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Forced Vibrations of the Infinite Shell of the Square Cross Section

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Abstract. The problem about steady-state forced vibrations of an infinite shell of the square cross section is investigated. The dispersion curves are given, the resonance frequencies are found. The stress distribution in a construction is investigated. In case of low-frequency vibrations the engineering formula for approximate calculation of the construction is offered. The graph of dependence of a relative accuracy on frequency is given.

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The thin-walled constructions of the square cross section have a wide application in construction, shipbuilding, bridge engineering and mechanical engineering. The theory and the methods of static and dynamic analysis of the box-like shells were studied in numerous works, which review is present in [1–4]. The simple harmonic motions of semi-infinite box-like shell of the rectangular cross section are surveyed in work [2], in which the homogeneous solutions were constructed. In the work [3] the dispersion equation for propagation of normal waves in the infinite box-like shell of the corner and the square cross section were obtained. Let's mark, that in the above-mentioned works, the resonance frequencies were not found also numerical calculations were not carried out. The present work is dedicated to study of these problems.

The plate-like construction consist of thin plates of thickness h and a width $2a$ (Fig. 1). The construction has square cross section. The identical transverse loading $q(x,y)e^{-i\omega t}$ symmetric concerning a medial line of a plate (in the further factor $e^{-i\omega t}$ we shall omit).

In a dimensionless form the boundary value problem that describe the combined planar and flexural state of a construction's plates will consist of the differential equation of vibrations of thin plates

$$D\Delta^2 w\left(x,y\right) - \omega^2\varepsilon^{-2} w\left(x,y\right) = q\left(x,y\right) \quad \left(0 < x < 1, |y| < \infty\right) \tag{1}$$

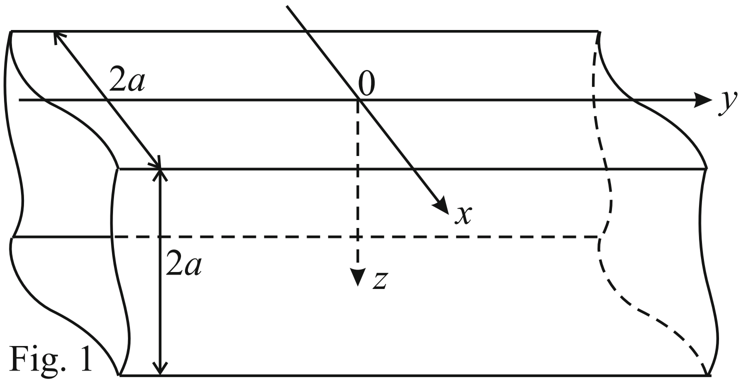

Fig. 1

the Lame equations, which describes the plain stressed state of the plate

$$\left\{ \begin{array}{l} G^{-1}\Delta u\left(x,y\right)+2\left(1-\mu\right)^{-1}\partial\theta\left(x,y\right)/\partial x+\omega^2 u\left(x,y\right)=0 \\ G^{-1}\Delta v\left(x,y\right)+2\left(1-\mu\right)^{-1}\partial\theta\left(x,y\right)/\partial y+\omega^2 v\left(x,y\right)=0 \end{array} \right. \tag{2}$$

$$(0<x<1, |y|<\infty)$$

boundary conditions taking into account symmetry, concerning an axis y

$$\partial w/\partial x|_{x=0}=0, \quad V_x|_{x=0}=0, \quad u|_{x=0}=0, \quad \tau_{xy}|_{x=0}=0 \tag{3}$$

boundary conditions, which describes the rigid joint of plates taking into account of symmetry to edges of a construction [4]

$$\partial w/\partial x|_{x=1}=0, \quad \tau_{xy}|_{x=1}=0, \quad w|_{x=1}=-\varepsilon^2 u|_{x=1}, \quad V_x|_{x=1}=\sigma_x|_{x=1}. \tag{4}$$

The dimensional quantities (they further will be marked by a sinuous line) are connected with dimensionless following relations $\tilde{x}=\tilde{a}x$, $\tilde{y}=\tilde{a}y$, $\tilde{h}=\tilde{a}\varepsilon$, $\tilde{D}=\tilde{E}\tilde{h}^3 D$, $\tilde{G}=\tilde{E}G$, $\tilde{q}=\tilde{E}q$, $\tilde{w}=\tilde{a}\varepsilon^{-3}w$, $\tilde{u}=\tilde{a}\varepsilon^{-1}u$, $\tilde{v}=\tilde{a}\varepsilon^{-1}v$, $\tilde{V}_{\tilde{x}}=\tilde{E}\tilde{a}V_x$, $\tilde{\sigma}_{\tilde{x}}=\tilde{E}\sigma_x$, $\tilde{\tau}_{\tilde{x}\tilde{y}}=\tilde{E}\tau_{xy}$, $\tilde{\omega}=\omega\tilde{T}^{-1}$, $\tilde{T}=\tilde{a}/\tilde{c}$, $\tilde{c}=\sqrt{\tilde{E}\Big/\tilde{\rho}}$; $\tilde{u}$, $\tilde{v}$, $\tilde{w}$ – the displacements of points of plates in the directions of axes $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ accordingly; $\tilde{V}_{\tilde{x}}=-\tilde{D}\left[\partial^3\tilde{w}/\partial\tilde{x}^3+(2-\mu)\,\partial^3\tilde{w}/\partial\tilde{x}\partial\tilde{y}^2\right]$, $\tilde{\sigma}_{\tilde{x}}=\tilde{F}\left(\partial\tilde{u}/\partial\tilde{x}+\mu\partial\tilde{v}/\partial\tilde{y}\right)$, $\tilde{\tau}_{\tilde{x}\tilde{y}}=\tilde{G}\left(\partial\tilde{u}/\partial\tilde{y}+\partial\tilde{v}/\partial\tilde{x}\right)$ – generalized transverse force, normal and tangential stresses; $\tilde{D}=\tilde{E}\tilde{h}^3\left[12\left(1-\mu^2\right)\right]^{-1}$-flexural rigidity of a plate; $\tilde{h}$ – thickness of a plate; $\tilde{\rho}$ – the plate density; $\tilde{E}$ - Young's modulus; μ – Poisson's ratio; $\Delta=\partial^2\big/\partial x^2+\partial^2\big/\partial y^2$ – the Laplace operator; $\tilde{G}=\tilde{E}\Big/[2\left(1+\mu\right)]$ – shear modulus; $\theta=\partial u/\partial x+\partial v/\partial y$; $\tilde{F}=\tilde{E}\Big/\left(1-\mu^2\right)$.

The solution of the problem (1)–(4) can be noted as the Fourier integral

$$f\left(x,y\right)=\frac{1}{2\pi}\int\limits_{\Gamma} f_\alpha\left(x\right)e^{-i\alpha y}d\alpha.$$

Where the contour of an integration Γ is picked by a principle of a limiting absorption [5, 6] and bypasses real poles of function $f_\alpha(x)$. The select of a contour of an integration enables to construct a unique solution of dynamic problems [5, 6]. The function $f_\alpha(x)$ represents the Fourier transforms of all required magnitudes of the problem:

$$u_\alpha(x) = \varphi'_\alpha(x) - i\alpha\psi_\alpha(x) = C_3\chi_1 sh(\chi_1 x) + C_4\alpha sh(\chi_2 x)/\chi_2$$

$$v_\alpha(x) = -i\alpha\varphi_\alpha(x) - \psi'_\alpha(x) = -i[C_3\alpha ch(\chi_1 x) + C_4 ch(\chi_2 x)]$$

$$\tau_{xy\alpha}(x) = -G\left[2i\alpha\varphi'_\alpha(x) + \left(2\alpha^2 - k_2^2\right)\psi_\alpha(x)\right]$$

$$= -iG\left[2C_3\alpha\chi_1 sh(\chi_1 x) + C_4\left(2\alpha^2 - k_2^2\right) sh(\chi_2 x)/\chi_2\right]$$

$$\sigma_{x\alpha}(x) = -F\left[\left(k_1^2 - (1-\mu)\alpha^2\right)\varphi_\alpha(x) + i\alpha(1-\mu)\psi'_\alpha(x)\right]$$

$$= -F\left[C_3\left(k_1^2 - (1-\mu)\alpha^2\right) ch(\chi_1 x) + C_4\alpha(1-\mu) ch(\chi_2 x)\right]$$

$$M_{x\alpha}(x) = -D\left[w''_\alpha(x) - \mu\alpha^2 w_\alpha(x)\right] = -D\left\{\left(\frac{d^2}{dx^2} - \mu\alpha^2\right) w^q_\alpha(x)\right.$$

$$\left. + C_1\left[(1-\mu)\alpha^2 + \gamma^2\right] ch(\lambda_1 x) + C_2\left[(1-\mu)\alpha^2 - \gamma^2\right] ch(\lambda_2 x)\right\}$$

$$w_\alpha(x) = w^q_\alpha(x) + C_1 ch(\lambda_1 x) + C_2 ch(\lambda_2 x)$$

$$\varphi_\alpha(x) = C_3 ch(\chi_1 x), \quad \psi_\alpha(x) = iC_4 sh(\chi_2 x)/\chi_2$$

$$\lambda_n = \sqrt{\alpha^2 - (-1)^n\gamma^2}, \quad \chi_n = \sqrt{\alpha^2 - k_n^2} \quad (n = 1, 2)$$

$$w^q_\alpha(x) = \frac{1}{D}\int_0^1 q_\alpha(\xi)\,\Phi_\alpha(x,\xi)\,d\xi$$

$$\Phi_\alpha(x,\xi) = e_\alpha(|x-\xi|) + e_\alpha(x+\xi)$$

$$e_\alpha(x) = \left(4\gamma^2\right)^{-1}\left[\lambda_1^{-1} sh(\lambda_1 x) - \lambda_2^{-1} sh(\lambda_2 x)\right]$$

$C_n = \Delta_n/\Delta, \quad n = \overline{1,4}, \quad$ – is the solution of the system

$$\begin{pmatrix} \lambda_1 \mathrm{sh}(\lambda_1) & \lambda_2 \mathrm{sh}(\lambda_2) & 0 & 0 \\ 0 & 0 & 2\alpha\chi_1 \mathrm{sh}(\chi_1) & \left(2\alpha^2 - k_2^2\right)\frac{\mathrm{sh}(\chi_2)}{\chi_2} \\ \mathrm{ch}(\lambda_1) & \mathrm{ch}(\lambda_2) & \varepsilon^2\chi_1 \mathrm{sh}(\chi_1) & \varepsilon^2\alpha\frac{\mathrm{sh}(\chi_2)}{\chi_2} \\ \frac{\lambda_1^3 \mathrm{sh}(\lambda_1)}{12} & \frac{\lambda_2^3 \mathrm{sh}(\lambda_2)}{12} & \left((1-\mu)\alpha^2 - k_1^2\right)\mathrm{ch}(\chi_1) & \alpha(1-\mu)\,\mathrm{ch}(\chi_2) \end{pmatrix}$$

$$\times \begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} -dw^q_\alpha(1)/dx \\ 0 \\ -w^q_\alpha(1) \\ -\frac{1}{12}d^3 w^q_\alpha(1)/dx^3 \end{pmatrix}$$

$$\Delta = \Delta_u\Delta_V - \Delta_\sigma\Delta_w$$

$$\Delta_\sigma = 2(1-\mu)\left[\alpha^2\chi_1 sh(\chi_1)\, ch(\chi_2) - \left(\alpha^2 - \frac{1}{2}k_2^2\right)^2 ch(\chi_1)\,\chi_2^{-1} sh(\chi_2)\right]$$

$$\Delta_u = \varepsilon^2 k_2^2 \chi_1 sh(\chi_1)\,\chi_2^{-1} sh(\chi_2), \quad \Delta_V = -\frac{k_1}{\sqrt{3}}\lambda_1\lambda_2 sh(\lambda_1)\, sh(\lambda_2)$$

$$\Delta_w = \lambda_1 sh(\lambda_1)\, ch(\lambda_2) - \lambda_2 sh(\lambda_2)\, ch(\lambda_1).$$

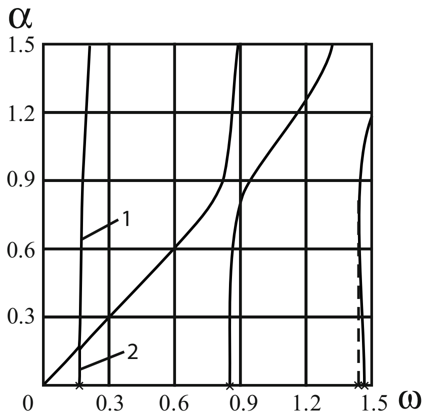

Fig. 2

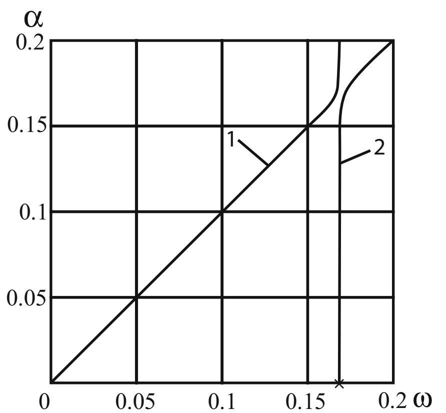

Fig. 3

In Fig. 2 the dispersion curves of the equation $\Delta = 0$ concerning dimensionless quantities α and ω are constructed with relative thickness of the shell $\varepsilon = 0.1$ and Poisson's ratio $\mu = 0.3$. For negative α the graph should symmetrically be reflected concerning an axis ω. In Fig. 3 the site of the graph Fig. 2 is figured in the enlarged aspect. We can see that curves 1 and 2 are not intersected. With a decrease of the parameter ε the dispersion curves are contracted to the origin of coordinates, along both axes. The values ω at which slope angle of a tangent to a dispersion curve is equal to 90 degrees are resonance frequencies [6].

Table 1

ε	ω				
0.01	0.017	0.091	0,225	0.419	0,670
0.1	0.168	0.852	1.444	1.463	1.948

In Table 1 the values of the first several resonance frequencies (in Figs. 2, 3 they are marked by dagger) with $\mu = 0.3$. Let's mark, that all frequencies which given in the table, except for $\omega = 0.444$ can be obtained from a solution of a problem about vibrations square frame if Young's modulus E to exchange on $E/\left(1-\mu^2\right)$.

In Fig. 4 the graph of amplitude values dimensionless maximum bending stresses $\sigma M = 6\tilde{a}^2 \tilde{M}_{\tilde{x}}\left(\tilde{x},\tilde{y}\right)\Big/\left(\tilde{P}\tilde{h}^2\right)$, $\left(\tilde{M}_{\tilde{x}}\left(\tilde{x},\tilde{y}\right) = -\tilde{D}\left(\partial^2\tilde{w}/\partial\tilde{x}^2 + \mu\partial^2\tilde{w}/\partial\tilde{y}^2\right)\right)$ at $\mu = 0.3$, $\varepsilon = 0.1$, $y = 0$, $\omega = 0.1$ (thus actual frequency $\tilde{\omega} \approx 52/\tilde{a}$ rad/sec) for a case of a concentrated force $\tilde{q}\left(\tilde{x},\tilde{y}\right) = \tilde{P}\delta\left(\tilde{x}\right)\delta\left(\tilde{y}\right)$ is given. Thus the values of plain stresses less then bending stresses, and greatest maximum bending stresses arise under a concentrated force (logarithmic singularity) and on an edge.

Let's mark, that in case of low-frequency vibrations the solution of a problem (1)–(4) practically coincides with a solution of a problem about vibrations of the

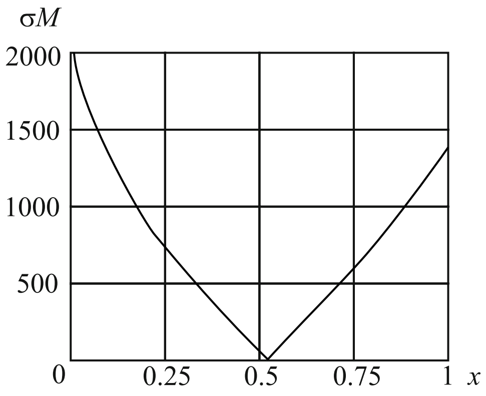

Fig. 4

clamped plate

$$D\Delta^2 w^* (x, y) - \omega^2 \varepsilon^{-2} w^* (x, y) = q (x, y) \quad (0 < x < 1, |y| < \infty) \tag{5}$$

$$\partial w^* / \partial x|_{x=0} = 0, \quad V_x^*|_{x=0} = 0, \quad \partial w^* / \partial x|_{x=1} = 0, \quad w^*|_{x=1} = 0. \tag{6}$$

The plain stresses and displacements thus can be neglected. Moreover if the solution of this problem is known at frequencies ω_0, ω_1 (in particular it is possible to take $\omega_0 = 0$, i.e., static case) approximate solution of a problem (1)–(4) present by the convenient formula for the engineering calculations

$$w_\omega (x, y) = L_0 (\omega) w_0^* (x, y) + L_1 (\omega) w_1^* (x, y) + O \left(\omega^4 \varepsilon^{-4}\right) \tag{7}$$

$$u_\omega = v_\omega = O \left(\varepsilon^2 w_\omega\right)$$

$$L_0 (\omega) = \left(\omega_1^2 - \omega^2\right) \left(\omega_1^2 - \omega_0^2\right)^{-1}, \quad L_1 (\omega) = \left(\omega^2 - \omega_0^2\right) \left(\omega_1^2 - \omega_0^2\right)^{-1}.$$

It is necessary to have in view, that this formula is valid for the small frequencies ($\omega/\varepsilon \ll 1$, i.e., $\tilde{\omega} \ll \tilde{h}\tilde{c} \big/ \tilde{a}^2$) smaller then first resonance frequency.

In Fig. 5 the graph of relative accuracies of maximum bending stresses on dimensionless frequency ω in the point of the edge $x = 1$, $y = 0$ is constructed, with $\omega_0 = 0$, $\omega_1 = 0.1$. The solid line shows an error of the formula (7), and dashed error for a problem (5)–(6). From the graph we can see that with $\omega < 0.12$ relative accuracy of the formula (7) less than 10%. The approximate solution of a problem (5)–(6) about the fastened plate gives good outcomes up to the first natural frequency.

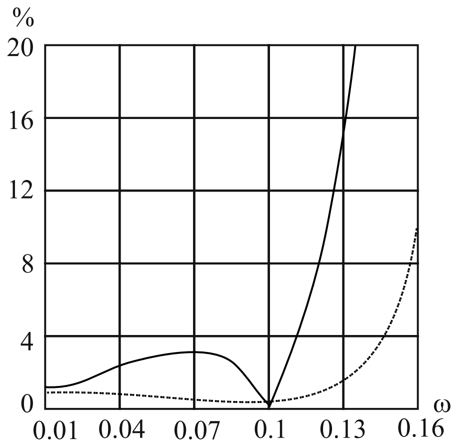

Fig. 5

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