\ v_{0y} = 0;$$

during the motion, the arrow's acceleration points along the positive y-axis, so

$$a_x = 0,\ a_y = +g.$$

Equations (1) take the form (*for the sake of simplicity, we omit the argument t in parentheses*):

x-axis: $$x = v_0 t, \qquad v_x = v_0 \tag{22}$$

y-axis: $$y = \frac{1}{2}gt^2, \qquad v_y = gt \tag{23}$$

(a), (b) Suppose that the arrow hits the target at a point P which is d m below the center, and this happens at $t = t^*$. We state:

$$x(t^*) = S, \quad y(t^*) = d$$

Equations (22) – (23), written for $t = t^*$, give:

$$S = v_0 t^*, \quad v_x(t^*) = v_0 \tag{24}$$

$$d = \frac{1}{2} g t^{*2}, \quad v_y(t^*) = g t^* \tag{25}$$

From the first equation in (24), the time taken by the arrow to reach the target equals $t^* = S / v_0 = 0.40\,\text{s}$.

From the first equation in (25), the deflection of the point of hitting from the center is

$$d = \frac{1}{2} g t^{*2} = \frac{1}{2} g \left(\frac{S}{v_0} \right)^2 = 0.78\,\text{m}$$

From the two other equations in (24) and (25), the components of the arrow's velocity just before the hitting are

$$v_x(t^*) = v_0 = 50.0\,\text{m/s}, \quad v_y(t^*) = \frac{gS}{v_0} = 3.92\,\text{m/s}$$

The velocity has the magnitude

$$v = \sqrt{v_x{}^2 + v_y{}^2} = 50.2\,\text{m/s}$$

and makes the angle

$$\beta = \tan^{-1}\left(\frac{v_y}{v_x} \right) = 4.48^\circ$$

below the horizontal. Since the velocity vector is always tangent to the object's path, the angle β represents the desired answer.

(c) Two objects, the arrow and the target, are involved into motion now. Their instantaneous positions are given by the following equations:

arrow's tip: $x = v_0 t, \quad y = \frac{1}{2} g t^2$

target's center: $X = S, \quad Y = \frac{1}{2} g t^2$

The arrow will hit the target at the center if, at a certain time $t = t^{**}$,

$$x(t^{**}) = X(t^{**}), \quad y(t^{**}) = Y(t^{**})$$

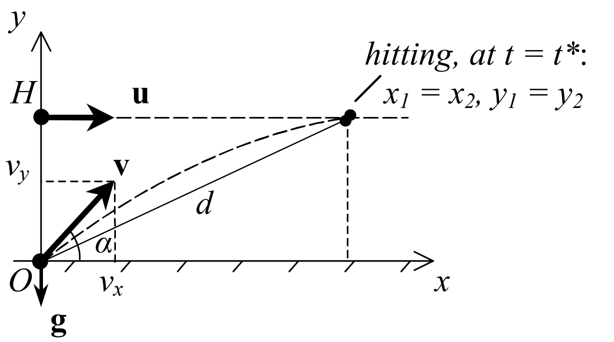
In more detail,

$$S = v_0 t^{**}, \quad \frac{1}{2} g t^{**2} = \frac{1}{2} g t^{**2}$$

The first equation gives $t^{**} = S/v_0 = 0.40\,\text{s}$. The other is an identity: the tip and the center have identical y-positions at all times. The arrow aimed initially at the center always hits it, no matter what the initial speed of the arrow is.

Make sure that this result remains valid when the arrow and the center are initially at different heights. In other words, the arrow's path curves just enough for it to hit the center of a freely falling target.

Example 5.3. A remote-controlled 'target' plane is flying horizontally at an altitude of H = 3.50 km with a speed of u = 210 m/s. When the plane is directly overhead, a projectile is fired from the ground with a speed of v = 300 m/s. (a) At what angle α above the ground must the projectile be fired in order to hit the target? (b) In what time and (c) how far from the launch point does hitting the target occur? (d) Study the above questions also for the case of v = 350 m/s.



Solution. Suppose that the x-axis is directed along the ground, the y-axis points in the upward direction, and the origin O is placed at the launch point of the projectile. We need to write out the equations describing the motions of the two objects, the projectile and the plane.

Projectile.

The general equations of motion are of the form (1).

The initial location: $x_{10} = 0$, $y_{10} = 0$.

The initial velocity components: $v_{10x} = v\cos\alpha$, $v_{10y} = +v\sin\alpha$.

The acceleration components: $a_{1x} = 0$, $a_{1y} = -g$.

The equations describing the projectile's motion under the above conditions:

$$x_1 = v\cos\alpha\, t\,, \qquad v_{1x} = v\cos\alpha \tag{26}$$

$$y_1 = v\sin\alpha\, t - \frac{1}{2}gt^2\,, \qquad v_{1y} = v\sin\alpha - gt \tag{27}$$

Plane.

Though the plane is not a projectile (explain why), it has a constant zero acceleration, and Equations (1) still apply.

The initial location: $x_{20} = 0$, $y_{20} = H$.

The initial velocity components: $v_{20x} = +u$, $v_{20y} = 0$.

The acceleration components: $a_{1x} = 0$, $a_{1y} = 0$.

The equations describing the plane's motion under the above conditions:

$$x_2 = ut\,, \qquad v_{2x} = u \tag{28}$$

$$y_2 = H\,, \qquad v_{2y} = 0 \tag{29}$$

(a) – (c) Suppose the target is hit at a certain time $t = t^*$. Then the positions of the objects coincide:

$$x_1(t^*) = x_2(t^*)\,, \qquad y_1(t^*) = y_2(t^*)$$

Of (26) – (29), written for $t = t^*$, the equations for coordinates give the following algebraic system:

$$v\cos\alpha\, t^* = ut^* \tag{30}$$

$$v\sin\alpha\, t^* - \frac{1}{2}gt^{*2} = H \tag{31}$$

From (30), after the cancellation of t^*, we obtain:

$$\cos\alpha = u/v \tag{32}$$

whence $\alpha = \cos^{-1}(u/v) = 45.6^\circ$. It is tempting, but premature to consider this value of α as the answer: it only guarantees that the condition $x_1 = x_2$ is met for some time t^*. We have to make sure that the moment t^* does exist, and that at that moment, the other condition $y_1 = y_2$ is met as well. This implies existence of a real root of the quadratic equation (31).

Equation (31) can be rewritten in the standard form

$$at^{*2} + bt^* + c = 0$$

where the coefficients are: $a = g/2$, $b = -v\sin\alpha$, and $c = H$. Its discriminant is

$$D = b^2 - 4ac = v^2\sin^2\alpha - 2gH$$

Using the identity $\sin^2\alpha = 1 - \cos^2\alpha$ and the result (32), we can represent D as

$$D = v^2 - u^2 - 2gH$$

Calculations give: $D = -22{,}700$. With D being negative, there is no real solution for t^*. Under the condition v = 300 m/s, the target will not be hit at all.

There is at least one real solution for t^* if D = 0; that is, if

$$v = \sqrt{u^2 + 2gH} = 336\,\text{m/s}$$

The initial speed of the projectile must be increased to this minimum level.

(d) In the case of v = 350 m/s, we have:

$$\alpha = \cos^{-1}(u/v) = 53.1^\circ$$

$$a = 4.90, \quad b = -280, \quad c = 3500$$

The discriminant D = 9800, therefore, Equation (31) has two real solutions, given by the quadratic formulas:

$$t_1^* = \frac{-b - \sqrt{D}}{2a} = 18.5\,\text{s}, \quad t_2^* = \frac{-b + \sqrt{D}}{2a} = 38.7\,\text{s}$$

In order to hit the target, the projectile must be fired at an angle of 53.1° above the ground. Hitting occurs at $t_1^* = 18.5\,\text{s}$, at the point with coordinates

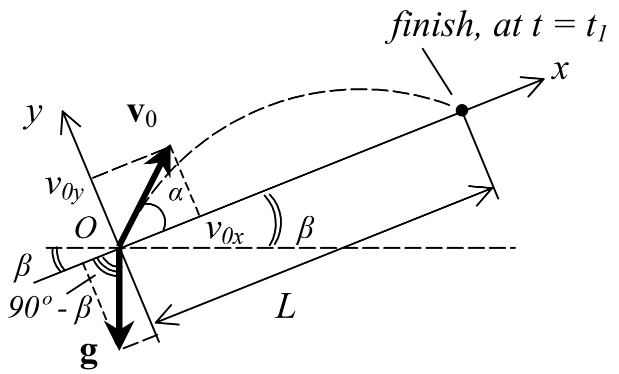
$$x_1 = 3880\,\text{m}, \quad y_1 = 3500\,\text{m}$$

at the distance

$$d = \sqrt{x_1^2 + y_1^2} = 5220\,\text{m}$$

from the launch point.

Example 5.4. A projectile is given an initial speed v_0 at an angle α above the surface of an incline. The incline itself makes an angle β above the horizontal. Find the distance, measured along the incline, from the launch point to where the object strikes the incline.



Solution. We can avoid some boring calculations by using the reference frame whose origin O is placed at the launch point, x-axis points along the surface of the incline, and y-axis is perpendicular to it.

The general equations of motion are given by (1). The object's initial location, initial velocity, and acceleration are as follows:

initially, the object is located at the origin, so $x_0 = 0$, $y_0 = 0$;

the initial velocity makes the angle α above the positive x-axis, so

$$v_{0x} = +v_0 \cos\alpha, \quad v_{0y} = +v_0 \sin\alpha$$

during the motion, the acceleration is directed downward and makes the angles of $90^\circ - \beta$ with the negative x-axis and β with the negative y-axis, respectively. Therefore,

$$a_x = -g \sin\beta, \quad a_y = -g\cos\beta$$

Equations (1) become

$$x = v_0 \cos\alpha\, t - \frac{1}{2} g \sin\beta\, t^2 \tag{33}$$

$$y = v_0 \sin\alpha\, t - \frac{1}{2} g \cos\beta\, t^2 \tag{34}$$

With no need to use the equations for velocity components, we left them out.

Suppose that the object strikes the incline at a distance L from the point O, and this occurs at some moment t_1. Then

$$x(t_1) = L, \quad y(t_1) = 0$$

Substituting these into Equations (33) and (34) gives an algebraic system involving the sought-for distance L:

$$L = v_0 \cos\alpha\, t_1 - \frac{1}{2} g \sin\beta\, t_1^{\,2} \tag{35}$$

$$0 = v_0 \sin\alpha\, t_1 - \frac{1}{2} g \cos\beta\, t_1^{\,2} \tag{36}$$

After being factored,

$$t_1\left(v_0 \sin\alpha - \frac{1}{2} g \cos\beta\, t_1 \right) = 0$$

Equation (36) yields two real roots: $t_1 = 0$, corresponding to the launch point, and $t_1 = 2v_0 \sin\alpha / g\cos\beta$, the time when the projectile returns to the incline. We substitute the latter into Equation (35) to calculate L:

$$L = v_0 \cos\alpha \frac{2v_0 \sin\alpha}{g\cos\beta} - \frac{1}{2} g \sin\beta \left(\frac{2v_0 \sin\alpha}{g\cos\beta} \right)^2 = 2v_0^{\,2} \sin\alpha \frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{g\cos^2\beta}$$

Now, making use of the trigonometric identity

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

we finally find:

$$L = \frac{2v_0^{\,2} \sin\alpha \cos(\alpha+\beta)}{g\cos^2\beta} \tag{37}$$

In particular, when $\beta = 0$, L becomes equal to the horizontal range R (see *Example 5.1c* and Formula (21)).

Had the x-axis been directed horizontally and the y-axis vertically, Equations (33) and (34) would have changed to

$$x = v_0 \cos(\alpha+\beta)\, t$$

$$y = v_0 \sin(\alpha+\beta)\, t - \frac{1}{2} g t^2$$

For the moment $t = t_1$, when the projectile strikes the incline, we would have written

$$x(t_1) = L\cos\beta\,, \qquad y(t_1) = L\sin\beta$$

Find L by making all necessary manipulations on your own. The answer, of course, remains the same.

6. ORBITAL MOTION

Circular or nearly circular motion of (A) artificial and natural satellites around a planet and (B) planets around the sun can be described in the framework of the **simplest version of the two-body problem.** This version is based on the following assumptions:

1) *The system consists of two distinctive objects, and the only forces that they exert on each other are gravitational forces.* These forces, attractive and acting along the line joining the centers of gravity of the objects, obey Newton's law of universal gravitation (5).

2) *The objects are treated as uniform spheres whose centers of gravity coincide with their geometric centers.* Consequently, r in Formula (5) is just their center-to-center distance. Furthermore, any of the objects whose size is very small in comparison with r is regarded as a particle (mathematical point of some mass) positioned at the object's geometric center.

3) *One of the objects, the planet in case* (A) *or the sun in* (B)*, is so massive* ($M >> m$) *as compared to the other one, 'satellite', that the satellite has no effect on it.* The center of the massive object remains fixed in place. At the same time, the massive object can rotate about its axis.

4) *The satellite is moving uniformly in a plane circular orbit around the center of the massive object.* The radius of the orbit is the distance r. The satellite has a constant speed v, but the direction of its velocity is continually changing, due to the centripetal acceleration

$$a_c = \frac{v^2}{r} \tag{38}$$

The acceleration a_c is caused by the only force which keeps the satellite in its orbit, namely, by the massive object's gravity. Newton's second law, $ma_c = F$, takes the form

$$m\frac{v^2}{r} = G\frac{Mm}{r^2} \tag{39}$$

The motion of a satellite is an example of *free fall*, with the acceleration due to gravity **g**. This acceleration points towards the center of the massive object and varies in magnitude with *r*: $g = F/m = GM/r^2$. It is also different for different heavenly objects, being equal to 9.80 m/s^2 only on the surface of the earth.

The mass *m* of the satellite is not a significant parameter, for it can be eliminated from Equation (39).

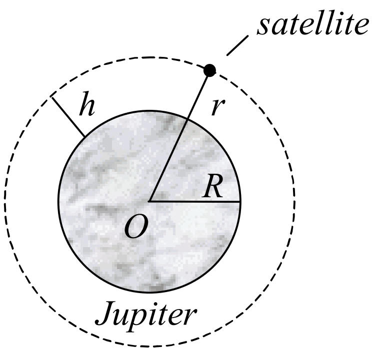
5) *A uniform circular motion is periodic*: starting at any point, the satellite returns to the same point over and over again in equal time intervals. The quantity most adequately reflecting this feature is the *period* T, the time required for the satellite to make one complete revolution. Since the distance covered in one revolution is the circumference $2\pi r$ of the orbit, and it is being traveled at a constant speed v, T is given by

$$T = \frac{2\pi r}{v} \tag{40}$$

The orbital radius r, the orbital speed v*, the period T, and the mass M make up the list of the major quantities we deal with in order to describe an orbital motion.* No more than two of them are usually unknown at a time. To determine those, we use two relations, Newton's second law (39) and definition (40) of period.

Example 6.1. A satellite is placed in an orbit $h = 6.00\times10^{+5}\text{ m}$ above the surface of Jupiter. Jupiter has a mass of $M = 1.90\times10^{+27}\text{ kg}$ and a radius of $R = 7.14\times10^{+7}\text{ m}$. Find (a) the orbital radius, (b) orbital speed, (c) acceleration, and (d) period of the satellite. Evaluate (e) the maximum orbital speed and (f) acceleration that a satellite could have if Jupiter had no atmosphere and were a perfectly smooth sphere. Compare these quantities with those for the earth ($R_E = 6.38\times10^{+6}\text{ m}$, $M_E = 5.98\times10^{+24}\text{ kg}$). (g) Why does not the satellite strike the planet?

Solution. We consider Jupiter as a uniform sphere of radius *R* and mass *M*, and the satellite as a particle of mass *m* in a circular orbit around the sphere.



(a) The orbital radius r is the distance between the center O of Jupiter and the satellite:

$$r = R + h = 7.20\times 10^{+7}\ \text{m}$$

(b), (e) The basic relations to be used are (39) and (40). The mass m and one r in (39) cancel out, to give $v^2 = GM/r$.

Thus, the orbital speed equals

$$v = \sqrt{\frac{GM}{r}} \tag{41}$$

Simple calculation gives: $v = 4.20\times 10^{+4}\ \text{m/s} = 42\,\text{km/s}$.

There is only one speed, given by (41)*, which a satellite can have in a circular orbit of fixed radius.* The closer the satellite to the surface of a planet, the smaller the orbital radius and the greater the orbital speed.

The *maximum speed* must be given to a satellite that is to orbit a planet near its surface, when $r = R$:

$$v_1 = \sqrt{\frac{GM}{R}} \tag{42}$$

For Jupiter, $v_1 = 4.21\times 10^{+4}\ \text{m/s}$.

In the case of the earth, $v_{1E} = \sqrt{GM_E/R_E} = 7.91\times 10^{+3}\ \text{m/s}$.

(c), (f) The acceleration of the satellite is just the acceleration due to Jupiter's gravity at the height h above the surface of Jupiter:

$$a_c = v^2/r = GM/r^2 = g = 24.5\,\text{m/s}^2$$

Its maximum value a_{c1} occurs near the surface of Jupiter:

$$a_{c1} = {v_1}^2/R = GM/R^2 = g = 24.9\,\text{m/s}^2$$

For the earth, $a_{c1E} = {v_{1E}}^2/R_E = GM_E/{R_E}^2 = g_E = 9.80\,\text{m/s}^2$.

(d) The period of the satellite is

$$T = 2\pi r/v = 1.08\times 10^{+4}\ \text{s} = 3.0\,\text{hours}$$

(g) When an object is projected with a great initial speed, the planet's curvature becomes significant: as the object falls, the planet curves away beneath it.

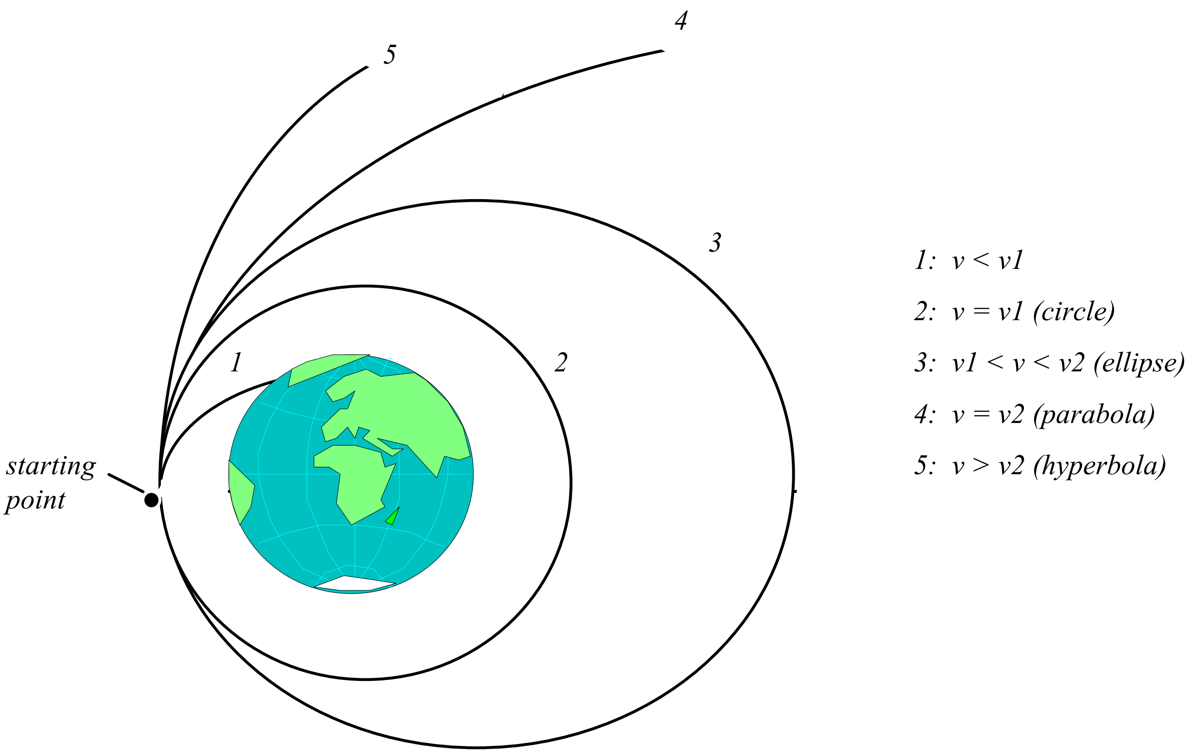


Fig. Trajectories of satellites

Nevertheless, the object returns to and strikes the planet (see Fig., path 1) unless its tangential speed attains the value given by (41). When launched at the speed (41), the object moves fast enough and misses the planet. It becomes a satellite and goes right on around the planet in a *circular orbit* (Fig., path 2).

Other results concerning orbital motion are as follows:

A) A satellite projected with an initial speed greater than (41), but less than

$$v_2 = \sqrt{\frac{2GM}{R}} \tag{43}$$

follows an *ellipse*, with the planet's center at one of the foci (Fig., path 3). Its instantaneous speed changes along the path. The motion is two-dimensional.

B) The speed v_2 is called the *escape speed*: when launched with v_2, a satellite never returns to the starting point, but escapes permanently from the planet. Its orbit is not closed anymore. It is a *parabola*, with the focus at the planet's center (Fig., path 4). The satellite travels with a variable instantaneous speed.

The escape speed is larger than the maximum orbital speed (42) by a factor of $\sqrt{2}$. In particular, $v_2 = 5.95\times10^{+4}$ m/s for Jupiter, and $v_{2E} = 1.12\times10^{+4}$ m/s for the earth.

The derivation of Formula (43) is based on the *law of conservation of energy*.

C) Launched with an initial speed greater than the escape speed, a satellite leaves the planet forever. It moves at a variable speed along a *hyperbola*, with the planet's center at one of its foci (Fig., path 5).

D) In general, when the masses *m* and *M* are comparable, the two-body problem can be resolved completely by using integral calculus. Both objects move with variable speeds in a fixed plane about their center of mass, which itself can move at a constant velocity. The individual paths followed by the objects are similar to each other. Depending on the initial conditions, their form is *one of the conic sections: a circle, an ellipse, a parabola, a hyperbolas, or a straight line*.

Example 6.2. An *isochronous* (*synchronous*) satellite of the earth is one that moves in the earth's equatorial plane, but, relative to a person on the ground, seems to stay above stationary. (a) What is the height above the ground at which all such satellites must be placed in orbit? (b) What must be the speed of a rocket that is used to place them in orbit? The earth has a mass of $M = 5.98\times10^{+24}$ kg and a radius of $R = 6.38\times10^{+6}$ m.

Solution. The period of an isochronous (synchronous) satellite is one day, the time that it takes for the earth to turn once about its axis:

$$T = 1\,\text{day} = 24\,\text{hours} = 8.64\times10^{+4}\ \text{s}$$

Since the mass *M* of the earth is also given, it remains for us to determine the orbital radius *r* and the orbital speed *v*. The desired height *h* is related to *r* by

$$h = r - R$$

As usual, Newton's second law (39) and definition (40) are used:

$$mv^2/r = GmM/r^2, \quad T = 2\pi r/v$$

(a) The common factor *m* cancels out. The radius *r* is found by excluding *v* from consideration. From the second equation, $v = 2\pi r/T$. Substituting this into the first equation gives:

$$4\pi^2 r^2/rT^2 = GM/r^2$$

Simple manipulations (multiplying both sides by r^2T^2, dividing both sides by $4\pi^2$, and reducing the fraction r^4/r to r^3) yield:

$$r^3 = \left(GM/4\pi^2\right)T^2$$

The orbital radius of an isochronous (synchronous) satellite is given by

$$r = \left(GM/4\pi^2\right)^{1/3} T^{2/3}$$

where *T*, being the period of the satellite, also represents the period of rotation of the planet about its axis.

In the case of the earth, $r = 4.23\times10^{+7}$ m. The height of the satellite above the ground is

$$h = r - R = 3.59\times10^{+7}\text{ m} = 35900\text{ km}$$

(b) The speed of the rocket that places the satellite in orbit equals the orbital speed of the satellite:

$$v = 2\pi r/T = 3.07\times10^{+3}\text{ m/s}$$

Example 6.3. The earth orbits the sun once a year at a distance of $r_1 = 1.50\times10^{+11}$ m. Pluto, the most distant planet, orbits the sun at a distance of $r_2 = 5.90\times10^{+12}$ m. (a) How long does it take for Pluto to make one complete revolution around the sun? (b) How many revolutions around the sun does the earth make during one Pluto's year? (c) What is the mass of Pluto?

Solution. We follow the *simplest model of planetary motion* in which the sun is the central stationary body, and planets move in circular orbits around it only under its gravity. The forces that the planets exert on one another and on the sun are ignored. In other words, the planets move independently around a common massive center. Their motion can be described by relations (39) and (40), written separately for each planet:

Earth: $m_1 v_1^2 / r_1 = G m_1 M / r_1^2$, $T_1 = 2\pi r_1 / v_1$

Pluto: $m_2 v_2^2 / r_2 = G m_2 M / r_2^2$, $T_2 = 2\pi r_2 / v_2$

(a), (b) Manipulations identical to those in *Example 6.2 (a)* yield:

$$r_1^3 = \left(GM/4\pi^2\right) T_1^2, \quad r_2^3 = \left(GM/4\pi^2\right) T_2^2$$

Dividing the left sides of these equations into each other, and then, in the same order, their right sides, we get:

$$r_1^3 / r_2^3 = T_1^2 / T_2^2 \tag{44}$$

Since $T_2^2 = T_1^2 \left(r_2^3 / r_1^3\right)$, taking the square roots of both sides gives:

$$T_2 = T_1 \left(r_2 / r_1 \right)^{3/2} = 247 \text{ years}$$

During one Pluto's year, the earth is expected to make 247 complete revolutions around the sun. In fact, the actual value for a Pluto's year is 248 years. The discrepancy between the predicted and experimental values is less than 1%. In your opinion, what factors cause this discrepancy?

(c) *The mass of Pluto cannot be determined from the data given.*

Relation (44) is referred to as **Kepler's third law of planetary motion**:

The ratio of the squares of the periods of any two planets revolving about the sun is equal to the ratio of the cubes of their mean distances from the sun.

Kepler's first and second laws *refine on* the simplest model of orbital motion. They state:

1) **The path of each planet about the sun is an ellipse with the sun at one focus.**

   Therefore, the distance between each planet and the sun varies as time passes.

2) **The orbital momentum of any planet is conserved.**

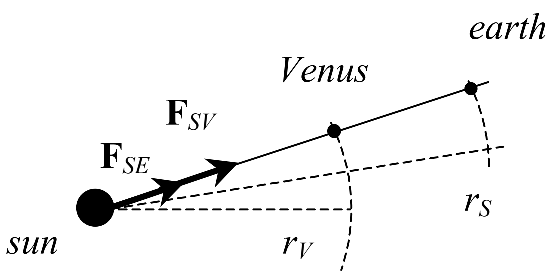
   That is, each planet follows its plane path nonuniformly, with a varying speed.

Example 6.4. The earth orbits the sun once a year, at a distance of $r_E = 1.50\times10^{+11}\,\text{m}$. Its mass is $m_E = 5.98\times10^{+24}\,\text{kg}$. Venus orbits the sun at a distance of $r_V = 1.08\times10^{+11}\,\text{m}$ and its mass is $m_V = 4.87\times10^{+24}\,\text{kg}$. The sun has a radius of $R = 6.96\times10^{+8}\,\text{m}$.

In order to partially substantiate the simplest model of planetary motion (see *Example 6.3)*, (a) show that the sun and the planets can be treated as particles, and estimate (b) the maximum acceleration and displacement of the sun due to the planets, (c) the maximum acceleration of each of the planets due to the other planet and the sun, and (d) the mass *M* of the sun.

Solution. (a) Among the given objects, the sun has the largest radius R. The distance $r_E - r_V = 4.2\times10^{+10}\,\text{m}$ between the earth and Venus is the shortest distance between any pair of the objects. Nevertheless, R is less than the earth-Venus distance by a factor of $(r_E - r_V)/R \approx 60$. All the objects can be treated as particles, with an accuracy of at least 2%. In fact, the accuracy is much better: we should compare quantities which are related to the same object. For instance, R is less than the sun-Venus distance by a factor of $r_V/R \approx 155$.

(b) The forces exerted on the sun by Venus and by the earth are $F_{SV} = Gm_V M/r_V^{\,2}$ and $F_{SE} = Gm_E M/r_E^{\,2}$, respectively. The sun experiences the greatest net force F_S and, therefore, the maximum acceleration a_S when the planets are aligned on a straight line on the same side of the sun. Then $F_S = F_{SV} + F_{SE}$, and the maximum acceleration of the sun equals:



$$a_S = F_S/M = Gm_V/r_V^{\,2} + Gm_E/r_E^{\,2} = 4.67\times10^{-8}\,\text{m/s}^2$$

Even if the sun had this acceleration permanently during one year, that is, for $t = T_E = 365.25\,\text{days} = 3.16\times10^{+7}\,\text{s}$, it would have displaced for that time interval by

$$S = \frac{1}{2}a_S t^2 = 2.32\times10^{+7}\,\text{m}$$

This amounts to about 3% of its radius: $S/R = 1/30$. In fact, ignoring S and considering the sun to be stationary introduces a much smaller error. Would you explain why?

(c) The maximum forces exerted on the earth by Venus and by the sun are $F_{EV} = Gm_E m_V / (r_E - r_V)^2$ and $F_{ES} = Gm_E M / r_E^2$, respectively. Venus causes the earth to gain the acceleration

$$a_{EV} = F_{EV} / m_E = Gm_V / (r_E - r_V)^2 = 1.84 \times 10^{-7} \text{ m/s}^2$$

The mass of the sun is not given. Nevertheless, the acceleration a_{ES} of the earth due to the sun can be estimated through the orbital speed v_E of the earth and its centripetal acceleration a_E:

$$v_E = 2\pi r_E / T_E = 2.98 \times 10^{+4} \text{ m/s}, \quad a_E = v_E^2 / r_E = 5.93 \times 10^{-3} \text{ m/s}^2$$

The net acceleration a_E of the earth is greater than its acceleration a_{EV} due to Venus by a factor of $a_E / a_{EV} = 32200$. Essentially, a_E is affected by the sun's gravity only. The identical result is also valid for Venus, which is even closer to the sun. This means that the interaction between the earth and Venus is negligibly small.

(d) The estimates carried out in (a) – (c) are consistent with the simplest model of planetary motion. Then, since a_E is caused by the sun only, Newton's second law, written for the earth, yields:

$$m_E a_E = Gm_E M / r_E^2$$

The mass of the sun is

$$M = a_E r_E^2 / G = 2.0 \times 10^{+30} \text{ kg}$$

APPENDIX. WHAT CALCULATOR DO YOU NEED?

Make sure that your calculator has capabilities to perform the following:

(a) *addition (+), subtraction (**-**), multiplication (×), and division (**:**);*

(b) *work in exponential (**EXP** or **EE**) notation;*

(c) *raise any number to any power (x^y or y^x, $x^{1/y}$ or $y^{1/x}$, ${}^y x$ or ${}^x y$). Separate keys may be designed to square and cube numbers (x^2, x^3), find square and cube roots ($\sqrt{x}$, $\sqrt[3]{x}$), calculate reciprocals (x^{-1} or **1/x**), etc.;*

(d) *work with trigonometric functions (**sin**, **cos**, **tan**) and their inverse functions (sin^{-1}, cos^{-1}, tan^{-1});*

(e) *work with logarithms (**log**, **ln**) and antilogs (10^x, e^x);*

(f) *statistical calculations*).*

Learn how to enter data into your calculator and perform math operations with the above arithmetic and function keys. Your owner's manual contains all necessary information, numerous hints and practical examples. On the other hand, you can always contact one of our staff members or tutors for assistance. We are here to help.

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*) This feature is needed if you plan to take courses on statistics.

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