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ON NUMBERS OF THE TYPE $n = (u^2 + dv^2)w$ IN ARITHMETIC PROGRESSION

Let us $R(n)$ denotes the number of representations of positive integers n by form $n = (u^2 + v^2)w,\ u, v \in \mathbb{Z},\ w \in \mathbb{N}$. The function $R(n)$ is an analogue of the divisor function $d_3(n)$. Summarize the Heath-Brown results on distribution of value of the divisor function $d_3(n)$ on an arithmetical progression $n \equiv a(modq),\ (a, q) = 1$, with increasing the arithmetical ratio together with x, an asymptotic formula for summatory function for $R(n)$ was being construct, which is a non-trivial for $q \to \infty$. The proof of this result use the truncated functional equation on the line $Res = \frac{1}{2} + \Delta,\ |\Delta| < \frac{1}{2}$ of the Hecke Zeta function with transport of an imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$.

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Introduction

Definition. Let denotes by $R(n)$ the number representations a positive integer n in the form $n = (u^2 + dv^2)w,\ u, v \in \mathbb{Z},\ w \in \mathbb{N},\ d$ is a free square positive integer. The function $R(n)$ you can consider as an analogue the arithmetic function $d_3(n)$ (a number of representations of n as a product of three natural numbers: $d_3(n) = \sum\limits_{n=n_1 n_2 n_3} 1$).

We denote

$$K(d) = \left\{ u + i\sqrt{d}v \,|\, u, v \in \mathbb{Z} \right\}.$$

For $\alpha \in K(d)$ we put $N(\alpha) = u^2 + dv^2,\ S_p(\alpha) = \lambda u = \lambda Re(\alpha)$. Our aim deduce an a asymptotic formula for summatory functions

$$F(x) = \sum_{n \leqslant x} R(n);$$

$$F(x, a, q) = \sum_{\substack{n \equiv a(modq) \\ n \leqslant x}} R(n).$$

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Notation. We will use the following notations:

- $G := \left\{ a + b\sqrt{d}i \,|\, a, b \in \mathbb{Z},\, i^2 = -1 \right\}$ is the ring of integer elements of the field $\mathbb{Q}(\sqrt{-d})$;
- G_γ is the ring of residues of G module γ;
- $G_\gamma^* = \{w \in G_\gamma,\, (w, \gamma) = 1\}$;
- $s \in \mathbb{C},\ s = \sigma + Res,\ t = Ims$;
- $\Gamma(z)$ is the Euler gamma function;
- by $f \ll g$ (or $f = O(g)$) for $x \in X$, where X is an arbitrary set on which f and g defined, we mean that exists a constant $C > 0$ such that $|f(x)| \leqslant cg(x)$ for all $x \in X$.

Let us denote shifting the Hecke function

$$Z_m(s; \delta_1, \delta_2) := \sum_{w \in G} e^{gmiarg(w+\delta_1)} \cdot e^{2\pi i Re(\delta_2 w)},$$

$Res > 1,\ \delta_1, \delta_2 \in \mathbb{Q}(\sqrt{-d}),\ d$ is a free-square, $d > 0$; (the pair (δ_1, δ_2) we call a shift of w).

In the domain $Res > 1$ the series for $Z_m(s; \delta_1, \delta_2)$ is defined by an absolutely convergent Dirichlet series.

1. Auxiliary arguments

Lemma 1. *The shifting Hecke zeta-function of the field $\mathbb{Q}(\sqrt{-d})$ satisfies the functional equation*

$$\pi^{-s}\Gamma\left(\frac{g|m|}{2} + s\right) Z_m(s; \delta_1, \delta_2) =$$
$$= \pi^{-(1-s)}\Gamma\left(\frac{g|m|}{2} + 1 - s\right) Z_m(1 - s; -\delta_2, \delta_1) e^{-2\pi i Re(\delta_2 \delta_1)}.$$

Moreover, $Z_m(s; \delta_1, \delta_2)$ is an entire function if $m \neq 0$. If $m = 0$ the for δ_2 not integer element from $\mathbb{Q}(\sqrt{-d})$ the $Z_m(s; \delta_1, \delta_2)$ is also entire function. For $m = 0$ and δ_2 is an integer element of $\mathbb{Q}(\sqrt{-d})$ the Hecke zeta-function is holomorphic except at $s = 1$, where it has a simple pole with residue π.

Proof. For $\delta_1 = \delta_2 = 0$ and $m = mg$, where g is a number of units in the ring G we get the well-known Hecke zeta-function $Z_m(s, G)$ of the first kind with the exponent m (see Hecke [3]). In [1] this lemma has be stated in case $d = 1$. But for the completeness of treatment we restore a proof of our statement.

We start from the relation

$$\Gamma(s) \cdot |w + \delta_1|^{-2s} = \int\limits_0^\infty exp(-x \cdot |w + \delta_1|^2) x^{s-1} dx.$$

For $Res > 1$ and $m \in \mathbb{Z}$ we have

$$\Gamma\left(\frac{g}{2}|m| + s\right) Z_m(s; \delta_1, \delta_2) = \int\limits_0^{\delta_2} \sum_{\substack{w \in G \\ w \neq -\delta_1}} e^{-x|w+\delta_1|^2} x^{s-1} dx.$$

Let us denote $\delta_j = \delta_{j1} + i\sqrt{d}\delta_{j2},\ j = 1, 2.$

Then grountruthing shows that the functions

$$f(v_1, v_2) = exp\left(-\frac{\pi^2}{x}\left[(\delta_{11} + v_1)^2 + d(\delta_{12} + v_2)^2\right]\right)$$

$$\hat{f}(v_1, v_2) = \frac{\pi}{x} exp\left(-\frac{\pi^2}{x}\left[(\delta_{11} + v_1)^2 + d(\delta_{12} + v_2)^2\right]\right)$$

satisfy the conditions of Poisson summation formula. Hence, putting

$$\Theta_m(x, \delta_1, \delta_2) =$$
$$= \sum_{w \in G} exp(-x(w + \delta_1)^2) \cdot (w + \delta_1)^{gm} exp\left(2\pi Re(\overline{\delta_2} w)\right)$$

and applying the Poisson formula, we find

$$\Theta_0(x, \delta_1, \delta_2) = \frac{\pi}{x}\Theta_0\left(\frac{\pi^2}{x}, \delta_2, -\delta_1\right) exp\left(-2\pi i Re(\delta_1 \overline{\delta_2})\right).$$

Consider the operator

$$\frac{d}{d\delta_1} := \frac{\partial}{\partial \delta_{11}} + i\sqrt{d}\frac{\partial}{\partial \delta_{12}}.$$

Then the following equalities hold for the $m \geqslant 0$

$$(-2x)^{gm}\Theta_m(x, \delta_1, -\delta_2) = \frac{d^m}{d\delta_1^m}\Theta_0(x, \delta_1, \delta_2)$$

and

$$\frac{\pi}{x}(-2\pi i)^{4m}\Theta_m\left(\frac{\pi^2}{x},\delta_1,-\delta_2\right)exp\left(-2\pi i(\delta_1\overline{\delta_2})\right) =$$
$$= \frac{d^m}{d\delta_1^m}\left(\frac{\pi}{x}\Theta_0\left(\frac{\pi^2}{x},\delta_2,-\delta_1\right)exp\left(-2\pi iRe(\delta_1\overline{\delta_2})\right)\right).$$

So, for any $m \in \mathbb{Z}$ the following functional equation

$$\Theta_m(x,\delta_1,\delta_2) = \left(\frac{\pi}{x}\right)^{gm+1}\Theta_m\left(\frac{\pi^2}{x},\delta_2,\delta_1\right)exp\left(-2\pi iRe(\delta_1\overline{\delta_2})\right) \qquad (1)$$

hold.

Now, applying reasoning used for the proof of functional equation for Riemann zeta–function (see [4]) by the functional equation for a theta–function Θ_m, we infer

$$\Gamma\left(\frac{g|m|}{2}+s\right)Z_m(s;\delta_1,\delta_2) = \pi^{-(1-2s)}exp\left(-2\pi iRe(\delta_1\overline{\delta_2})\right)I_m(\delta_1,\delta_2),$$

where

$$I_m(\delta_1,\delta_2) =$$
$$= \int_0^\infty \sum_{\substack{w\in G\\ w\neq-\delta_1}} exp(-x|w+\delta_1|^2)(w+\delta_1)^{gm}\cdot exp(2\pi iRe(\overline{\delta_2}w))x^{s+2+\frac{1}{2}gm-1}dx =$$
$$= \int_0^\pi + \int_\pi^\infty := I_{m1} + I_{m2}.$$

In integral I_m we apply the functional equation (1) for $\Theta_m(x,\delta_1,\delta_2)$ and make substitution $x = \pi^2 y^{-1}$. We have

$$\Gamma\left(\frac{1}{2}g|m|+s\right)Z_m(s;\delta_1,\delta_2) = \pi^{2s-1}exp\left(-2\pi iRe(\delta_1\overline{\delta_2})\right)\times$$
$$\times\int_\pi^\infty \sum_{\substack{w\in G\\ w\neq-\delta_2}} exp(-x|w+\delta_1|^2)(w+\delta_2)^{gm}exp(-2\pi iRe(\overline{\delta_1}w))x^{-s+\frac{1}{2}|gm|}dx+$$
$$+\int_\pi^\infty \sum_{\substack{w\in G\\ w\neq-\delta_1}} exp(-x|w+\delta_1|^2)(w+\delta_1)^{gm}exp(-2\pi iRe(\overline{\delta_2}w))x^{s-1+\frac{g}{2}|m-1|}dx+$$
$$+\varepsilon(m,\delta_2)\frac{\pi^s}{s-1} - \varepsilon(m,\delta_1)exp(-2\pi iRe(\delta_1\overline{\delta_2}))\cdot\frac{\pi^s}{s}, \qquad (2)$$

where

$$\varepsilon(m,\delta) = \begin{cases} 1 & \text{if } m=0 \text{ and } a\in G,\\ 0 & \text{otherwise.}\end{cases}$$

The relation (2) was obtained for $Res > 1$. However, the right part of this equality is an analytic function in all–complex s-plaines except maybe the points $s = 0$ and $s = 1$, which can be the poles.

Finally multiplying (2) be $exp(2\pi i Re(\overline{\delta_1}\delta))\pi^{-2s+1}$ and making the substitution $s \to 1-s,\ \delta_1 \to \delta_2,\ \delta_2 \to \delta_1$, we obtain that the right part doesn't vary, and hence proved the following functional equation for $m > 0$

$$\pi^{-s}\Gamma\left(\frac{g}{2}|m| + s\right) Z_m(s;\delta_1,\delta_2) =$$
$$= \pi^{-(1-s)}\Gamma\left(\frac{g}{2}|m| + 1 - s\right) Z_{-m}(1-s;-\delta_2,\delta_1) exp(-2\pi i Re(\overline{\delta_1}\delta_2)).$$

For $m = -m',\ m' > 0$, we put $\delta_1 = -\delta_1',\ \delta_2 = -\delta_2'$, and then we obtain

$$Z_m(s;\delta_2,\delta_1) = Z_{m'}(s;-\delta_2,-\delta_1) \text{ and } Z_{m'}(1-s;\delta_1,-\delta_2) = Z_m(1-s;-\delta_1,\delta_2).$$

Thus, for any $m \in \mathbb{Z}$

$$\pi^{-s}\Gamma\left(\frac{g|m|}{2} + s\right) Z_m(s;\delta_2,\delta_1) =$$
$$= \pi^{-(1-s)}\Gamma\left(\frac{g|m|}{2} + 1 - s\right) Z_{-m}(1-s;-\delta_1,\delta_2) exp(-2\pi i Re(\overline{\delta_1}\delta_2)) =$$
$$= \pi^{-(1-s)}\Gamma\left(\frac{g|m|}{2} + 1 - s\right) Z_{-m}(1-s;\delta_1,-\delta_2) exp(-2\pi i Re(\overline{\delta_1}\delta_2)). \qquad \square$$

Consequence 1. *For $\delta_2 \notin G$ (but $\delta_2 \in \mathbb{Q}(\sqrt{-d})$), then $Z_0(0;\delta_1,\delta_2) = 0$.*

Consequence 2. *In the strip $\varepsilon \leqslant Res \leqslant 1+\varepsilon$ we have*

$$(s-1)Z_m(s;\delta_1,\delta_2) \ll (|t|+3)(t^2+m^2)^{k_1} q^{k_2},$$

where $k_1 = \dfrac{(1-2\sigma)(1-\sigma+\varepsilon)}{1+2\varepsilon}$, $k_2 = -\dfrac{\sigma+\varepsilon}{1+2\varepsilon}$, $\varepsilon > 0$ an arbitrary little number, holds.

This follows of once if we employ by the Phragmén–Lindelöf principle and the estimates for $Z_m(s;\delta_1,\delta_2)$ on the band edge $-\varepsilon \leqslant Res \leqslant 1+\varepsilon$.

For $q \in \mathbb{N}$ let us denote χ a multiplicative character of the group G_q^* : $\chi(\alpha) := \chi(N(\alpha))$. We have for $\delta_1 = \dfrac{l_1}{q},\ \delta_2 = \dfrac{l_2}{q},\ l_1, l_2 \in \mathbb{Z}_q$:

$$Z_m\left(s;\chi,\frac{l_2}{q}\right) := \sum_{w\in G} \frac{e^{gmiargw}}{N(w)^s}\chi_2\left(N(w)\right) e^{2\pi i Re\left(\frac{l_2}{q}w\right)},$$

$$Z_m\left(s;\frac{l_1}{q},\frac{l_2}{q}\right)=\frac{1}{\varphi(q)}\sum_{\chi}Z_m\left(s;\chi,\frac{l_2}{q}\right).$$

In [3] we have the following truncated functional equation.

Lemma 2. *Let $q\in\mathbb{N}$, $m\in\mathbb{Z}$, $d>0$ be a free-square rational integer, R is an ideal class of the field $\mathbb{Q}(\sqrt{-d})$; $s\in\mathbb{C}$, $s=\sigma+it$, $\tau\in\mathbb{C}$, $arg\tau=actg\dfrac{t}{\sigma+\frac{g}{2}}$,*

$$g=\begin{cases}4 & \text{if } d=1;\\ 2 & \text{if } d=3;\\ 1 & \text{in other cases;}\end{cases}$$

$$x=\frac{dq^2\left(t^2+\left(\frac{g}{2}|m|+\sigma\right)^2\right)^{\frac12}}{2\pi|\tau|};\qquad y=\frac{dq^2\left(t^2+\left(\frac{g}{2}|m|+\sigma\right)^2\right)^{\frac12}}{2\pi|\tau^{-2}|}$$

$$X=\left(1+\frac{2(M+5)}{g|m|}\log x\right);\qquad Y=\left(y+\frac{2(M+5)}{g|m|}\right).$$

Then for $t^2+gm^2\geqslant const$ the following truncated functional equation for

$$Z_m\left(s;\chi_q,\frac{l_2}{q}\right)=$$
$$=\sum_{\substack{w\in R\\ N(w)\leqslant\chi}}\frac{e^{gmiargw}}{N(w)^s}\cdot\left(\chi_q(N(w))e^{2\pi iRe\left(\frac{l_2}{q}w\right)}\times\Gamma^*\left(s+\frac{g}{2}|m|,\frac{2\pi\tau N(w)}{\sqrt{d}q}\right)\right)+$$
$$+\left(\frac{dq^2}{4\pi^2}\right)^{\frac12-\epsilon}\left(\frac{\Gamma\left(\frac{g}{2}|m|+1-s\right)}{\Gamma\left(s+\frac{g}{2}|m|\right)}\sum_{\substack{w\in\overline{R}\\ N(w)\leqslant\chi}}\frac{\overline{\chi_2}(N(w))}{N(w)^{1-s}}e^{-gmiargw}e^{-2\pi iRe\left(\frac{l}{q}w\right)}\times\right.$$
$$\left.\times\Gamma^*\left(1-s+\frac{g}{2}|m|,\frac{2\pi\tau^{-1}N(w)}{\sqrt{d}q}\right)\right)+O\left(X^{-M}+Y^{-M}\right),$$

where $M>0$ is an arbitrary number.

$$\Gamma^*\left(z+\frac{g}{2}|m|,\frac{2\sigma iN(w)}{\sqrt{d}q}\tau_r\right)=\Gamma\left(z+\frac{g}{2}|m|,\frac{2\pi N(w)}{\sqrt{d}q}\tau_r\right)\cdot\Gamma\left(z+\frac{g}{2}|m|\right)^{-1}.$$

Moreover $\Gamma^*\left(z+\frac{g}{2}|m|,\frac{2\sigma iN(w)}{\sqrt{d}q}\tau_r\right)$ in all indicated parameters have the estimation

$$\ll exp\Bigg(\frac{-N(w)+z}{z}\left(\frac{g}{2}|m|+\sigma\right)\times$$

$$\times\left(\frac{N(w)}{z}\right)^{\frac{g}{2}|m|+Rew}\left(t^2+\left(\frac{g}{2}|m|+\sigma\right)^2\right)\times\left(t^2+\left(\frac{g}{2}|m|+\sigma\right)^2\right)^{\frac{1}{q}}+$$

$$+\left(\frac{N(w)}{z}-\left(t^2+\left(\frac{g}{2}|m|+\sigma^2\right)^{-1}\right)^{\frac{q}{2}}\right)^{-1}\Bigg).$$

Similarly truncated equation is true for $Z_m\left(s;\frac{l_1}{q},\frac{l_2}{q}\right)$, where $l_1,l_2\in G_q$.

We shall need

Lemma 3. *The zeta–function Gurwits $\xi(s,u)$ determined by the relation for $Res>1$*

$$\xi(s,u)=\sum_{n=0}^{\infty}\frac{1}{(n+u)^s},\quad (0<u\leqslant 1)$$

is an analytic for all $s\in\mathbb{C}$ (except $s=1$), where it has a prime pole with residue 1. Moreover $\xi(s,u)$ satisfies the following Gurwits relation

$$\xi(s,u)=\frac{2\Gamma(1-s)}{(2\pi)^{1-s}}\left\{\sin\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi a}{n^{1-s}}+\cos\frac{\pi s}{2}\sum_{n=1}^{\infty}\frac{\sin 2n\pi a}{n^{1-s}}\right\}.$$

Lemma 4. *Let $s=\sigma+it,\ |\sigma|\leqslant 2,\ \tau\in\mathbb{C},\ arg\tau=\left(\frac{\pi}{2}+|t|^{-1}\right)sgn(Ims)$. There exists a constant $t_0>1$ such that uniformly at $|t|>t_0,\ \tau$, we have the truncated functional equation*

$$\xi(s,u)=$$

$$=\frac{1}{2}\sum_{|n+u|\leqslant xlogx}\frac{F\left(s;(n+u)\tau^{\frac{1}{2}}\sqrt{\pi}\right)+a_nF\left(1-s;(n+u)\tau^{\frac{1}{2}}\sqrt{\pi}\right)}{(n+u)^s}+$$

$$+\pi^{-\frac{1-2s}{2}}\frac{\Gamma\left(\frac{1}{2}(1-s)\right)}{2\Gamma\left(\frac{1}{2}s\right)}\sum_{|n|\leqslant y\log y}\frac{F\left(1-s;n\tau^{-\frac{1}{2}}\sqrt{\pi}\right)e^{-2\pi inu}}{u^{1-s}}+$$

$$+\pi^{-\frac{1-2s}{2}}\frac{\Gamma\left(\frac{1}{2}(1-s)+\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}s+\frac{1}{2}\right)}\sum_{|n|\leqslant y\log y}\frac{b_nF\left(-s;n\tau^{-\frac{1}{2}}\sqrt{\pi}\right)}{n^{1-s}}+$$

$$+ O\left(x^{-M} + y^{-M}\right),$$

where

$$a_n = \begin{cases} sgn(n) & \text{if } n \neq 0; \\ 1 & \text{if } n = 0; \end{cases} \qquad b_n = -isign(n)e^{2\pi inu},$$

$$x = |t|^{\frac{1}{2}}(\sqrt{2}|\tau|)^{-1}, \quad y = |t|^{\frac{1}{2}}|\tau|(\sqrt{2})^{-1}, \quad M > 0 \text{ – arbitrary constant.}$$

Moreover, uniformly in all parameters

$$F(w, Z) = l + O\left(exp\left\{-\frac{|Z|^2}{|t|}\right\} \cdot \left(\frac{|Z|}{|t|^{\frac{1}{2}}}\right)^{Rew} \times \left(1 + \left|\frac{1}{2}|t|^{\frac{1}{2}} - \frac{|Z|}{|t|^{\frac{1}{2}}}\right|\right)^{-1}\right)$$

where $l = 1$ if $|n + u| \leqslant x$ and $|n| \leqslant y$, and $l = 0$ in other cases.

Lemma 5. *Consider a Dirichlet polynomial*

$$P(s; l, q, N) := \sum_{n=1}^{N} \frac{a_n}{n^s}.$$

For any real values T_0 and T_1 we have

$$\int_{T_0}^{T_0+\tau} |P(s; l, q, N)|^2 dt \ll T + \frac{4\pi}{\sqrt{3}} \cdot \frac{N}{q} \sum_{\substack{n=1 \\ n \equiv l (mod q)}}^{N} |a_n|^2$$

(It is some generalization of Montgomery Theorem for an integrals at the Dirichlet polynomials).

Main Results

Let us C denotes the following conditions

$$\begin{cases} \delta = \dfrac{l_1 + il_2\sqrt{d}}{q}, \ l_1, l_2 \in \mathbb{Z}_q, \ q \in \mathbb{N}, \ q > 1; \\ (l_1^2 + l_2 \cdot d)l_0 \equiv a(modq) \\ a, l_0 \in \mathbb{Z}_q, \ (a, q) = 1. \end{cases}$$

Then the generating series for $R(mq + a)$ have the form

$$F(s; a, q) = \frac{1}{N(q)} \sum_{(C)} Z_0\left(s; \frac{\delta}{q}, 0\right) \xi\left(s, \frac{l_0}{q}\right), \quad Res > 1, \tag{3}$$

where $\xi\left(s,\dfrac{l_0}{q}\right)$ is Gurwits zeta-function.

Thus the Perron formula for an arithmetic progression gives

$$\sum_{\substack{n\equiv a(modq)\\ n\leqslant x}} R(n)=\frac{1}{2\pi i}\int\limits_{C-iT}^{C+iT}\left(F(s;a,q)-\sum_{n\in B}\frac{R(n)}{n^s}\right)\cdot\frac{x^s}{s}ds+$$

$$+O_\varepsilon\left(\frac{x^{1+\epsilon}}{Tq}\right)+O(x^\varepsilon),\quad (4)$$

where $B:=\{a,a(1\pm qN(w)|w=\pm1,\pm i\}$.

The Gurwits function $\xi\left(s,\dfrac{l_0}{q}\right)$ is an analytic on all completely s-plane except at the point $s=1$ with residue 1. At a point $s=1$ there is expansion in series

$$\xi\left(s,\frac{l_2}{q}\right)=\frac{1}{s-1}+c_0\left(\frac{l_2}{q}\right)+c_1\left(\frac{l_2}{q}\right)(s-1)+\dots\,,\qquad (5)$$

where $c_0\left(\dfrac{l_2}{q}\right)=E+\left(\dfrac{q}{l_2}\right)$, E it Euler constant.

Moreover, $Z_m(s;\delta_1,\delta_2)$ is an analytic function on all plane of complex numbers except of the case $m=0,\ \delta_2\in G$, when $Z_m(s;\delta_1,\delta_2)$ have first polarized:

$$Z_0(s;\delta,0)=\frac{\pi}{s-1}+a_0(\delta)+a_1(\delta)\cdot(s-1)+\dots\,,\qquad (6)$$

where

$$a_0(\delta)=E+L'(1,\chi_4)+b_0(\delta)+\sum_{\beta\in B}N^{-1}(\delta+\beta),\ \delta\neq0,\qquad (7)$$

E is the Euler constant, $B=\{0,\pm1,\pm i\}$;

χ_4 is the Dirichlet L-function with the non-principal character module 4;

$|b_0(\delta)|\leqslant$ an absolute constant, $b_0(\delta)=4+O\left(N^{\frac{1}{2}}(\delta)\right)$. (see [3], [6])

Applying the Phragmén–Lindelöf principal and the estimations $\xi\left(s,\dfrac{b}{q}\right)$ and $Z\left(s;\dfrac{\delta}{q},0\right)$ on the boundary of the strip $-\varepsilon\leqslant Res\leqslant1+\varepsilon$ may be calculated for $Ims=t,\ |t|\geqslant t_0>3$

$$F(s;a,q)\ll\left(q^{\frac{1}{2}+\varepsilon}|t|^{\frac{3}{2}+\varepsilon}\right)^{\frac{1+\varepsilon-\delta}{1+2\varepsilon}}\left(q^{-1+\varepsilon}\right)^{\frac{\sigma+\varepsilon}{1+2\varepsilon}}\qquad (8)$$

(with constant in symbol " $\ll$ " is an absolute constant).

Let us calculate $\underset{s=1}{res}\left\{F(s;a,q)\cdot\frac{x^s}{s}\right\}$. We have use the expanding (5) and (6):

$$\underset{s=1}{res}\left\{F(s;a,q)\frac{x^s}{s}\right\}=\frac{\pi\chi\rho(a,q)}{q^2}\left(\log\frac{x}{q^2}+E-1\right)+ \\ +\frac{x}{q^2}\sum_{\substack{\alpha_0\in G_q^*\\ N(\alpha_0)\equiv a(modq)}} a_0\left(\frac{\alpha_0}{q}\right)-\frac{xr(a)}{a}, \quad (9)$$

where $\rho(a,q)$ is the number of solutions of the congruence $u^2+v^2\equiv a(modq)$, $(a,q)=1$, and $a_0\left(\frac{\alpha_0}{q}\right)$ from (7) for $\delta=\frac{\alpha_0}{q}$.

Hence (7) gives

$$\frac{x}{q^2}\sum_{\substack{\alpha_0\in G_q\\ N(\alpha_0)\equiv a(modq)}} a_0\left(\frac{\alpha_0}{q}\right)=\frac{\chi\rho(a,q)}{q^2}\left(\pi E+L'(1,\chi_4)+O(1)\right)+ \\ +\frac{x}{q^2}\sum_{\substack{\alpha_0\in G_q\\ N(\alpha_0)\equiv a(modq)}}\sum_{\beta\in B}N^{-1}\left(\beta+\frac{\alpha_0}{q}\right). \quad (10)$$

Next,

$$\frac{x}{q^2}\sum_{\substack{\alpha_0\in G_q\\ N(\alpha_0)\equiv a(modq)}}\sum_{\beta\in B}N^{-1}\left(\beta+\frac{\alpha_0}{q}\right)=O\left(\frac{x}{q^{\frac{3}{2}}}\log x\right) \quad (11)$$

$$\rho(a,q)=c(a,q)q\prod_{p|q}\left(1-\frac{\chi_4(p)}{p}\right),\quad 0<c(a,q)\leqslant 2$$

(see [2]).

So,

$$\underset{s=1}{res}\left\{F(s;a,q)\frac{x^s}{s}\right\}=\frac{\pi\chi\rho(a,q)}{q^2}\left(\log\frac{x}{q^2}+E-1\right)+O\left(\frac{x}{q^{\frac{3}{2}}}\log x\right) \quad (12)$$

Now we are in a position to prove the main theorem.

Theorem. *Let us $a,q\in\mathbb{N}$, $(a,q)=1$. Then the asymptotic formula*

$$\sum_{\substack{n\equiv a(mod q)\\ n\leqslant x}} R(n) = c(a,q)\frac{x}{q}\prod_{p|q}\left(1-\frac{\chi_4(p)}{p}\right)\left(\log\frac{x}{q^2}+E-1\right)+ + O\left(\frac{x^{\frac{3}{5}}}{q^{\frac{1}{5}}}\log^3 x\right),$$

holds.

Proof. Consider the rectangle with the vertexes in points

$$c-iT,\ c+iT,\ \frac{1}{2}+iT,\ \frac{1}{2}-iT$$

($T>1$ and its precise meaning be determined).

We have

$$\frac{1}{2\pi i}\int\limits_{c-iT}^{c+iT}\left(F(s;a,q)-\sum\frac{R(n)}{n^s}\right)\frac{x^s}{s}ds = res\left\{\left(F(s;a,q)-\sum_{n\in B}\frac{R(n)}{n^s}\right)\frac{x^s}{s}\right\}+ + \left\{\frac{1}{2\pi i}\int\limits_{\frac{1}{2}-iT}^{c-iT} - \frac{1}{2\pi i}\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} - \frac{1}{2\pi i}\int\limits_{\frac{1}{2}-iT}^{\frac{1}{2}+iT}\right\}\left(F(s;a,q)-\sum_{n\in B}\frac{R(n)}{n^s}\right)\frac{x^s}{s}ds = = \underset{s=1}{res}\left\{\left(F(s;a,q)-\sum_{n\in B}\frac{R(n)}{n^s}\right)\frac{x^s}{s}\right\}+I_1-I_2-I_3 \quad (13)$$

is say.

In the integrals I_1 and I_3 we apply an estimate under the integral function by (8). So we have

$$I_1, I_2 \ll \frac{x^c}{Tq}+\frac{x^{\frac{1}{2}}T^{\frac{2}{3}}}{q^{\frac{1}{2}}}. \quad (14)$$

Next

$$I_3 \ll \left|\int\limits_{-T_0}^{T_0}\right| + \left|\int\limits_{T_0}^{T}\right| + \left|\int\limits_{-T}^{-T_0}\right| := J_0+J_1+J_2. \quad (15)$$

We put $T_0=q^{\varepsilon}$, $\varepsilon>0$ an arbitrary constant.

The integral J_1 and J_2 estimate in the same manner.

We shall estimate J_1.

It is well known that

$$\int_{T_0}^{T} \left| \xi\left(\frac{1}{2} + it, u\right) - \frac{1}{u^{\frac{1}{2}+it}} \right|^2 dt \ll T \log^2 (qT). \tag{16}$$

The truncated functional equation for $Z_m(s, \delta_1, 0)$ forth $m = 0$. We can write $\left(\text{for } \frac{\alpha}{q},\ \alpha \in G_q,\ s = \frac{1}{2} + it\right)$:

$$Z_0(s; \delta_1, 0) := Z(s; \delta_1, 0) = N(q)^s \left\{ \sum_{\substack{w \equiv a (mod q) \\ N(w) \leqslant X_1}} N(w)^{-\frac{1}{2} - it} + \right.$$

$$\left. + \frac{\pi^{-2it}}{N^{\frac{1}{2}+it}(q)} \cdot \frac{\Gamma\left(\frac{1}{2} - it\right)}{\Gamma\left(\frac{1}{2} + it\right)} \cdot \sum_{N(w) \leqslant Y_1} e^{-2\pi i Re\left(\frac{\alpha w}{q}\right)} N^{-\frac{1}{2}+it}(w) \right\} +$$

$$+ O\left(\frac{\log X_1 Y_1}{N(q)}\right) + O\left(|t|^{-M+2}\right)$$

$(X_1 = x, Y_1 = y$ in designation of Lemma 2$) =$

$$= \sum\nolimits_1 + \sum\nolimits_2 + O\left(\frac{\log X_1 Y_1}{N(q)}\right) + O\left(|t|^{-M+2}\right) \tag{17}$$

its say.

Now using (9), the Cauchy inequality and the relation (16) we obtain

$$\left| \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} F(s; a, q) - \sum_{n \in B} \frac{R(n)}{n^s} \frac{x^s}{s} ds \right| \ll$$

$$\ll \int_{T_0}^{T} \left| \sum\nolimits_1 + \sum\nolimits_2 \right| \cdot \left| \xi\left(\frac{1}{2} + it, u\right) - \frac{1}{u^{\frac{1}{2}+it}} \right| \frac{dt}{t} \ll$$

$$\ll \left(\int_{T_0}^{T} \left(\left|\sum\nolimits_1\right|^2 + \left|\sum\nolimits_2\right|^2 \right) \frac{dt}{t} \cdot \int_{T_0}^{T} \left| \xi\left(\frac{1}{2} + it, u\right) - \frac{1}{u^{\frac{1}{2}+it}} \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \cdot x^{\frac{1}{2}} +$$

$$+ O\left(\frac{\log T}{N(q)}\right) + O\left(qT^{\frac{3}{2}} \log^3 T\right) \tag{18}$$

Hence, putting $T = \frac{x^{\frac{1}{5}}}{q^{\frac{4}{5}}}$ we from (13), (14), (16), (18) obtain the statement theorem.

Conclusion

Having used $Z_m(s;\delta_1,0)$ rather than $Z_0(s;\delta_1,0)$ may be achieved the asymptotic formula of distribution of values of the function $R(n)$ in arithmetic progression and in narrow sectors.

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Числа виду $n=(u^2+dv^2)w$ в арифметичній прогресії

Резюме

Нехай $R(n)$ означає кількість зображень натурального n у вигляді $n=(u^2+v^2)w$, $u,v\in\mathbb{Z},\ w\in\mathbb{N}$. Функція $R(n)$ є аналогом функції дільників $d_3(n)$. Узагальнюючи результат Хіз-Брауна про розподіл значень функції $d_3(n)$ на арифметичній прогресії $n\equiv a(modq),\ (a,q)=1$, зі зростаючою разом з x різницею прогресії q, побудована асимптотична формула для суматорної функції для $R(n)$, яка нетривіальна для $q\le x^{\frac{1}{2}}log^{-3}x$. При доведенні цього результату використовується скорочене функціональне рівняння дзета-функції Гекке з уявного квадратичного поля $\mathbb{Q}(\sqrt{-d})$ з зсувом на прямій $Res=\frac{1}{2}+\Delta,\ |\Delta|<\frac{1}{2}$.

Ключові слова: уявне квадратичне поле, дзета-функція Гекка, ряд Діріхле, функціональне рівняння, суматорна функція.

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