МІНІСТЕРСТВО ОСВІТИ І НАУКИ УКРАЇНИ

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МЕХАНІКА.

Частина 2. РОБОТА, ЕНЕРГІЯ, ІМПУЛЬС

Методичні вказівки з механіки

англійською мовою

для студентів фізичного факультету

Одеса – 2002

Рекомендовано до друку
Вченою радою фізичного факультету
Одеського національного університету ім. І. І. Мечникова
Протокол № 5 від 29 січня 2002 р.

Детально аналізуються поняття роботи, енергії та імпульсу та їх застосування до розв’язування ряду типових задач механіки. Вказівки зорієнтовані як на студентів першого курсу, що тільки приступають до систематичного вивчення механіки в рамках загального курсу фізики у вищих навчальних закладах, так і на студентів четвертого курсу, що слухають курс з методики викладання фізики та готуються до державного іспиту з фізики.

Виклад (американською) англійською мовою надає студентам можливість ознайомитися зі стандартною фізико-математичною термінологією та типовими лексичними зворотами, потрібними фізикам для професійного спілкування англійською мовою. Вказівки можуть використовуватися як зручне джерело англомовних технічних текстів.

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ЗМІСТ

1. LAWS OF CONSERVATION

Newton's laws give a general approach to a wide variety of mechanics problems. Together with the equations of kinematics, they, in principle, provide us with the most complete description of mechanical motions, which includes information about the acceleration, velocity, and displacement. The major steps to follow within this approach are as follows:

1) if necessary, divide the entire path of the object under study into smaller segments, with a fixed set of forces acting on the object over each;

2) write out Newton's second law equations for each segment;

3) solve the equations in order to find the accelerations for the segments;

4) write out kinematics equations for each segment;

5) analyze the behavior of the object's velocity on the successive segments, starting from the object's initial positions and going toward the final one.

In practice, however, we can succeed on this way only if all acting forces are known and they are changing in a predictable manner. This is not necessarily the case. But even when the forces are known, the realization of the above approach is often too difficult to be practical.

On the other hand, we are often interested not in all, moment-to-moment, details of the motion, but only in some of them. In circumstances such as these, several rigorous identities called the *laws of conservation are of great importance.*

The laws of conservation deal with a restricted number of mechanical quantities which, under quite general conditions, remain constant throughout the motion. We say that these quantities are conserved or that they are the integrals of motion. Some examples are: energy, linear momentum, and angular momentum.

Significantly, the integrals of motion are certain combinations of objects' velocities, relative positions, and also physical parameters, such as masses or moments of inertia. In order to determine, let's say, the velocities for many different configurations of the objects involved, it is enough to know them only for one particular. There is also no need to analyze the objects' accelerations and the equations of kinematics.

Using the laws of conservation is another way to attack practical problems, especially valuable in dealing with systems of many objects, when detailed information about the forces involved is absent. Moreover, caused by very profound properties of nature, these laws can be generalized and extended to every area of physics, not only to mechanics. No violation of the laws of conservation has ever been found. They are fundamental and universal.

In general courses on Mechanics, three kinds of laws of conservation are usually considered. They are stated below, along with pertaining concepts and brief comments on their applications:

1. Law of conservation of energy: whatever processes occur within an isolated system, its total energy, including all forms of energy, remains constant.

The major concepts behind the law are *work, kinetic energy, work-energy theorem, conservative force, and potential energy.* It is effective in determining the speeds and relative locations of objects interacting by means of gravitational and elastic forces; or, should these kinematic quantities be given, in finding some of the physical parameters of the objects, such as their masses, spring constants, etc.

2. Law of conservation of linear momentum: the total linear momentum of an isolated system remains constant during any processes within the system.

The basic concepts involved are *linear momentum and impulse.* The law is invaluable for analysis of the velocities of a system's constituents; or, if the velocities are known, in finding the masses of the constituents. The most significant examples are *inelastic and elastic collisions.*

3. Law of conservation of angular momentum: the total angular momentum of an isolated system remains constant despite any processes within the system.

The related concepts are *angular momentum and angular impulse*. The law can be used to find the angular velocities of a system's objects; or, if these are known, the objects' physical parameters, such as their moments of inertia, masses, sizes, etc.

2. WORK

Suppose that an object A is acted upon by a constant force $\mathbf{F}$. The object moves along a fixed straight line and makes a displacement $\mathbf{S}$.

No matter whether $\mathbf{S}$ *is caused by* $\mathbf{F}$ *alone or in combination with other forces, the work done by* $\mathbf{F}$ *on* A *is defined as the product*

$$\mathrm{W} = FS\cos\theta \tag{1}$$

where F and S are the magnitudes of $\mathbf{F}$ and $\mathbf{S}$, respectively, and θ stands for the angle between the vectors $\mathbf{F}$ and $\mathbf{S}$.

Work is a scalar quantity that can take positive, negative, or zero values. Generally, these values vary, even for the same force and object, from one reference frame to another. The SI unit for work is *Joule* (J).

Only being applied to it, can a force do work on an object. When saying that an object A does work on another object B, we mean the following: A exerts some force on B, and that force does work on B.

If A is acted upon by a number of constant forces $\mathbf{F}_1$, $\mathbf{F}_2$, ... , $\mathbf{F}_n$, then force $\mathbf{F}_i$ contributes $\mathrm{W}_i = F_i S\cos\theta_i$, θ_i being the angle between $\mathbf{F}_i$ and $\mathbf{S}$ ($i = 1,2,...,n$), to the total work W_{tot} done on A by all the forces. Therefore,

$$\mathrm{W}_{tot} = \mathrm{W}_1 + \mathrm{W}_2 + ... + \mathrm{W}_n \tag{2}$$

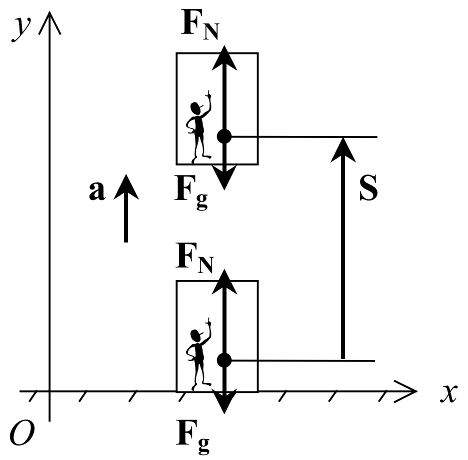
and this is equal to the work W_{net} done on A by the resultant $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + ... + \mathbf{F}_n$ ($\mathrm{W}_{net} = RS\cos\theta_{net}$, where θ_{net} is the angle between $\mathbf{R}$ and $\mathbf{S}$):

$$\mathrm{W}_{net} = \mathrm{W}_{tot} \tag{3}$$

To calculate work in a particular problem, be sure to specify:

1) *the object under study;*
2) *the reference frame used to describe the motion of the object;*
3) *the force(s) performing work;*
4) *the displacement of the object relative to the chosen frame of reference;*
5) *the angle(s) between the force(s) and the displacement.*

Example 2.1. A man of mass m stands on the floor of an elevator that is moving upward. The elevator rises a distance l with a constant acceleration a. (a) Calculate the work done on the man by each acting force and find the total work done on the man, (b) compare the total work with that done on the man by the net force, and (c) find the work done by each acting force as seen from the elevator.



Solution. (a) The object under study is the man. His motion is studied relative to the ground. The displacement **S** of the man is directed upward, its magnitude being *l*.

The forces acting on the man are:

1) the gravitational force $\mathbf{F_g}$; exerted by the earth, it has a magnitude of $F_\mathbf{g} = mg$ and points downward, making an angle of $\theta_1 = 180^\circ$ with the vector **S**. The work done by $\mathbf{F_g}$ on the man, also called the work done by the earth, is

$$\mathrm{W}_\mathbf{g} = F_\mathbf{g}\, l \cos 180^\circ = -mgl$$

2) the normal force $\mathbf{F_N}$; exerted by the floor, it has a magnitude of $F_\mathbf{N} = m(g+a)$ and points upward, making an angle of $\theta_2 = 0^\circ$ with $\mathbf{S}$. The work done by $\mathbf{F_N}$ on the man, also called the work done by the floor, is

$$\mathrm{W}_\mathbf{N} = F_\mathbf{N}\, l \cos 0^\circ = m(g+a)l$$

The total work done on the man equals

$$\mathrm{W}_{tot} = \mathrm{W}_\mathbf{g} + \mathrm{W}_\mathbf{N} = mal$$

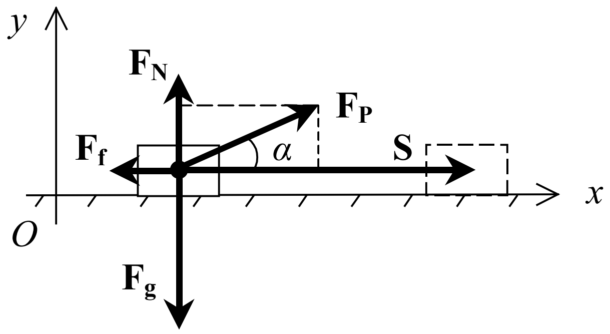
(b) The magnitude R of the net force $\mathbf{R} = \mathbf{F_g} + \mathbf{F_N}$ acting on the man is $R = F_\mathbf{N} - F_\mathbf{g} = ma$. In this particular problem, the vectors **R** and **S** point in the same direction, making zero angle with each other. Therefore,

$$\mathrm{W}_{net} = Rl \cos 0^\circ = \mathrm{W}_{tot}$$

(c) With respect to the elevator $\mathbf{S} = 0$, so the forces $\mathbf{F_g}$ and $\mathbf{F_N}$ do no work.

Keep in mind: the displacement of an object depends not only upon the net force, but also on the object's initial position, velocity, and duration of motion. In general, the vectors **R** *and* **S** *have different directions – calculating work, do not forget about the angular factor* $\cos\theta$.

Example 2.2. A child is pulling a sled of mass M across the snow by exerting a force $\mathbf{F_P}$ at an angle α above the horizontal. The sled moves along the snow a distance L. The kinetic friction coefficient is μ_k. Find the work done on the sled by each acting force and also the net work done on the sled.



Solution. The object under study is the sled. Its motion is studied with respect to the ground. The displacement **S** of the sled is directed along the ground; its magnitude is L.

The forces acting are as follows:

1) the pulling force $\mathbf{F_P}$; exerted by the child, it has a known magnitude of $F_\mathbf{P}$ and makes a known angle of $\theta_1 = \alpha$ with **S.** Its work is positive:

$$\mathrm{W}_\mathbf{P} = F_\mathbf{P}\, L\cos\theta_1 = F_\mathbf{P}\, L\cos\alpha$$

2) the gravitational force $\mathbf{F_g}$; exerted by the earth, it has a magnitude of $F_\mathbf{g} = Mg$ and points downward, making an angle of $\theta_2 = 90^\circ$ with **S.** Its work is zero:

$$\mathrm{W}_\mathbf{g} = F_\mathbf{g}\, L\cos\theta_2 = MgL\cos 90^\circ = 0$$

3) the normal force $\mathbf{F_N}$; exerted by the snow (ground) upward, it makes an angle of $\theta_3 = 90^\circ$ with **S**. The vertical components of the forces are balanced, therefore, its magnitude equals $F_\mathbf{N} = F_\mathbf{g} - F_\mathbf{P}\sin\alpha$. This force does no work:

$$\mathrm{W}_\mathbf{N} = F_\mathbf{N}\, L\cos\theta_3 = (Mg - F_\mathbf{P}\sin\alpha)L\cos 90^\circ = 0$$

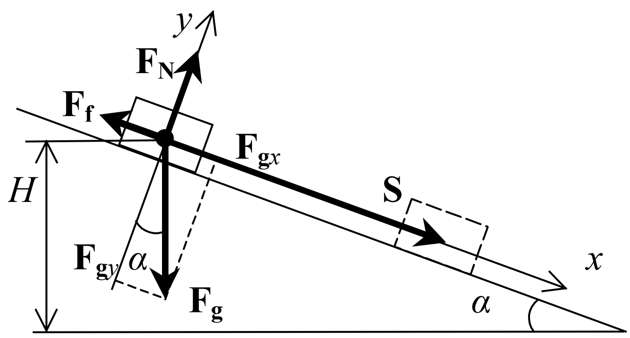
4) the frictional force $\mathbf{F_f}$; exerted by the snow horizontally, it has a magnitude of $F_\mathbf{f} = \mu_k F_\mathbf{N}$ and makes an angle of $\theta_4 = 180^\circ$ with **S**. Its work is negative:

$$W_{\mathbf{f}} = F_{\mathbf{f}}\, L\cos\theta_4 = \mu_k(Mg - F_{\mathbf{P}}\sin\alpha)L\cos 180^\circ = -\mu_k(Mg - F_{\mathbf{P}}\sin\alpha)L$$

The total work done on the sled equals

$$W_{tot} = W_{\mathbf{P}} + W_{\mathbf{g}} + W_{\mathbf{N}} + W_{\mathbf{f}} = F_{\mathbf{P}}L\cos\alpha + 0 + 0 - \mu_k(Mg - F_{\mathbf{P}}\sin\alpha)L$$

Example 2.3. A skier of mass M skis down a hill that has a constant angle of inclination θ. The skier starts from a point at height *H*. The coefficient of kinetic friction between the skis and the snow is μ_k. Find the work done on the skier by each of the acting forces.



Solution. The object under consideration is the skier. His motion is studied with respect to the hill. His resultant displacement **S** is directed along the hill and has a magnitude of $S = H/\sin\alpha$.

The forces acting on the skier are the force of gravity $\mathbf{F_g}$, the normal force $\mathbf{F_N}$, and the frictional force $\mathbf{F_f}$. Since there is no motion along the *y*-axis, the force $\mathbf{F_N}$ and the *y*-component of $\mathbf{F_g}$ balance each other: $F_{\mathbf{N}} = F_{\mathbf{g}}\cos\alpha$. The forces $\mathbf{F_f}$ and $\mathbf{F_N}$ are related by $F_{\mathbf{f}} = \mu_k F_{\mathbf{N}}$.

Therefore, the magnitudes of the forces are

$$F_{\mathbf{g}} = Mg, \quad F_{\mathbf{N}} = Mg\cos\alpha, \quad F_{\mathbf{f}} = \mu_k Mg\cos\alpha$$

and the angles that the forces make with **S** are

$$\theta_1 = 90^\circ - \alpha, \quad \theta_2 = 90^\circ, \quad \theta_3 = 180^\circ$$

respectively. The forces make the following contributions to the total work done on the skier:

$$W_{\mathbf{g}} = F_{\mathbf{g}}\, S\cos\theta_1 = Mg(H/\sin\alpha)\cos(90^\circ - \alpha) = MgH$$

$$W_{\mathbf{N}} = F_{\mathbf{N}}\, S\cos\theta_2 = Mg\cos\alpha(H/\sin\alpha)\cos 90^\circ = 0$$

$$W_{\mathbf{f}} = F_{\mathbf{f}}\, S\cos\theta_3 = \mu_k\, Mg\cos\alpha(H/\sin\alpha)\cos 180^\circ = -F_{\mathbf{f}}S$$

The above results reflect some general facts:

1) The work done by the gravity does not depend on the path (inclination of the hill), but only on the initial and final positions (heights above the ground) of the object. *Gravitational force belongs to conservative forces.*

2) Of the components of a force, *only the component* $F_s = F\cos\theta$ *along the displacement contributes to work:* $\mathrm{W} = F_s\, S$. *The perpendicular components do no work* (as the normal force in *Examples 2.2, 2.3,* or the magnetic force acting on a moving charge).

3) *Friction is a nonconservative force*, for its work is dependent of the path (distance) traveled by the object.

Example 2.4. An object is acted upon by several forces. Justify that their total work on the object equals the work done on the object by their resultant.

Solution. Let the displacement of the object be directed along the *x*-axis. Then only the *x*-components of the forces contribute to the total work:

$$\mathrm{W}_{tot} = F_{1x}\, S + F_{2x}\, S + \ldots + F_{nx}\, S = \left(F_{1x}\, S + F_{2x}\, S + \ldots + F_{nx}\right) S$$

Taking into account that the expression in the parentheses is the component of the net force along the displacement, we immediately obtain (3).

Formula (1) may be treated as the dot (scalar) product of the vectors **F** and **S**:

$$\mathrm{W} = \mathbf{F} \bullet \mathbf{S} \tag{4}$$

Then (3) is just a simple consequence of the distributivity of dot product:

$$\left(\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n\right) \bullet \mathbf{S} = \mathbf{F}_1 \bullet \mathbf{S} + \mathbf{F}_2 \bullet \mathbf{S} + \ldots + \mathbf{F}_n \bullet \mathbf{S}$$

3. KINETIC ENERGY AND WORK-ENERGY THEOREM

As you will have known, the general approach to mechanics problems based on Newton's laws is not always possible to realize. In this connection, some rigorous identities generated by Newton's laws become of great importance. They let us not only introduce a number of fundamental concepts, but also considerably simplify working out particular problems. The idea is very simple: we try to advance in finding one quantity at the sacrifice of others.

The work-energy theorem, one of those identities, is especially effective when speed is to be found. Stating that *the speed, v, of an object, of mass m, changes only when the net force does nonzero work on the object,* it relates the values of the combination

$$\mathrm{KE} = \frac{1}{2}mv^2 \tag{5}$$

for two different times, initial (earlier) t_i and final (later) t_f, to the total work W_{tot} done on the object during the time interval $t_f - t_i$:

$$\mathrm{KE}_f - \mathrm{KE}_i = \mathrm{W}_{tot} \tag{6}$$

Note that a force may act and, therefore, do work on an object either over the entire path or just along different parts of the motion.

Formula (5) is the definition of *kinetic energy*. The terms $\mathrm{KE}_i = \frac{1}{2}mv_i^2$ and $\mathrm{KE}_f = \frac{1}{2}mv_f^2$, with v_i and v_f standing for the initial and final speeds, are referred to as the initial and final kinetic energies of the object. Equation (6) represents the *work-energy theorem* itself.

Kinetic energy is a scalar quantity whose values are always *nonnegative. These values do not depend on the direction, but only on the magnitude of the velocity.* They may vary, even for the same object, from one reference frame to another. The SI unit for kinetic energy is *Joule* (J).

Only an object that is in motion possesses nonzero kinetic energy. Its amount *is determined by the instantaneous speed of the object, no matter whether the object is being acted upon by a force or is moving on its own. An object at rest has zero kinetic energy. To change the kinetic energy of an object, work must be done on the object.*

The kinetic energy KE_{sys} of a system is the sum of the kinetic energies of the system's constituents:

$$\mathrm{KE}_{sys} = \mathrm{KE}_1 + \mathrm{KE}_2 + \ldots + \mathrm{KE}_n \tag{7}$$

The work-energy theorem for a system,

$$\mathrm{KE}_{sys,f} - \mathrm{KE}_{sys,i} = \mathrm{W}_{tot1} + \mathrm{W}_{tot2} + \ldots + \mathrm{W}_{tot\,n} \tag{8}$$

relates any change in KE_{sys} to the total work that *both internal and external forces* do on the constituents. The symbol $\mathrm{W}_{tot\,j}$ in (8) represents the total work done on object j of the system.

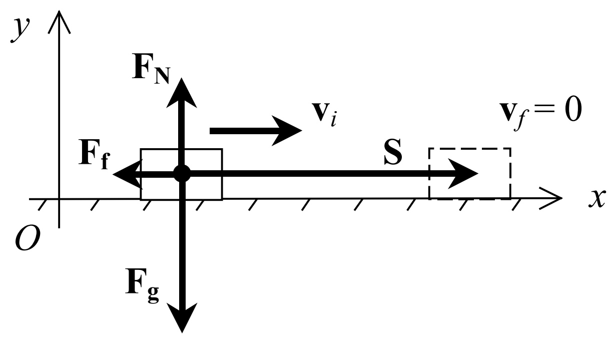
Based on Newton's laws, the work-energy theorem is valid only in inertial frames of reference. The reference frame must be specified in each particular case.

When solving a particular problem, be sure to specify:

1) *the object being studied;*
2) *the path and displacement of the object;*
3) *the object's initial position, speed, and kinetic energy; then the final ones;*
4) *the forces exerted on the object and where (over the entire path or just on some of its parts) they are acting;*
5) *the work done by each force and the total work.*

Example 3.1. In screeching to a halt, a car leaves skid marks of length S. The coefficient of kinetic friction between the tires and the road is μ_k. How fast was the car moving before the driver applied the brakes?

Solution. The object under study is the car. The motion occurs on the ground. The displacement $\mathbf{S}$ of the car is directed along the ground, and its magnitude is S.



With respect to the ground, the final velocity $\mathbf{v}_f$ of the car is zero.

In the process of screeching, the *car's kinetic energy is changing* from the initial value $\mathrm{KE}_i = \frac{1}{2}mv_i^2$ to the final one $\mathrm{KE}_f = \frac{1}{2}mv_f^2 = 0$. This is *due to the total work done on the car* by the gravitational $\mathbf{F_g}$, normal $\mathbf{F_N}$, and frictional $\mathbf{F_f}$ forces, which all act over the entire path of the car:

$$\mathrm{W}_{tot} = \mathrm{W}_\mathbf{g} + \mathrm{W}_\mathbf{N} + \mathrm{W}_\mathbf{f}$$

In fact, $\mathbf{F_g}$ and $\mathbf{F_N}$ both do zero work, for they are perpendicular to $\mathbf{S}$:

$$\mathrm{W}_\mathbf{g} = F_\mathbf{g}\, S\cos 90^\circ = 0, \qquad \mathrm{W}_\mathbf{N} = F_\mathbf{N}\, S\cos 90^\circ = 0$$

The forces $\mathbf{F_g}$ and $\mathbf{F_N}$ balance each other (no motion occurs in the vertical direction), so $F_\mathbf{N} = mg$. The frictional force has a magnitude of $F_\mathbf{f} = \mu_k F_N = \mu_k mg$ and makes an angle of 180° with $\mathbf{S}$. Its work is

$$\mathrm{W}_\mathbf{f} = F_\mathbf{f}\, S\cos 180^\circ = \mu_k mgS\cos 180^\circ = -\mu_k mgS$$

The work-energy theorem takes the form

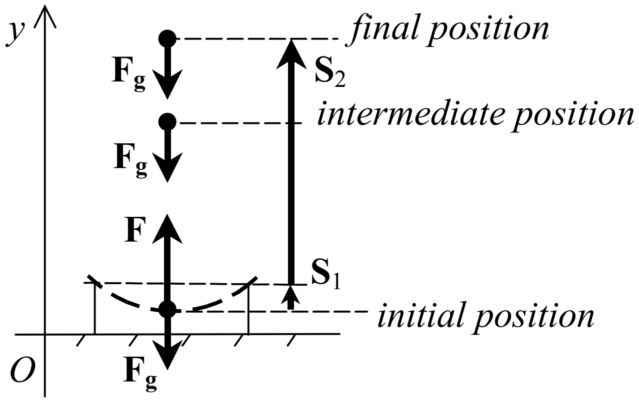
$$0 - \frac{1}{2}mv_i^2 = 0 + 0 + (-\mu_k mgS)$$

The initial speed was

$$v_i = \sqrt{2\mu_k gS}$$

Example 3.2. A gymnast of mass m is bouncing on a trampoline. On the upward part of the motion, the mat of the trampoline pushes on the gymnast with a nonconservative force over a distance of S_1. After leaving the mat, the gymnast rises into the air for an additional distance of S_2 before falling back down. What average force does the mat exert on the gymnast?

Solution. The object under study is the gymnast He moves upward along a



straight line and makes the displacement $\mathbf{S}_1 + \mathbf{S}_2$ with respect to the ground. His initial $\mathbf{v}_i$ and final $\mathbf{v}_f$ velocities are both equal to zero; consequently, so are his initial and final kinetic energies.

The total work done *on the gymnast* over his way upward is

$$\mathrm{W}_{tot} = \mathrm{W}_{\mathbf{g}} + \mathrm{W}_{\mathbf{F}}$$

Here, the first term represents the work done by the gravity $\mathbf{F_g}$ during the gymnast's entire displacement, whose magnitude is $S_1 + S_2$ and which makes an angle of 180° with $\mathbf{F_g}$. The second term is the work done by the nonconservative force $\mathbf{F}$, acting only over the first part S_1 of the entire path and pointing along the displacement $\mathbf{S}_1$:

$$\mathrm{W}_{\mathbf{g}} = mg(S_1 + S_2)\cos 180^\circ = -mg(S_1 + S_2), \qquad \mathrm{W}_{\mathbf{F}} = FS_1 \cos 0^\circ = FS_1$$

The work-energy theorem yields:

$$0 - 0 = FS_1 - mg(S_1 + S_2)$$

Thus,

$$F = mg\frac{S_1 + S_2}{S_1}$$

Keep in mind:

1) As long as we are interested in finding just the speed of an object and can disregard the direction of the velocity, the work-energy theorem is a very powerful method. With its help, we can avoid tedious calculations typical for the standard approach based on Newton's laws and the equations of kinematics.

2) Nonzero net force always causes acceleration (change in velocity). When the net force does nonzero work, it changes the speed and, perhaps, the direction of the velocity of the object. When the net force does no work, it changes merely the direction of the object's velocity.

4. CONSERVATIVE FORCE AND POTENTIAL ENERGY

The concept of potential energy is introduced only for conservative forces. In order to determine whether a force $\mathbf{F}$ is conservative, two equivalent ways can be used.

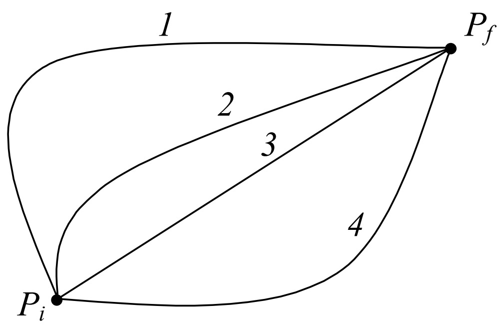


Fig. 1. Conservative force – for every path, its work is the same: $W_1 = W_2 = W_3 = W_4 \equiv W_{con}$

1) Suppose that an object A, acted upon by $\mathbf{F}$, is allowed to move between two arbitrarily fixed positions, initial P_i and final P_f. The motion can occur along different curves joining P_i and P_f (for instance, *1*, *2*, *3*, and *4* in Fig. 1).

$\mathbf{F}$ *is conservative if the work,* W_{con}, *that it does in moving* A *from* P_i *to* P_f *depends only on these positions, and not upon the path followed by* A.

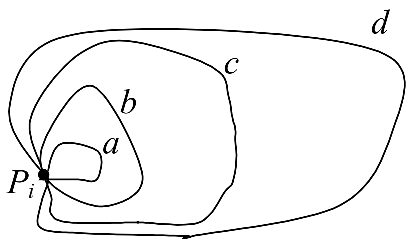


Fig. 2. Conservative force – for every closed path, its work is zero: $W_a = W_b = W_c = W_d = 0$

2) Suppose that an object A, acted upon by $\mathbf{F}$, can move around different closed paths (such as *a*, *b*, *c*, and *d* in Fig. 2), starting from an arbitrarily fixed position.

$\mathbf{F}$ *is conservative if the work that it does in moving* A *around any closed path, starting and finishing at the same point, is always zero.*

The major idea behind the concept of potential energy is this. As usual, we can calculate W_{con} by definition (1) of work. Since only the points P_i and P_f are of significance, we expect W_{con} to be a function of their coordinates, with its values vanishing when P_i and P_f coincide. Assuming that every point available for the object's motion can be assigned some value of this function, we anticipate that W_{con}

is uniquely determined by the values PE_i and PE_f assigned to P_i and P_f.

The potential energy of an object A *in the field of a conservative force* $\mathbf{F}$ *is a function,* PE*, such that the work that* $\mathbf{F}$ *does in moving* A *between two points* P_i *and* P_f *is the difference of the values of* PE *at those:*

$$\mathrm{W}_{\mathrm{con}} = -(\mathrm{PE}_f - \mathrm{PE}_i) \tag{9}$$

The values PE_i and PE_f are referred to as the potential energy of the object A at point P_i and that at point P_f, respectively.

Potential energy is a scalar function. Its values may be positive, negative, or equal to zero; for a given force, they depend only upon the object's *relative* location, but not upon its velocity or acceleration. The SI unit for potential energy is *Joule* (J).

The concept of potential energy is inseparable from interaction between objects. An object possesses potential energy only due to its interaction, by means of a conservative force, with another object. In fact, it is a system of objects that possesses potential energy.

So, stating that an object A possesses potential energy, you should be able to identify another object B, interacting with A, and the conservative force acting between A and B. The amount of potential energy that an object A possesses in the presence of an object B depends upon the relative position of A with respect to B, not just on the position of A.

The work $\mathrm{W}_{\mathrm{con}}$ is positive if $\mathrm{PE}_i > \mathrm{PE}_f$. A conservative force $\mathbf{F}$ tends to move an object from regions of higher potential energy towards regions where the potential energy is lower; that is, $\mathbf{F}$ tends to reduce the object's potential energy. The negative sign in definition (9) reflects this fact.

To increase the potential energy of an object in the field of $\mathbf{F}$, somebody or something else, called an *agent*, must exert another force to perform some work on the object 'against' $\mathbf{F}$.

Potential energy is not a unique function: any function $\mathrm{PE}' = \mathrm{PE} + C$, where C is an arbitrary constant, also obeys definition (1) and may be equally treated as the potential energy of A in the field of $\mathbf{F}$. This means that *not potential energy itself,* *but its changes are physically important. Taken with the negative sign, the change in an object's potential energy is equal to the work done by* $\mathbf{F}$ *on the object.*

The value of C must be specified for each particular problem. For this purpose, the potential energy is set zero at a certain point (or level), called the *zero reference point (level)*. We are free to choose it arbitrarily. Once the zero point (level) is chosen, we must keep it fixed while solving the problem.

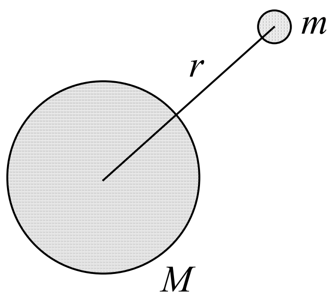
When several different conservative forces $\mathbf{F}_1, \mathbf{F}_2, ..., \mathbf{F}_n$ are simultaneously acting on an object, the object's potential energy is defined as the sum of the potential energies

$$\mathrm{PE} = \mathrm{PE}_{\mathbf{F}_1} + \mathrm{PE}_{\mathbf{F}_2} + ... + \mathrm{PE}_{\mathbf{F}_n} \tag{10}$$

associated with these forces.

In Mechanics we deal with two kinds of potential energy, *gravitational* and *elastic*. *Electrostatic potential energy* is the subject of study in *Electrostatic*

1. *Gravitational potential energy* is associated with the gravitational interaction which exists, according to Newton's law of gravitation, between any two masses in the universe. The gravitational forces that the masses exert on each other are conservative.



Let m and M be the masses of two uniform spheres (or particles), and r be their center-to-center distance. The gravitational potential energy of their interaction is given by

$$\mathrm{PE}_{\mathrm{grav}} = -G\frac{mM}{r} + C \tag{11}$$

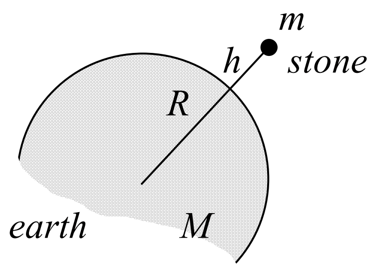
where G is the universal gravitational constant.

Formula (11) is derived by using definition (9) and integral calculus. The negative sign, which arises because gravitational forces are always attractive, is very important: the larger the distance between interacting objects, the greater the potential energy of their interaction. The gravitational forces tend to reduce the distance between the objects and bring them closer together.

The standard way of fixing C consists in setting the potential energy of two faraway objects equal to zero: when separated by a very large distance, they practically do not interact. As r approaches infinity, expression (11) changes to $0 = 0 + C$, whence $C = 0$.

Formula (11) gives the most general expression for the gravitational potential energy due to the interaction between two uniform spheres.

Very often, we deal with the situation when the mass and size of one object are very small as compared to those of the other, for instance, a stone near the surface of the earth. If the stone is treated as a particle of mass m that is located at a height of h above the earth's surface, then $r = R + h$, and the potential energy is



$$\mathrm{PE}_{\mathrm{grav}} = -G\frac{mM}{(R+h)} + C$$

where M and R stand for the earth's mass and radius.

For h small in comparison with R, the linear approximation $\dfrac{1}{R+h} \approx \dfrac{1}{R} - \dfrac{h}{R^2} + \ldots$ lets us rewrite (11) as

$$\mathrm{PE}_{\mathrm{grav}} = -G\frac{mM}{R} + m\left(G\frac{M}{R^2}\right)h + C$$

The quantity inside the parentheses is the acceleration g due to gravity. The value of C can be fixed by setting $\mathrm{PE}_{\mathrm{grav}}$ equal to zero on the ground: $\mathrm{PE}_{\mathrm{grav}} = 0$ as $h = 0$. Then $C = G\dfrac{mM}{R}$, and we conclude that *the gravitational potential energy of a small object near the surface of the earth is given by*

$$\mathrm{PE}_{\mathrm{grav}} = mgh \tag{12}$$

Formula (12) is precisely the work done by an agent against the earth's gravity in raising (at a constant speed) an object of mass m from the ground to a height of h. By doing so, the agent increases the gravitational potential energy of the object. On the other hand, (12) represents the work that the earth's gravity does on an object in returning it from a height of h to the ground. This work is, of course, independent of the path followed by the object.

Within the same accuracy, h in expression (12) can be measured from a horizontal reference level other than the ground. As a result, h can be either *positive or negative*: h is positive for objects located above the zero reference level, and negative if objects are below it.

2) *Elastic potential energy* is associated with elastic forces, which make themselves evident when an object is deformed – compressed or stretched.

The elastic potential energy of an object is given by

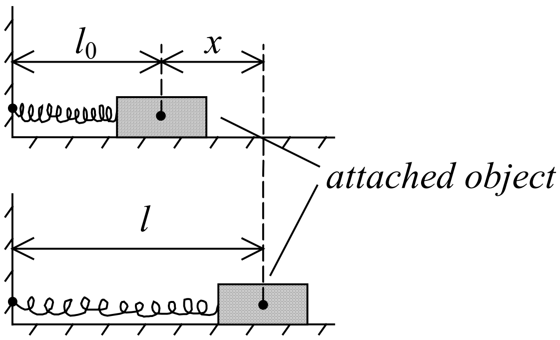
$$\mathrm{PE}_{\mathrm{elast}} = \frac{1}{2}kx^2 \tag{13}$$

where k is a parameter depending upon the object's physical properties, and x is the *deformation*, that is, the amount by which the object is compressed or stretched from its normal size. The deformation is assumed small in comparison with the object's corresponding size.

The constant C in (13) is already set equal to zero: when an object is not deformed ($x = 0$), it has no elastic potential energy.

For a spring stretched or compressed to a length of l, the deformation is

$$x = l - l_0 \qquad (x << l_0)$$



where l_0 is the spring's *unstrained* length; the coefficient k is called the spring constant or stiffness. The energy (13) is then precisely the work done by an agent against the elastic force in stretching or compressing (at a constant speed) the spring from its unstrained length l_0 by the amount x. By doing so, the agent increases the elastic potential energy of the spring. On the other hand, (13) represents the work that the spring does on an attached object in restoring the normal length l_0.

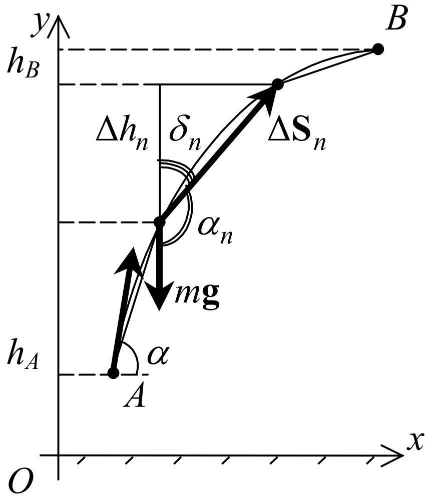
3) *Electrostatic potential energy* is associated with the electrostatic forces that act between stationary electric charges.

The electrostatic potential energy of two point charges q_1 and q_2 is

$$\mathrm{PE}_{\mathrm{elec}} = k\frac{q_1 q_2}{r} + C \tag{14}$$

where the proportionality constant $k = 8.988 \times 10^{+9}\ \mathrm{N \cdot m^2/C^2}$, r stands for the distance between the charges, and C is usually taken to be zero: the potential energy of two distant charges vanishes for they practically do not interact.

Example 4.1. A ball of mass $m = 0.2\,\mathrm{kg}$ is thrown from point A at a height of $h_A = 1.0\,\mathrm{m}$ at an angle of $\alpha = 70^\circ$ above the horizontal. (a) Prove that the gravitational force is conservative, and derive the expression for the gravitational potential energy. Find the work done by the gravitational force when (b) the ball has risen to point B at a height of $h_B = 8.0\,\mathrm{m}$ and (c) returned to the ground, hitting it at point D. How would these values change if (d) the ball were thrown straight up or (e) air resistance were significant?



Solution. (a) The ball is a projectile that follows a parabola. In order to calculate the work done by the constant downward gravitational force $\mathbf{F_g} = m\mathbf{g}$ over the path *AB*, we should take into account that the angle between this force and the ball's instantaneous displacement is continually changing. To be able to apply the definition of work via force and displacement, we approximate the curve *AB* by N small line segments $\Delta\mathbf{S}_n$. The elementary work done by $\mathbf{F_g}$ on the nth segment $\Delta\mathbf{S}_n$ is

$$\Delta \mathrm{W_n} = mg\Delta S_n \cos\alpha_n$$

where α_n stands for the angle between the vectors $\mathbf{F_g}$ and $\Delta\mathbf{S}_n$. From the drawing,

$$\Delta S_n \cos\alpha_n = \Delta S_n \cos\left(180^\circ - \delta_n\right) = -\Delta S_n \cos\delta_n = -\Delta h_n$$

Δh_n being the vertical component of the ball's displacement for the segment $\Delta\mathbf{S}_n$.

The work done by $\mathbf{F_g}$ over the entire path AB is the sum

$$\mathrm{W}_{AB} = -mg\left(\Delta h_1 + \Delta h_2 + \ldots + \Delta h_n\right)$$

The expression in the parentheses represents the total vertical displacement $\left(h_B - h_A\right)$ of the ball. Then:

$$\mathrm{W}_{AB} = -\left(mgh_B - mgh_A\right)$$

This result remains valid when the number N of the segments approaches infinity – in this case the segments $\Delta\mathbf{S}_n$ become smaller and smaller and fit the actual path better and better. Moreover, it does not depend on the path at all, but only on the heights of the end points. Therefore, the force $\mathbf{F_g}$ is conservative, and comparison with definition (9) reveals that the associated potential energy is given by $\mathrm{PE} = mgh + C$, where h is the height of the object above the ground.

(b), (c) The work can be found by using definition (9) of potential energy.

Let the ground be the zero reference level: $\mathrm{PE} = 0$ when $h = 0$. The values of the ball's potential energy at different points are:

point A: $\mathrm{PE}_A = mgh_A = 1.96\,\mathrm{J}$

point B: $\mathrm{PE}_B = mgh_B = 15.7\,\mathrm{J}$

point D: $\mathrm{PE}_D = mgh_D = 0\,\mathrm{J}$

The work done by the gravity is:

from A to B: $\mathrm{W}_{AB} = -(\mathrm{PE}_B - \mathrm{PE}_A) = -14\,\mathrm{J}$

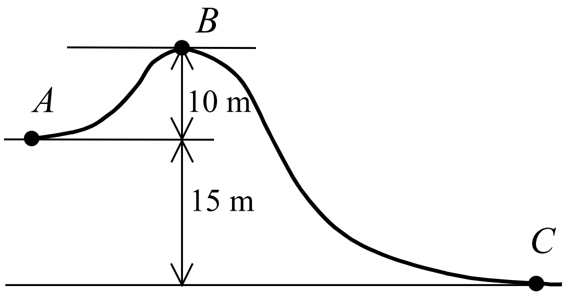
from A to D: $\mathrm{W}_{AD} = -(\mathrm{PE}_D - \mathrm{PE}_A) = +2.0\,\mathrm{J}$

These results have been rounded off to two significant figures.

(d) The values W_{AB} and W_{AD} do not change – they depend only on the initial and final heights of the ball, which remain the same.

(e) The values W_{AB} and W_{AD} do not change, for the work done by the gravitational force does not depend upon the presence of other forces.

Example 4.2. A car of mass $m = 1000\,\text{kg}$ moves from point A to point B at the top of a hill and then downhill to point C. What is its potential energy (a) at points B and C relative to point A, (b) at A and C relative to point B, and (c) at A and B relative to point C? What are the changes in potential energy and the work done by the gravity when the car goes (d) from A to B, (e) from B to C, and (f) from A to C?



Solution. Since the slight irregularities of the earth's surface due to the hill are negligible as compared to the earth's radius $R = 6.38 \times 10^{+6}\,\text{m}$, the earth can still be considered as a uniform sphere which causes any object to gain the acceleration $g = 9.80\,\text{m/s}^2$. The car can be thought of as a particle moving in the field of the earth's gravity. Then the car's potential energy due to the interaction with the earth is given by (12).

We need to specify the zero reference level in each case and take into account the sign of the object's height h: it is positive if the object is above the zero level, and negative if below.

(a) The zero level passes through point A horizontally. Point B is located at a height of $h_{AB} = +10\,\text{m}$ above it (the first subscript indicates the position of the car, and the second one the reference level), so the car's potential energy at B is

$$\text{PE}_{BA} = mgh_{BA} = +1.0 \times 10^{+5}\ \text{J}$$

Point C is located 15 m below the zero level, so $h_{CA} = -15\,\text{m}$. The potential energy of the car at C is

$$\text{PE}_{CA} = mgh_{CA} = -1.5 \times 10^{+5}\ \text{J}$$

(b), (c) The answers are:

$$\text{PE}_{AB} = -1.0 \times 10^{+5}\ \text{J}, \qquad \text{PE}_{CB} = -2.5 \times 10^{+5}\ \text{J}$$

$$\text{PE}_{AC} = +1.5 \times 10^{+5}\ \text{J}, \qquad \text{PE}_{BC} = +2.5 \times 10^{+5}\ \text{J}$$

(d) The change ΔPE in potential energy and the work W_{con} done by the gravity in moving the car from point $P_i = A$ to point $P_f = B$ are given (see (9)) by

$$\Delta\text{PE} = \text{PE}_f - \text{PE}_i, \qquad \text{W}_{\text{con}} = -(\text{PE}_f - \text{PE}_i) = -\Delta\text{PE}$$

The values PE_i and PE_f must be calculated relative to the same reference level, which itself can vary.

Relative to point A:

$$\text{PE}_i = \text{PE}_{AA} = 0\,\text{J}, \qquad \text{PE}_f = \text{PE}_{BA} = +1.0\times10^{+5}\,\text{J}$$

$$\Delta\text{PE} = +1.0\times10^{+5}\,\text{J}, \qquad \text{W}_{\text{con}} = -1.0\times10^{+5}\,\text{J}$$

Relative to point B:

$$\text{PE}_i = \text{PE}_{AB} = -1.0\times10^{+5}\,\text{J}, \qquad \text{PE}_f = \text{PE}_{BB} = 0\,\text{J}$$

$$\Delta\text{PE} = +1.0\times10^{+5}\,\text{J}, \qquad \text{W}_{\text{con}} = -1.0\times10^{+5}\,\text{J}$$

Relative to point C:

$$\text{PE}_i = \text{PE}_{AC} = +1.5\times10^{+5}\,\text{J}, \qquad \text{PE}_f = \text{PE}_{BC} = +2.5\times10^{+5}\,\text{J}$$

$$\Delta\text{PE} = +1.0\times10^{+5}\,\text{J}, \qquad \text{W}_{\text{con}} = -1.0\times10^{+5}\,\text{J}$$

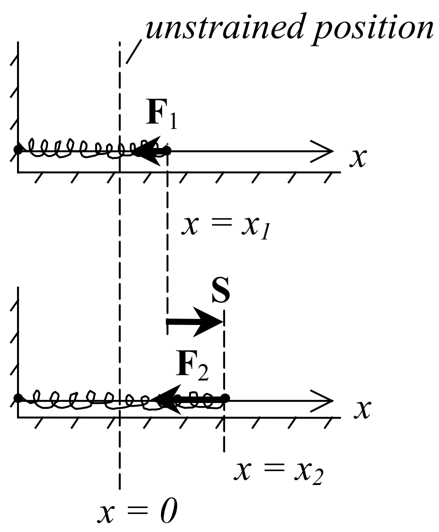
The change in potential energy and the work W_{con} *are independent of the choice of the zero reference level, even though the potential energy values are not.*

(e) $\Delta\text{PE} = -2.5\times10^{+5}\,\text{J}, \qquad \text{W}_{\text{con}} = +2.5\times10^{+5}\,\text{J}$

(f) $\Delta\text{PE} = -1.5\times10^{+5}\,\text{J}, \qquad \text{W}_{\text{con}} = +1.5\times10^{+5}\,\text{J}$

Example 4.3. A force of $P = 1000\,\text{N}$ stretches a spring a distance of $d_1 = 0.10\,\text{m}$. (a) Prove that the elastic force is conservative and derive the expression for the elastic potential energy. Find the work done by the elastic force (b) when the spring has been stretched from d_1 to $d_2 = 0.20\,\text{m}$, and (c) then compressed to $d_3 = -0.05\,\text{m}$. (d) What is the work done by the elastic force if the spring, initially stretched by d_1, has been compressed to $d_4 = -0.10\,\text{m}$?

Solution. (a) Suppose that the extension x of the spring from its unstrained length is not too great. Then, according to Hooke's law, the magnitude of the elastic force $\mathbf{F}$ exerted by the spring on its movable end is directly proportional to x. The direction of $\mathbf{F}$ is towards the unstrained position.



It is convenient to direct the x-axis along the spring away from its fixed end, place the origin O at the spring's unstrained position, and consider the extension x as the x-coordinate of the movable end. The x-component of $\mathbf{F}$ is then given by

$$F_x = -kx$$

the coefficient k being the spring constant.

When the movable end displaces from position x_1 to position x_2, the x-component of its displacement $\mathbf{S}$ is

$$S_x = (x_1 - x_2)$$

The component F_x varies from $F_{x1} = -kx_1$ to $F_{x2} = -kx_2$, so its average is

$$F_{x,av} = \frac{F_{x2} + F_{x1}}{2} = -\frac{1}{2}k(x_2 + x_1)$$

The work done by the elastic force is the dot product of the vectors $\mathbf{F}_{av}$ and $\mathbf{S}$. In terms of their x-components, we have:

$$\mathrm{W} = F_x \cdot S_x = -\frac{1}{2}k\left(x_2{}^2 - x_1{}^2\right)$$

This work depends only on the initial and final positions of the movable end – the elastic force is conservative. Comparison with (9) reveals that the elastic potential energy is given by (13), provided the energy in the unstrained state is zero.

(b), (c) Once again, we use definition (9). Since the forces $\mathbf{P}$ and $\mathbf{F}$ balance each other, $P = kd_1$, and the spring constant is

$$k = P / d_1 = 10000\,\text{N/m}$$

The potential energies of the spring for points d_1, d_2, and d_3 are:

$$\mathrm{PE}_1 = \frac{1}{2}kd_1{}^2 = 50\,\text{J}, \quad \mathrm{PE}_2 = \frac{1}{2}kd_2{}^2 = 200\,\text{J}, \quad \mathrm{PE}_3 = \frac{1}{2}kd_3{}^2 = 12.5\,\text{J}$$

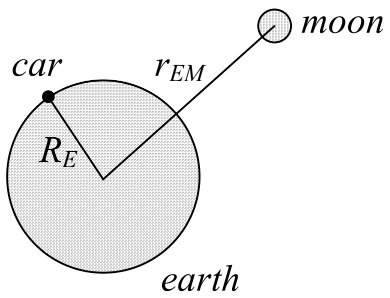
The work done by $\mathbf{F}$ is:

from d_1 to d_2: $\quad \mathrm{W}_{12} = -(\mathrm{PE}_2 - \mathrm{PE}_1) = -250\,\mathrm{J}$

from d_2 to d_3: $\quad \mathrm{W}_{23} = -(\mathrm{PE}_3 - \mathrm{PE}_2) = +186\,\mathrm{J}$

(d) The answer is $\mathrm{W}_{14} = 0\,\mathrm{J}$. Would you explain why?

Example 4.4. Compare the gravitational potential energies due to the interaction between the earth and the moon and between the earth and a car of mass $m = 1000\,\mathrm{kg}$. What is the gravitational potential energy of the earth-moon-car system? The mass and radius of the earth are $M_E = 5.98\times10^{+24}\,\mathrm{kg}$ and $R_E = 6.38\times10^{+6}\,\mathrm{m}$, respectively, and those of the moon are $M_M = 7.35\times10^{+22}\,\mathrm{kg}$ and $R_M = 1.74\times10^{+6}\,\mathrm{m}$; the earth-moon distance is $r_{EM} = 3.85\times10^{+8}\,\mathrm{m}$.



Solution. The gravitational potential energy of two objects is given by (11). We can fix $C = 0$. Once we have chosen the zero reference point, we must keep it unchanged. This means that formula (12) cannot be applied to the earth-car system.

The desired gravitational potential energies are:

earth-moon: $$\mathrm{PE}_{EM} = -G\frac{M_E M_M}{r_{EM}} = -7.62\times10^{+28}\,\mathrm{J}$$

earth-car: $$\mathrm{PE}_{Ec} = -G\frac{M_E m}{R_E} = -6.25\times10^{+10}\,\mathrm{J}$$

The gravitational potential energy of the earth-moon-car system is determined by all the interactions: earth-moon, earth-car, and moon-car:

$$\mathrm{PE}_{sys} = \mathrm{PE}_{EM} + \mathrm{PE}_{Ec} + \mathrm{PE}_{Mc}$$

Only the first term PE_{EM} is significant; the others, PE_{Ec} and PE_{Mc}, are negligibly small. So,

$$\mathrm{PE}_{sys} = -7.62\times10^{+28}\,\mathrm{J}$$

5. LINEAR MOMENTUM AND IMPULSE

The linear momentum, $\mathbf{p}$*, of an object is defined as the product of the object's mass and instantaneous velocity:*

$$\mathbf{p} = m\mathbf{v} \tag{15}$$

Momentum is also called the quantity of motion, for it is a property that an object possesses due to motion. The more the momentum of an object, the harder it is to stop the object, and the greater the effect it has on other objects in collision.

Momentum is a vector quantity that points along the instantaneous velocity of the object. The SI unit for momentum is $\text{kg} \cdot \text{m/s}$.

A force is required to change the momentum of an object. If the net force $\mathbf{F}_{net}$ is constant and acts during a time interval $\Delta t = t_f - t_i$, the change in the object's momentum equals

$$\mathbf{p}_f - \mathbf{p}_i = \mathbf{F}_{net}\Delta t \tag{16}$$

where $\mathbf{p}_i$ and $\mathbf{p}_f$ stand for the momentum at an initial (earlier) time t_i (usually $t_i = 0$) and that at a final (later) time t_f, respectively. The product $\mathbf{F}_{net}\Delta t$ is called the *impulse* $\mathbf{J}$ of the net force $\mathbf{F}_{net}$.

If several forces affect an object during different parts of Δt, not the entire Δt, then $\mathbf{J}$ is easier to compute as the vector sum of the individual impulses of the forces: $\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 + ... + \mathbf{J}_m$. *The change in an object's momentum is caused by the impulse due to all the forces acting on the object:*

$$\mathbf{p}_f - \mathbf{p}_i = \mathbf{J} \tag{17}$$

This is another statement, known as the *impulse-momentum theorem*, of Newton's second law.

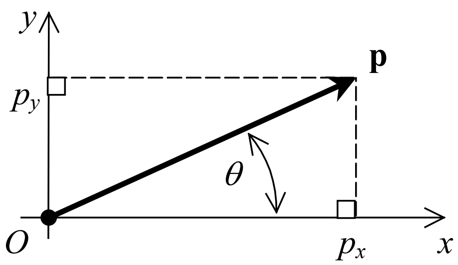
Impulse is a vector quantity that remains the same with respect to different inertial frames of reference; and so is a change in momentum. We say that both impulse and change in momentum are invariant relative to inertial frames of reference. The SI unit for impulse is $\text{N} \cdot \text{s} (= \text{kg} \cdot \text{m/s})$.

For a system of objects, its total momentum is defined as the vector sum of the momenta of the individual objects:

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 + ... + \mathbf{p}_n = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + ... + m_n\mathbf{v}_n \tag{18}$$

where $\mathbf{p}_i = m\mathbf{v}_i$ is the momentum of object i. Generally, the momentum of a system and those of its constituents are different.

In dealing with momentum and impulse, it is crucial to remember that they (A) *are vectors and* (B) *obey vector equations.*



A) Like any vector, $\mathbf{p}$ has magnitude p and direction. It can be represented by an arrow whose length is proportional to p and whose direction is determined by some angles with the axes of the reference frame. If, for instance, the motion occurs in a plane, it is enough to use the angle θ between $\mathbf{p}$ and the positive x-axis. On the other hand, the vector $\mathbf{p}$ is completely determined by its components, p_x and p_y. These are related to p and θ by $p_x = p\cos\theta$, $p_y = p\sin\theta$.

B) Vector operations must be used to calculate, for instance, the momentum $\mathbf{P}$ of a system. Using components is usually the simplest way. After specifying the reference frame,

a) write down the vector sum (18);

b) resolve this vector sum into components by rewriting it (for a plane) as

x-axis: $\quad P_x = mv_{1x} + mv_{2x} + ... + m_n v_{nx}$

y-axis: $\quad P_y = mv_{1y} + mv_{2y} + ... + m_n v_{ny}$

c) compute the components on the right in terms of the magnitudes and directional angles of the momenta of the individual objects;

d) find the magnitude and direction of $\mathbf{P}$ by

$$P = \sqrt{P_x^{\,2} + P_y^{\,2}} \tag{19}$$

$$\alpha = \tan^{-1}\left(\frac{P_y}{P_x}\right) \tag{20}$$

A similar procedure applies to equation (18) when a momentum $\mathbf{p}_i$ is to be found instead of $\mathbf{P}$; and to vector equations (16) and (17).

Be aware that direct calculations of angles by formula (20) *may lead to a wrong answer: this equation has two solutions lying between* 0° *and* 360°*, but most calculators give only one. To select the correct value:*

1) sketch the vector studied and identify the quadrant in which the vector lies;

2) use (20) to calculate the angle for the absolute value of the quotient P_y/P_x (that is, leave out the negative sign in this quotient if any);

3) this angle, always less than 90°, will be the one between the *x*-axis (positive or negative) and the vector of interest.

When solving a particular problem, be sure to:

1) specify the object or the system of objects under study;

2) sketch a picture of the situation studied; label the masses and velocities of the objects, and, if needed, the forces acting; if the situation is studied for different times, reflect this in the picture or draw separate pictures for each time;

3) pick a convenient frame of reference;

4) write out necessary vector equations;

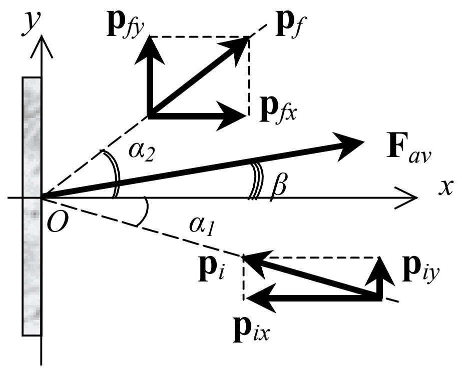
5) resolve the vector equations into components;

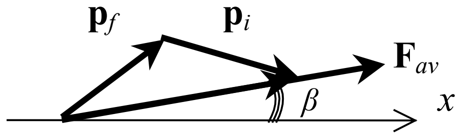
6) calculate the components;

7) solve the equations for the desired quantities.

Example 5.1. A ball of mass $m = 0.20\,\text{kg}$ hits a vertical rough wall at a speed of $v_1 = 30\,\text{m/s}$ and rebounds at a speed of $v_2 = 25\,\text{m/s}$. Its velocity just before and just after hitting makes angles of $\alpha_1 = 30^\circ$ and $\alpha_2 = 45^\circ$ above the horizontal. Find (a) the change in the momentum of the ball, (b) impulse of the force, and (c) the average force exerted on the ball by the wall, assuming a collision time $t = 0.01\,\text{s}$.

Solution. Let the *x*-axis of the reference frame point to the right horizontally, the y-axis point upward, and the origin O be placed at the point of hitting.





$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$

The change $\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$ in the ball's momentum and the impulse $\mathbf{J}$ of the force exerted on the ball by the wall obey vector equation (17). In components, this reads:

$$x-\text{axis}: \qquad p_{fx} - p_{ix} = J_x$$
$$y-\text{axis}: \qquad p_{fy} - p_{iy} = J_y \tag{21}$$

The components of the initial (just before striking) momentum $\mathbf{p}_i = m\mathbf{v}_1$ and final (just after) momentum $\mathbf{p}_f = m\mathbf{v}_2$ of the ball are:

$$p_{ix} = -mv_1 \cos\alpha_1, \qquad p_{iy} = +mv_1 \sin\alpha_1$$

$$p_{fx} = +mv_2 \cos\alpha_2, \qquad p_{fy} = +mv_2 \sin\alpha_2$$

(a) The components of the vector $\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$ are:

$$\Delta p_x = p_{fx} - p_{ix} = mv_2 \cos\alpha_2 + mv_1 \cos\alpha_1 = 8.7\,\text{kg}\cdot\text{m/s}$$

$$\Delta p_y = p_{fy} - p_{iy} = mv_2 \sin\alpha_2 - mv_1 \sin\alpha_1 = 0.54\,\text{kg}\cdot\text{m/s}$$

The magnitude and direction of $\Delta\mathbf{p}$ are:

$$\Delta p = \sqrt{\Delta p_x{}^2 + \Delta p_y{}^2} = 8.7\,\text{kg}\cdot\text{m/s},$$

$$\alpha = \tan^{-1}\left(\Delta p_y / \Delta p_x\right) = 3.5^\circ$$

(b) The impulse $\mathbf{J}$ equals the vector $\Delta\mathbf{p}$ and, therefore, has the same components, magnitude and directional angle:

$$J_x = \Delta p_x = 8.7\,\text{N}\cdot\text{s}, \qquad J_y = \Delta p_y = 0.54\,\text{N}\cdot\text{s}$$

$$J = \Delta p = 8.7\,\text{N}\cdot\text{s}, \qquad \alpha = 3.5^\circ$$

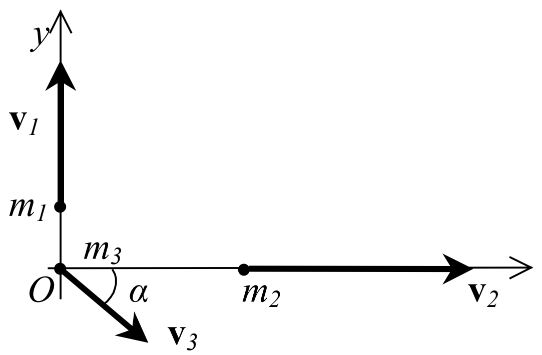
(c) With $\Delta t = t$, the average force $\mathbf{F}_{av}$ is related to $\mathbf{J}$ by $\mathbf{F}_{av} = \mathbf{J}/t$. Its components, magnitude, and directional angle above the positive x-axis are:

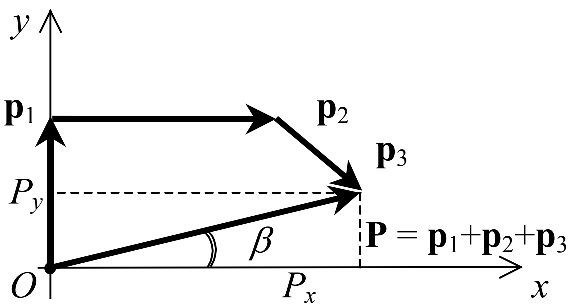
$$F_{av,x} = J_x/t = 870\,\text{N}, \qquad F_{av,y} = J_y/t = 54\,\text{N}$$

$$F_{av} = \Delta p = 870\,\text{N}, \qquad \beta = 3.5^\circ$$

Example 5.2. A car of mass $m_1 = 1200\,\text{kg}$ is moving due north at a speed of $v_1 = 20.0\,\text{m/s}$. A second car of mass $m_2 = 1200\,\text{kg}$ is moving due east at a speed of $v_2 = 30.0\,\text{m/s}$. A third car of mass $m_3 = 900\,\text{kg}$ is heading $\alpha = 40.0^\circ$ south of east at a speed of $v_3 = 15.0\,\text{m/s}$. Find (a) the total linear momentum (magnitude and direction) of the three-car system, (b) the total kinetic energy of the system. (c) What must the velocity (magnitude and direction) of a fourth car of mass $m_4 = 1800\,\text{kg}$ be so that the total momentum of the four cars is zero? At first, solve problem (c) without referring to problem (a), then compare the results.

Solution. To begin with, we sketch a picture of the situation studied and then choose the frame of reference. Let its *x*-and *y*-axis be directed due east and due north, respectively.





(a) The momentum of the three-car system is given by

$$\mathbf{P} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3$$

This vector equation is equivalent to two scalar equations in the *x*- and *y*-components:

x-axis: $P_x = m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x}$

y-axis: $P_x = m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x}$

The components of the velocities through the magnitudes and directional angles are:

car *1*: $v_{1x} = 0$, $v_{1y} = +v_1$

car *2*: $v_{2x} = +v_2$, $v_{2y} = 0$

car *3*: $v_{3x} = +v_3 \cos\alpha$, $v_{3y} = -v_3 \sin\alpha$

The components of $\mathbf{P}$ equal:

$$P_x = m_1 \cdot 0 + m_2 v_2 + m_3 v_3 \cos\alpha = 5.53 \times 10^{+4}\,\text{kg} \cdot \text{m/s}$$

$$P_y = m_1 \cdot v_1 + m_2 \cdot 0 - m_3 v_3 \sin\alpha = 1.53 \times 10^{+4}\,\text{kg} \cdot \text{m/s}$$

The magnitude and direction of $\mathbf{P}$ are:

$$P = \sqrt{P_x^2 + P_y^2} = 5.57 \times 10^{+4} \text{ kg} \cdot \text{m/s}$$

$$\beta = \tan^{-1}\left(P_x / P_y\right) = 15.5^\circ \text{ north of east}$$

(b) The kinetic energies of the cars and that of the system are as follows:

$$\text{KE}_1 = \frac{1}{2} m_1 v_1^2, \quad \text{KE}_2 = \frac{1}{2} m_2 v_2^2, \quad \text{KE}_3 = \frac{1}{2} m_3 v_3^2$$

$$\text{KE}_{sys} = \text{KE}_1 + \text{KE}_2 + \text{KE}_3 = 1.02 \times 10^{+6} \text{ J}$$

c) The total momentum of the four-car system is

$$\mathbf{P} = m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2 + m_3 \mathbf{v}_3 + m_4 \mathbf{v}_4$$

Resolving into components gives:

x-axis: $\quad P_x = m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x} + m_4 v_{4x}$

y-axis: $\quad P_y = m_1 v_{1y} + m_2 v_{2y} + m_3 v_{3y} + m_4 v_{4y}$

The velocity components of cars 1, 2, and 3 are found in (a). The momentum components of the four-car system are

$$P_x = 0, \qquad P_y = 0$$

The components v_{4x} and v_{4y} of the fourth car's velocity $\mathbf{v}_4$ are determined from the equations

$$m_1 \cdot 0 + m_2 v_2 + m_3 v_3 \cos\alpha + m_4 v_{4x} = 0$$

$$m_1 v_1 + m_2 \cdot 0 - m_3 v_3 \sin\beta + m_4 v_{4y} = 0$$

Algebraic manipulations and numerical calculations yield:

$$m_4 v_{4x} = -(m_2 v_2 + m_3 v_3 \cos\alpha) = -5.53 \times 10^{+4} \text{ kg} \cdot \text{m/s}$$

$$m_4 v_{4y} = -(m_1 v_1 - m_3 v_3 \sin\alpha) = -1.53 \times 10^{+4} \text{ kg} \cdot \text{m/s}$$

The momentum $\mathbf{p}_4 = m_4 \mathbf{v}_4$ is exactly the total momentum, taken with the negative sign, of the three-car system. You might have predicted this result without going such a long way (how?). Nevertheless, it is essential for you to get used to manipulations like those above: when dealing with topics such as vectors, Newton's second law, conservation of linear momentum, etc., you will most likely need them.

The components of $\mathbf{v}_4$ are:

$$v_{4x} = -30.7\,\text{m/s}, \qquad v_{4y} = -8.5\,\text{m/s}$$

The vector $\mathbf{v}_4$ has the magnitude

$$v_4 = \sqrt{{v_{4x}}^2 + {v_{4y}}^2} = 31.9\,\text{m/s}$$

and makes the angle

$$\alpha_4 = \tan^{-1}\left(v_{4y}/v_{4x}\right) = 15.5^\circ$$

below the negative x-axis: in order to make the total momentum of the four cars zero, the fourth car must head 15.5° south of east at a speed of 31.9 m/s.

Example 5.3. A hockey puck of mass $m = 0.10\,\text{kg}$ moves along the x-axis on a frictionless ice surface with an initial speed of $v_0 = 20.0\,\text{m/s}$. Find its final velocity in each of the following cases: (a) a force $F_{1x} = 0.20\,\text{N}$ acts for $t_1 = 5\,\text{s}$; (b) a force $F_{1x} = 0.20\,\text{N}$ acts for $t_1 = 5\,\text{s}$, and then a force $F_{2x} = -0.40\,\text{N}$ acts for the next $t_2 = 5\,\text{s}$; (c) forces $F_{1x} = 0.20\,\text{N}$ and $F_{2x} = -0.40\,\text{N}$ act for $t_1 = 5\,\text{s}$, and then the force F_{2x} acts for the next $t_2 = 5\,\text{s}$; (d) a force $F_{1x} = 0.20\,\text{N}$ acts for $t_1 = 5\,\text{s}$, and then a force $F_{2y} = -0.40\,\text{N}$ acts for the next $t_2 = 5\,\text{s}$; (e) forces $F_{1x} = 0.20\,\text{N}$ and $F_{2y} = -0.40\,\text{N}$ act for $t_1 = 5\,\text{s}$, and then the force F_{2y} acts for the next $t_2 = 5\,\text{s}$.

Solution. The change $\mathbf{p}_f - \mathbf{p}_i$ in the momentum and, consequently, velocity of the puck is caused by the total impulse $\mathbf{J}$ of the forces. These quantities are related by equations (17) and (21).

The components of the puck's initial (at $t_i = 0$, just before the puck is driven by forces) and final (at $t = t_f$, just after the forces are off) momenta are:

$$p_{ix} = +mv_0, \qquad p_{iy} = 0$$

$$p_{fx} = mv_x, \qquad p_{fy} = mv_y$$

where v_x and v_y are the components of the desired final velocity $\mathbf{v}$. By substituting the above expressions into (21), we find:

$$v_x = v_0 + \frac{J_x}{m}, \qquad v_y = \frac{J_y}{m} \tag{22}$$

The problem reduces to calculating the impulse components J_x and J_y.

(a) The constant force F_{1x} acts from $t_i = 0$ to $t_f = t_1$; therefore, its impulse is $\mathbf{J} = \mathbf{F}_1 t_1$. In components: $J_x = F_{1x} t_1 = 1.0\,\mathrm{N \cdot s}$, $J_y = F_{yx} t_1 = 0\,\mathrm{N \cdot s}$. Then, according to (22), $v_x = 30\,\mathrm{m/s}$, $v_y = 0\,\mathrm{m/s}$. After 5 s elapsed, the puck is moving at a speed of 30 m/s along the positive *x*-axis.

(b) The time interval when at least one force is involved lasts from $t_i = 0$ to $t_f = t_1 + t_2$. The constant force $\mathbf{F}_1$ acts from $t_i = 0$ to $t = t_1$; so, its impulse equals $\mathbf{J}_1 = \mathbf{F}_1 t_1$. The constant force $\mathbf{F}_2$ acts from $t = t_1$ to $t_f = t_1 + t_2$; therefore, its impulse is $\mathbf{J}_2 = \mathbf{F}_2 t_2$. The total impulse over the time interval $(t_f - t_i)$ is the vector sum

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 = \mathbf{F}_1 t_1 + \mathbf{F}_2 t_2$$

In components: $J_x = F_{1x} t_1 + F_{2x} t_2 = -1.0\,\mathrm{N \cdot s}$, $J_y = F_{1y} t_1 + F_{2y} t_2 = 0\,\mathrm{N \cdot s}$. By (22), the velocity components are: $v_x = 10\,\mathrm{m/s}$, $v_y = 0\,\mathrm{m/s}$.

After 10 s elapsed, the puck is moving with a speed of 10 m/s along the positive *x*-axis.

(c) $J_x = F_{1x} t_1 + F_{2x}(t_1 + t_2) = -3.0\,\mathrm{N \cdot s}$, $\quad J_y = F_{1y} t_1 + F_{2y}(t_1 + t_2) = 0\,\mathrm{N \cdot s}$

$$v_x = -10\,\mathrm{m/s}, \qquad v_y = 0\,\mathrm{m/s}$$

10 s later, the puck is moving at a speed of 10 m/s along the negative *x*-axis.

(d) $v_x = 30\,\mathrm{m/s}$, $v_y = -20\,\mathrm{m/s}$. The pucks' final velocity has a magnitude of $36\,\mathrm{m/s}$ and points at an angle of 34° below the positive *x*-axis.

(e) $v_x = 30\,\mathrm{m/s}$, $v_y = -40\,\mathrm{m/s}$. The puck's final velocity has a magnitude of $50\,\mathrm{m/s}$ and points at an angle of 53° below the positive *x*-axis.

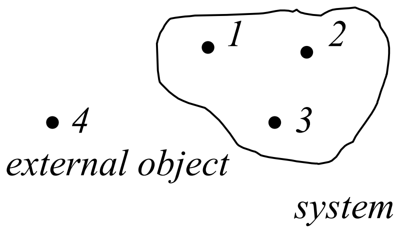
6. CONSERVATION OF LINEAR MOMENTUM

The concept of linear momentum is particularly useful in situations when two or more objects interact during some time interval, but detailed information about the forces that they exert on one another is not available. Just before and just after the interaction, the objects are moving with certain velocities, initial and final, and we are usually interested in determining those.

Applications of the concept are based on the following key definitions and statements.

1) *A system is any collection of objects.* We are free to include particular objects into a system or not. The forces that the objects of a system exert on one another are called *internal*.

Objects located outside a system are called *external*, and so are the forces that they exert on the objects of the system.



If, for instance, a system comprises three objects *1*, *2*, and *3*, whereas object *4* is situated outside, then the forces $\mathbf{F}_{12}$, $\mathbf{F}_{13}$, $\mathbf{F}_{21}$, $\mathbf{F}_{23}$, $\mathbf{F}_{31}$, and $\mathbf{F}_{32}$ that objects *1*, *2*, and *3* exert on one another are internal; and the forces $\mathbf{F}_{14}$, $\mathbf{F}_{24}$, and $\mathbf{F}_{34}$ that object *4* exerts on objects *1*, *2*, and *3* are external.

2) *The (linear) momentum* $\mathbf{P}$ *of a system is the vector sum, such as* (18)*, of the momenta of the constituent objects.*

According to the momentum-impulse theorem, *any change in* $\mathbf{P}$ *is caused by a nonzero impulse* $\mathbf{J}_{ext}$ *due to the net external force* $\mathbf{F}_{net,ext}$*:*

$$\mathbf{P}_f - \mathbf{P}_i = \mathbf{J}_{ext} \tag{23}$$

where $\mathbf{P}_i$ and $\mathbf{P}_f$ are the system's initial (at an initial time t_i) and final (at a final time t_f) momenta, respectively. *The internal forces do not affect* $\mathbf{P}$ *at all; they can change only the momenta of the constituents.*

In the above example, $\mathbf{F}_{net,ext} = \mathbf{F}_{14} + \mathbf{F}_{24} + \mathbf{F}_{34}$. If $\mathbf{F}_{net,ext}$ remains constant during the time $\Delta t = t_f - t_i$, then $\mathbf{J}_{ext} = \mathbf{F}_{net,ext}\Delta t$. Equation (23) takes the form

$$\mathbf{P}_f - \mathbf{P}_i = \mathbf{F}_{net,ext}\Delta t \tag{24}$$

Relative to a three-dimensional Cartesian coordinate system, (24) is equivalent to three scalar equations in components:

$$P_{fx} - P_{ix} = F_{net,ext,x}\Delta t \tag{25}$$

$$P_{fy} - P_{iy} = F_{net,ext,y}\Delta t \tag{26}$$

$$P_{fz} - P_{iz} = F_{net,ext,z}\Delta t \tag{27}$$

In the case of a two-dimensional motion in the *xy*-plane, equation (24) is equivalent to the first two, (25) and (26). For a one-dimensional motion along the *x*-axis, equation (24) reduces to (25) alone.

3) *Under certain conditions, the momentum* $\mathbf{P}$ *is conserved (that is, does not change in time):*

$$\mathbf{P}_f = \mathbf{P}_i \tag{28}$$

In components:

$$P_{fx} = P_{ix}, \quad P_{fy} = P_{iy}, \quad P_{fz} = P_{iz} \tag{29}$$

Equations (28) and (29) hold when the right side of (24) vanishes. The factors behind this and, therefore, the conditions resulting in the conservation of $\mathbf{P}$ may be different:

a) the system is *isolated* – there is no external force acting on it. Then $\mathbf{F}_{net,ext} = 0$;

b) the system is not isolated, but the net external force is zero. Once again, $\mathbf{F}_{net,ext} = 0$;

c) the system is not isolated, $\mathbf{F}_{net,ext} \neq 0$, but the time interval Δt is so short that the impulse of $\mathbf{F}_{net,ext}$ vanishes. One example is an object that breaks into smaller pieces. At least approximately, the total momentum of the pieces is conserved during the time Δt.

4) *The vector nature of momentum is extremely significant for applications. The following fact is widely used.*

Suppose that a nonzero $\mathbf{F}_{net,ext}$ is acting on a system, but its *x*-component is zero: $F_{net,ext,x} = 0$. Formulas (25) – (27) then yield:

$$P_{fx} - P_{ix} = 0, \qquad P_{fy} - P_{iy} = F_{net,ext,y}\Delta t, \qquad P_{fz} - P_{iz} = F_{net,ext,z}\Delta t$$

Since $\mathbf{F}_{net,ext} \neq 0$, the system's momentum is not conserved: $\mathbf{P}_f \neq \mathbf{P}_i$. Nevertheless, its *x*-component remains constant during the motion:

$$P_{fx} = P_{ix} \quad \text{if} \quad F_{net,ext,x} = 0 \tag{30}$$

A similar conclusion is valid for any other direction along which $\mathbf{F}_{net,ext}$ has no component. For instance, we can write

$$P_{fy} = P_{iy} \quad \text{if} \quad F_{net,ext,y} = 0 \tag{31}$$

If there is a direction in which the net external force is zero, the component of the system's momentum along that direction remains constant, even though the system's total momentum may not be conserved.

We often use this consequence of the momentum-impulse theorem, but loosely call it the law of conservation of momentum. Not the vector **P** of the momentum, but only some of its components are conserved.

5) As a result, several kinds of problems are customarily treated in terms of the conservation of momentum:

i) *isolated systems or systems that experience zero net external force*: **P** and, therefore, its components are conserved;

ii) *systems that experience a nonzero net external force during infinitesimal time intervals*: at least approximately, **P** and, therefore, its components are conserved during the intervals;

iii) *systems that experience a nonzero net external force with no components in particular directions*: in general, **P** is changing, but its components along those directions are conserved.

In order to solve the above problems, you may follow these steps:

1) specify the objects constituting the system 'before' (at an initial time) and 'after' (at a final time);

2) sketch pictures of the situations 'before' and 'after'; label the masses and velocities of the objects; write out the system's momenta 'before' and 'after';

3) identify external forces acting on the system, and the directions in which the net external force has no components;

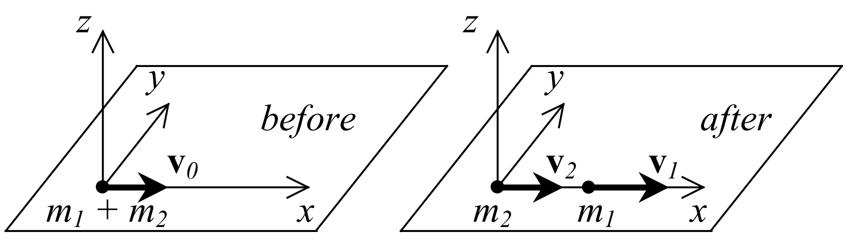
4) choose a reference frame with its axes along those directions;

5) calculate and equate the components of the initial and final momenta of the system along those directions;

6) find the desired quantities.

Example 6.1. A two-stage rocket moves in outer space at a constant speed of $v_0 = 4900\,\text{m/s}$. The two stages, upper with a mass of $m_1 = 1200\,\text{kg}$ and lower with a mass of $m_2 = 2400\,\text{kg}$, are then separated by an explosive charge placed between them. After the explosion, the upper stage keeps moving in the same direction at a speed of $v_1 = 5700\,\text{m/s}$. (a) What is the velocity of the lower stage after the explosion? (b) What work was done by the explosive charge?

Solution. The motion can be studied relative to distant stars, for an attached



three-dimensional Cartesian coordinate system is an inertial frame of reference. Let the *x*-axis of the frame be directed along the initial direction of the motion.

(a) Since the motion occurs in outer space, no external forces are acting on the rocket (more precisely, the gravitational forces due to distant heavenly bodies are negligibly small and can be ignored). The stages of the rocket constitute an isolated two-object system.

The system 'before' includes the two unseparated stages moving as a whole with the velocity $\mathbf{v}_0$. Their combined mass is $m_1 + m_2$.

The initial momentum of the system is

$$\mathbf{P}_i = (m_1 + m_2)\mathbf{v}_0$$

Its components equal:

$$P_{ix} = (m_1 + m_2)v_{0x} = (m_1 + m_2)v_0$$

$$P_{iy} = 0, \qquad P_{iz} = 0$$

The system 'after' consists of the two separated stages. One of mass m_1 moves at the given velocity $\mathbf{v}_1$, so its momentum is $\mathbf{p}_1 = m_1\mathbf{v}_1$. The other of mass m_2 moves with an unknown velocity of $\mathbf{v}_2$; its momentum is $\mathbf{p}_2 = m_2\mathbf{v}_2$.

The final momentum of the system is

$$\mathbf{P}_f = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$$

Its components equal:

$$P_{fx} = m_1v_{1x} + m_2v_{2x} = m_1v_1 + m_2v_{2x}$$

$$P_{fy} = m_1v_{1y} + m_2v_{2y} = 0 + m_2v_{2y}$$

$$P_{fz} = m_1v_{1z} + m_2v_{2z} = 0 + m_2v_{2z}$$

The components v_{2x}, v_{2y}, and v_{2z} are the quantities to be determined.

The momentum of an isolated system is conserved. Then formulas (28) and (29) yield:

$$m_1v_1 + m_2v_{2x} = (m_1 + m_2)v_0, \qquad m_2v_{2y} = 0, \qquad m_2v_{2z} = 0$$

Algebraic manipulations give:

$$v_{2x} = \frac{(m_1 + m_2)v_0 - m_1v_1}{m_2} = 4500\,\text{m/s}, \qquad v_{2y} = 0\,\text{m/s}, \qquad v_{2z} = 0\,\text{m/s}$$

The lower stage keeps moving in the original direction at a speed of 4500 m/s.

(b) According to the work-energy theorem, the change in the kinetic energy of a system equals the total work W done on the system by all the forces, both external and internal: $\mathrm{KE}_f - \mathrm{KE}_i = \mathrm{W}$. In our case, W is done by the explosive charge. The initial and final kinetic energies of the two-stage system are (we retain here more significant figures than necessary):

$$\mathrm{KE}_i = \frac{1}{2}(m_1 + m_2){v_0}^2 = 4.3218 \times 10^{+10}\ \mathrm{J}$$

$$\mathrm{KE}_f = \frac{1}{2}m_1{v_1}^2 + \frac{1}{2}m_2{v_2}^2 = 4.3794 \times 10^{+10}\ \mathrm{J}$$

The work done by the charge equals:

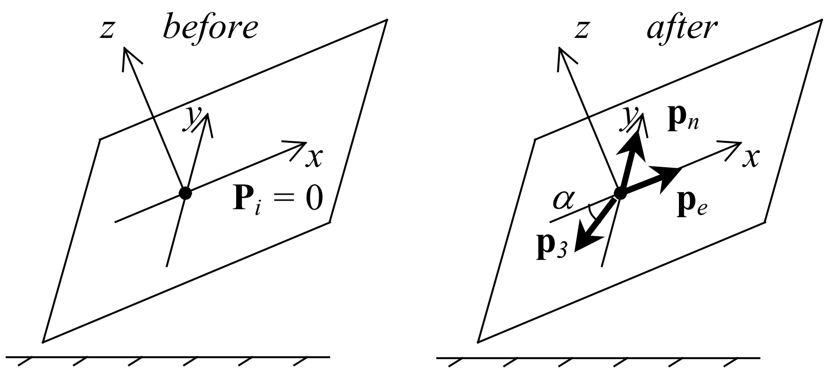
$$\mathrm{W} = \mathrm{KE}_f - \mathrm{KE}_i = 5.76 \times 10^{+8}\ \mathrm{J}$$

So, the initial and final kinetic energies of the system are $4.3 \times 10^{+10}\ \mathrm{J}$ and $4.4 \times 10^{+10}\ \mathrm{J}$, respectively; the explosive charge does $5.8 \times 10^{+8}\ \mathrm{J}$ of work.

Note that the kinetic energy of a system can change because of the work done by the internal forces. These, however, cannot change the system's total momentum.

Example 6.2. A radioactive nucleus at rest decays into a new nucleus, an electron, and a neutrino. The electron and the neutrino are emitted at right angle and have momenta $p_e = 8.60 \times 10^{-23}\ \mathrm{kg \cdot m/s}$ and $p_n = 6.20 \times 10^{-23}\ \mathrm{kg \cdot m/s}$. What is the magnitude and direction of the momentum of the recoiling nucleus?

Solution. We study the radioactive decay relative to the ground. Let the *x*- and the *y*-axes of the coordinate system point along the velocities of the electron and the neutrino, respectively, and let the *z*-axis be perpendicular to the *xy*-plane.



Radioactive decays occur during very brief time intervals, $10^{-13}\ \mathrm{s}$ or less. The total impulse due to various external forces (gravitational and electromagnetic) acting on the particles during such a time interval is extremely small. It does not cause an appreciable change in the momentum – the law of conservation of momentum holds.

The system 'before' is the original nucleus. Being at rest, the nucleus has zero momentum:

$$\mathbf{P}_i = 0$$

In components:

$$P_{ix} = 0, \quad P_{iy} = 0, \quad P_{iz} = 0$$

The system 'after' consists of three distinct particles: the electron with the momentum $\mathbf{p}_e$, the neutrino with the momentum $\mathbf{p}_n$, and the new nucleus with a momentum $\mathbf{p}_3$. The latter is to be determined.

The final momentum of the system is

$$\mathbf{P}_f = \mathbf{p}_e + \mathbf{p}_n + \mathbf{p}_3$$

Its components equal:

$$P_{fx} = p_{ex} + p_{nx} + p_{3x} = p_e + 0 + p_{3x}$$

$$P_{fy} = p_{ey} + p_{ny} + p_{3y} = 0 + p_n + p_{3y}$$

$$P_{fz} = p_{ez} + p_{nz} + p_{3z} = 0 + 0 + p_{3z}$$

The momentum of the system is conserved; that is, formulas (28) and (29) hold true. With the use of the above expressions for the components, (29) gives:

$$p_e + p_{3x} = 0, \quad p_n + p_{3y} = 0, \quad p_{3z} = 0$$

The components of the momentum of the new nucleus are:

$$p_{3x} = -p_e, \quad p_{3y} = -p_n, \quad p_{3z} = 0$$

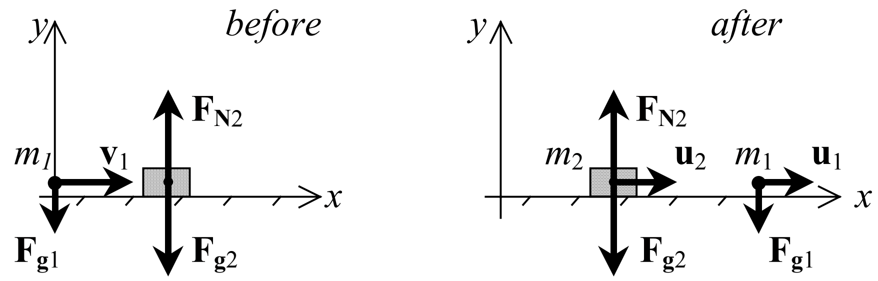
The three products of the decay are moving in one plane, generated by the vectors $\mathbf{p}_e$ and $\mathbf{p}_n$. The magnitude and direction (below the negative x-axis) of $\mathbf{p}_3$ are given by

$$p = \sqrt{p_e{}^2 + p_n{}^2} = 1.06 \times 10^{-23}\ \text{kg} \cdot \text{m/s}$$

$$\alpha = \tan^{-1}\left(\frac{p_n}{p_e}\right) = 35.8^\circ$$

Example 6.3. A projectile of mass $m_1 = 0.150\,\text{kg}$ is fired with a speed of $v_1 = 715\,\text{m/s}$ at a wooden block. The block has a mass of $m_2 = 2.00\,\text{kg}$ and rests on a frictionless table. Immediately after the projectile passes through the block, the block has a speed of $u_2 = 40.0\,\text{m/s}$ and is moving along the original direction of the velocity $\mathbf{v}_1$. Find (a) the horizontal velocity $\mathbf{u}_1$ with which the projectile exits from the block, (b) the initial and final kinetic energies of the projectile-block system, and the decrease in the kinetic energy. What causes this decrease?

Solution. It is convenient to study the motion relative to the ground. The



motion occurs in a vertical plane. Let the x-axis be directed along the vector $\mathbf{v}_1$ and the y-axis point upward.

(a) The system 'before' includes two objects, the projectile and the stationary block. Their momenta are $\mathbf{p}_{1i} = m_1\mathbf{v}_1$ and $\mathbf{p}_{2i} = m_2\mathbf{v}_2 = 0$, respectively. The initial momentum of the system equals

$$\mathbf{P}_i = m_1\mathbf{v}_1$$

The system 'after' includes the same objects, but now the momentum of the projectile is $\mathbf{p}_{1f} = m_1\mathbf{u}_1$ and that of the block is $\mathbf{p}_{2f} = m_2\mathbf{u}_2$. The final momentum of the system equals

$$\mathbf{P}_f = m_1\mathbf{u}_1 + m_2\mathbf{u}_2$$

The projectile-block system is not isolated because there are three external forces acting on it: the gravitational force $\mathbf{F}_{\text{g1}} = m_1\mathbf{g}$ exerted on the projectile by the earth; the gravitational force $\mathbf{F}_{\text{g2}} = m_2\mathbf{g}$ exerted on the block also by the earth; and the normal force $\mathbf{F}_{\text{N2}}$ exerted on the block by the ground. All the forces are parallel to the y-axis, and so is their resultant $\mathbf{F}_{net,ext} = \mathbf{F}_{\text{g1}} + \mathbf{F}_{\text{g2}} + \mathbf{F}_{\text{N2}}$.

We expect that $\mathbf{F}_{\mathbf{g}2}$ and $\mathbf{F}_{\mathbf{N}2}$ balance each other. However, the following is more important. As the force $\mathbf{F}_{net,ext}$ is not zero, the system's momentum is not conserved. However, since $\mathbf{F}_{net,ext}$ has zero component along the *x*-axis, the *x*-component of the system's momentum remains constant (see formula (30)):

$$P_{fx} = P_{ix}$$

These components equal:

$$P_{ix} = m_1 v_{1x} = m_1 v_1, \quad P_{fx} = m_1 u_{1x} + m_2 u_{2x} = m_1 u_1 + m_2 u_2$$

Then:

$$m_1 v_1 = m_1 u_1 + m_2 u_2$$

The answer is

$$u_1 = \frac{m_1 v_1 - m_2 u_2}{m_1} = 182\,\mathrm{m/s}$$

(b) The initial and final kinetic energies are:

$$\mathrm{KE}_i = \frac{1}{2} m_1 v_1{}^2 + \frac{1}{2} m_2 v_2{}^2 = \frac{1}{2} m_1 v_1{}^2 = 3.83 \times 10^{+4}\ \mathrm{J}$$

$$\mathrm{KE}_f = \frac{1}{2} m_1 u_1{}^2 + \frac{1}{2} m_2 u_2{}^2 = 4.08 \times 10^{+3}\ \mathrm{J}$$

The change in the kinetic energy equals

$$\mathrm{KE}_f - \mathrm{KE}_i = -3.43 \times 10^{+4}\ \mathrm{J}$$

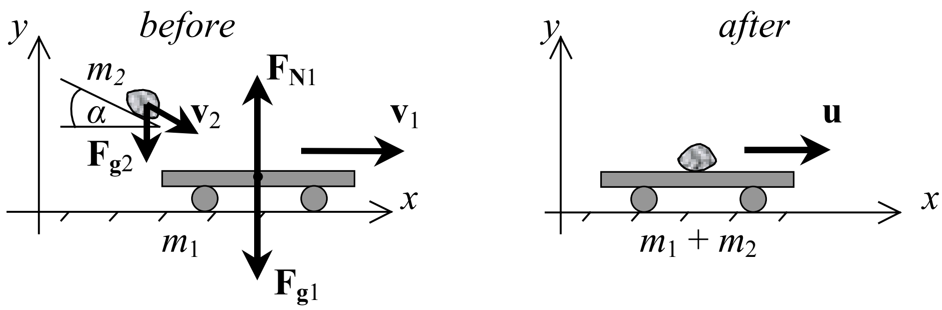
This change is negative – the kinetic energy of the system decreases. The loss is caused by the negative work done by the friction between the projectile and the block while the projectile was traveling within the block.

Note that the friction here is an internal force. *Once again, internal forces, even friction, cannot change the total momentum of a system, though they can change the kinetic energy of the system. We can apply the law of conservation of linear momentum to situations when mechanical energy is not conserved. Inelastic collisions exemplify those.*

It is only external forces that affect the total momentum of a system.

Example 6.4. A mine car, whose mass is $m_1 = 440\,\text{kg}$, rolls at a speed of $v_1 = 0.50\,\text{m/s}$ on a horizontal track. A chunk of coal, with a mass of $m_2 = 150\,\text{kg}$, has a speed of $v_2 = 0.80\,\text{m/s}$ when it leaves a declined chute. The chute makes an angle of $\alpha = 25^\circ$ with the horizontal level. Determine the velocity of the car-coal system after the coal has come to rest in the car.

Solution. The motion occurs in a vertical plane. It is convenient to study it



relative to the ground, and direct the *x*-axis horizontally and the *y*-axis upward.

The system ‘before’ includes two objects, the car and the chunk. Their momenta are $\mathbf{p}_{1i} = m_1\mathbf{v}_1$ and $\mathbf{p}_{2i} = m_2\mathbf{v}_2$, respectively. The initial momentum of the system equals

$$\mathbf{P}_i = m_1\mathbf{v}_1 + m_2\mathbf{v}_2$$

The system ‘after’ also includes the car and the chunk, which now move horizontally as a whole with some velocity $\mathbf{u}$. Their combined mass is $m_1 + m_2$. The final momentum of the system is

$$\mathbf{P}_f = (m_1 + m_2)\mathbf{u}$$

The velocity $\mathbf{u}$ is to be determined. The car-coal system is not isolated because there are three vertical and one horizontal external forces acting on it: the gravitational force $\mathbf{F}_{g1} = m_1\mathbf{g}$ exerted on the car by the earth, the normal force $\mathbf{F}_{N1}$ exerted on the car by the ground ($\mathbf{F}_{g1}$ and $\mathbf{F}_{N1}$ are not necessary equal in magnitude), the gravitational force $\mathbf{F}_{g2} = m_2\mathbf{g}$ exerted on the chunk by the earth, and the friction $\mathbf{F}_f$ between the car and the track. Since the car is rolling, we expect the force $\mathbf{F}_f$ to be small enough so as to neglect it.

The net external force $\mathbf{F}_{net,ext} = \mathbf{F}_{\mathbf{g}1} + \mathbf{F}_{\mathbf{N}1} + \mathbf{F}_{\mathbf{g}2}$ is not zero; therefore, the momentum of the car-coal system is not conserved. However, since $\mathbf{F}_{net,ext}$ is directed along the vertical and has no component along the x-axis, it does not affect the x-component of the momentum:

$$P_{fx} = P_{ix}$$

Taking into account that

$$P_{ix} = m_1 v_{1x} + m_2 v_{2x} = m_1 v_1 + m_2 v_2 \cos\alpha$$

$$P_{fx} = (m_1 + m_2) u_x = (m_1 + m_2) u$$

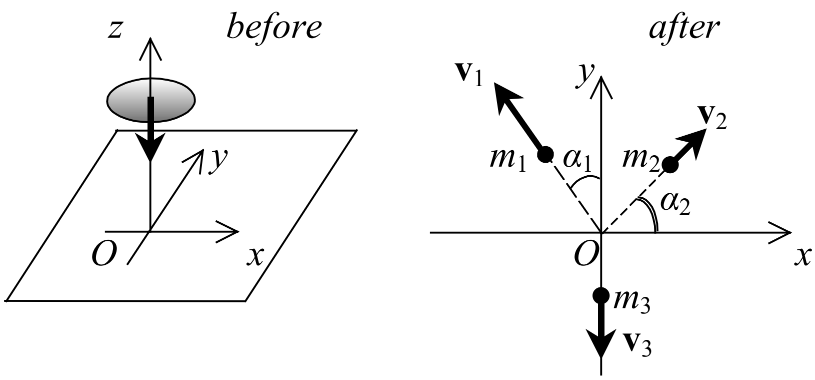
we get:

$$(m_1 + m_2) u = m_1 v_1 + m_2 v_2 \cos\alpha$$

The final speed of the car-coal system is

$$u = \frac{m_1 v_1 + m_2 v_2 \cos\alpha}{m_1 + m_2} = 0.56\,\text{m/s}$$

Example 6.5. By accident, a large plate is dropped vertically down and breaks into three pieces. The pieces fly apart parallel to the floor as shown in the drawing. Find the masses m_1 and m_2 by using the following data: $v_1 = 3.00\,\text{m/s}$, $\alpha_1 = 25^\circ$, $v_2 = 1.79\,\text{m/s}$, $\alpha_2 = 45^\circ$, $m_3 = 1.30\,\text{kg}$, $v_3 = 3.07\,\text{m/s}$.



Solution. We study the event with respect to the floor. Let the x- and y-coordinate axes lie in the plane of the floor and the z-axis be perpendicular to it.

The system 'before' is the plate; the system 'after' includes the three pieces.

The external forces vary during the process. They are: the gravitational force $\mathbf{F_g} = m\mathbf{g}$ exerted on the plate by the earth, the normal force $\mathbf{F_N}$ exerted on the plate by the floor, and the gravitational forces $\mathbf{F_{g1}} = m_1\mathbf{g}$, $\mathbf{F_{g2}} = m_2\mathbf{g}$, and $\mathbf{F_{g3}} = m_3\mathbf{g}$ exerted on the pieces by the earth. Significantly, all the forces are parallel to the z-axis and have no components along the floor. In other words, their x- and y-components are zero. The corresponding momentum components are conserved (see also (30) and (31)):

$$P_{fx} = P_{ix}, \qquad P_{fy} = P_{iy}$$

The x- and y-components of the initial momentum are zero: while the plate is falling, its velocity is parallel to the z-axis:

$$P_{ix} = 0, \qquad P_{iy} = 0$$

The x- and y-components of the final momentum are (see the drawing above):

$$P_{fx} = m_1 v_{1x} + m_2 v_{2x} + m_3 v_{3x} = -m_1 v_1 \sin\alpha_1 + m_2 v_2 \cos\alpha_2 + 0$$

$$P_{fy} = m_1 v_{1y} + m_2 v_{2y} + m_3 v_{3y} = +m_1 v_1 \cos\alpha_1 + m_2 v_2 \sin\alpha_2 - m_3 v_3$$

Formulas (30) and (31) now yield:

$$-m_1 v_1 \sin\alpha_1 + m_2 v_2 \cos\alpha_2 = 0$$

$$m_1 v_1 \cos\alpha_1 + m_2 v_2 \sin\alpha_2 - m_3 v_3 = 0$$

From the former equation:

$$m_2 = \frac{m_1 v_1 \sin\alpha_1}{v_2 \cos\alpha_2}$$

Substitution of this into the latter equation gives:

$$m_1 v_1 \cos\alpha_1 + m_1 v_1 \sin\alpha_2 \tan\alpha_2 = m_3 v_3$$

Then

$$m_1 = \frac{m_3 v_3}{v_1(\cos\alpha_1 + \sin\alpha_2 \tan\alpha_2)}$$

Finally, calculations yield:

$$m_1 = 1.00\,\text{kg}, \qquad m_2 = 1.00\,\text{kg}$$

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