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ON THE EXISTENCE OF AN INTEGRAL MANIFOLD OF A SPECIAL TYPE OF A SOME NONLINEAR DIFFERENTIAL SYSTEM

A nonlinear system of the differential equations is considered. This system is a linear extension of the nonlinear differential equation on a circle. The coefficients of the system belong to the class F, which are the functions that can be represented in the form of the absolutely and uniformly convergent Fourier series on the entire axis with slowly varying coefficients in a certain sense and frequencies. The basic properties of this functions' class are investigated. This system is expected as a close one to an inhomogeneous linear system of the differential equations with a diagonal matrix of the slowly varying coefficients and free terms, belonging to the class F. The transformations with the coefficients of class F, causing the system to a system in which the frequency is determined from the nonlinear equations with slowly varying coefficients, are constructed using the contraction mapping principle. As a result, the conditions, under which the original system has an integral manifold of class F, are considered.
MSC: 34A34, 34A25.
Key words: integral manifold, differential, slowly-varying, Fourier series.

Introduction. One of the powerful methods of the study of nonlinear systems of differential equations is the method of integral manifolds [1–3]. Particularly important role it plays in the research of multi-frequency oscillations, in particular, in systems containing the slowly varying parameters [4]. An important object of study in the same time and are single-frequency system [5]. This paper is a continuation of research initiated in paper [6–9] on the existence of the particular solutions and integral manifolds of the linear and quasilinear differential systems, are represented as an absolutely and unifirmly convergent Fourier-series with slowly verying in a certain sense coefficients.

Notation. Let

$$G(\varepsilon_0) = \{t, \varepsilon : \ t \in \mathbf{R}, \ \varepsilon \in [0, \varepsilon_0], \ \varepsilon_0 \in \mathbf{R}^+\}$$

Definition 1. We say, that a function $f(t, \varepsilon)$, in general a complex-valued, belong to the class $S_m(\varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$, if
1) $f : G(\varepsilon_0) \to \mathbf{C}$; 2) $f(t, \varepsilon) \in C^m(G(\varepsilon_0))$ with respect t;
3) $d^k f(t, \varepsilon)/dt^k = \varepsilon^k f_k^*(t, \varepsilon)$ $(0 \leqslant k \leqslant m)$,

$$\|f\|_m \stackrel{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |f_k^*(t, \varepsilon)| \ < +\infty$$

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Definition 2. We say, that a function $f(t,\varepsilon,\theta)$ belong to the class $F^{\theta}_{m,\infty}(\varepsilon_0)$ ($m \in \mathbf{N} \cup \{0\}$), if this function can be represented in form:

$$f(t,\varepsilon,\theta) = \sum_{n=-\infty}^{\infty} f_n(t,\varepsilon)e^{in\theta},$$

and
1) $f_n(t,\varepsilon) \in S_m(\varepsilon_0)$, $\theta \in \mathbf{R}$;
2)

$$\|f_0\|_m + \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} |n|^l \cdot \|f_n\|_m \ < \ +\infty \ (l = 0,1,2,...).$$

If the function $f(t,\varepsilon,\theta)$ is real, then $f_{-n}(t,\varepsilon) = \overline{f_n(t,\varepsilon)}$.
We denote

$$\|f\|_{m,\theta} = \sum_{n=-\infty}^{\infty} \|f_n\|_m.$$

For any $f \in F^{\theta}_{m,\infty}(\varepsilon_0)$ we introduce the linear operators:

$$\Gamma_n[f(t,\varepsilon,\theta)] = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)e^{-in\theta}d\theta, \ n \in \mathbf{Z},$$

in particular

$$\Gamma_0[f(t,\varepsilon,\theta)] = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)d\theta;$$

$$I[f(t,\varepsilon,\theta)] = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n[f(t,\varepsilon,\theta)]}{in}\ e^{in\theta}.$$

We note the some properties of the functions of the class $F^{\theta}_{m,\infty}(\varepsilon_0)$. Let $u(t,\varepsilon,\theta)$, $v(t,\varepsilon,\theta)$ belongs to class $F^{\theta}_{m,\infty}(\varepsilon_0)$, and $k = const$.
1) $u(t,\varepsilon,\theta)$ is 2π-periodic with respect θ.
2)

$$\frac{\partial^l u(t,\varepsilon,\theta)}{\partial\theta^l} \in F^{\theta}_{m,\infty}(\varepsilon_0) \ (l = 0,1,2,\ldots),$$

$$\frac{\partial^k u(t,\varepsilon,\theta)}{\partial t^k} \in F^{\theta}_{k-1,\infty}(\varepsilon_0) \ (k = \overline{1,m}).$$

3) $\Gamma_n[u(t,\varepsilon,\theta)] \in S_m(\varepsilon_0)$ $(n \in \mathbf{Z})$.
4)

$$I[u(t,\varepsilon,\theta)] \in F^{\theta}_{m,\infty}(\varepsilon_0), \ I\left[\frac{\partial u(t,\varepsilon,\theta)}{\partial\theta}\right] = u(t,\varepsilon,\theta) - \Gamma_0[u(t,\varepsilon,\theta)] \in F^{\theta}_{m,\infty}(\varepsilon_0).$$

5) $\|ku\|_{m,\theta} = |k| \cdot \|u\|_{m,\theta}$.
6) $\|u+v\|_{m,\theta} \leqslant \|u\|_{m,\theta} + \|v\|_{m,\theta}$.

7)

$$\|u\|_{m,\theta} = \sum_{k=0}^{m} \left\| \frac{1}{\varepsilon^k} \frac{\partial^k u}{\partial t^k} \right\|_{0,\theta}.$$

8)

$$\|uv\|_{m,\theta} \leqslant 2^m \|u\|_{m,\theta} \cdot \|v\|_{m,\theta}.$$

We prove the property 8). For $m = 0$ we have:

$$\|uv\|_{0,\theta} \leqslant \|u\|_{0,\theta} \cdot \|v\|_{0,\theta}.$$

By virtue of the property 7):

$$\|uv\|_{m,\theta} = \sum_{k=0}^{m} \left\| \frac{1}{\varepsilon^k} \frac{\partial^k (uv)}{\partial t^k} \right\|_{0,\theta} \leqslant \sum_{k=0}^{m} \frac{1}{\varepsilon^k} \sum_{j=0}^{k} C_k^j \left\| \frac{\partial^j u}{\partial t^j} \right\|_{0,\theta} \cdot \left\| \frac{\partial^{k-j} v}{\partial t^{k-j}} \right\|_{0,\theta} \leqslant$$

$$\leqslant 2^m \left[\sum_{j=0}^{m} \frac{1}{\varepsilon^j} \left\| \frac{\partial^j u}{\partial t^j} \right\|_{0,\theta} \right] \cdot \left[\sum_{j=0}^{m} \frac{1}{\varepsilon^j} \left\| \frac{\partial^j v}{\partial t^j} \right\|_{0,\theta} \right] = 2^m \|u\|_{m,\theta} \cdot \|v\|_{m,\theta}.$$

We can say that the space $F^{\theta}_{m,\infty}(\varepsilon_0)$ is the Banach algebra [10].

9) If u, v are real, then $u(t, \varepsilon, \theta + v(t, \varepsilon, \theta)) \in F^{\theta}_{m,\infty}(\varepsilon_0)$.

We prove the property 9). We denote: $w(t, \varepsilon, \theta) = u(t, \varepsilon, \theta + v(t, \varepsilon, \theta))$. Since $u, v \in F^{\theta}_{m,\infty}(\varepsilon_0)$, then

$$\sup_{t,\varepsilon \in G(\varepsilon_0)} \sup_{\theta \in \mathbf{R}} \left| \frac{\partial^{s+l} u(t, \varepsilon, \theta)}{\partial t^s \partial \theta^l} \right| < +\infty, \quad \sup_{t,\varepsilon \in G(\varepsilon_0)} \sup_{\theta \in \mathbf{R}} \left| \frac{\partial^{s+l} v(t, \varepsilon, \theta)}{\partial t^s \partial \theta^l} \right| < +\infty \qquad (1)$$

$$(s = \overline{0, m}; \ l = 0, 1, 2, \ldots).$$

Converting the expression

$$\Gamma_n[w(t, \varepsilon, \theta)] = \frac{1}{2\pi} \int_0^{2\pi} w(t, \varepsilon, \theta) e^{-in\theta} d\theta \quad (n \neq 0)$$

by the formula $(l+2)$-fold integration by the parts, we obtain:

$$\Gamma_n[w(t, \varepsilon, \theta)] = \frac{1}{2\pi (in)^{l+2}} \int_0^{2\pi} p(t, \varepsilon, \theta) e^{-in\theta} d\theta \quad (n \neq 0),$$

where $p(t, \varepsilon, \theta)$ is polynom of degree $l + 3$ with coefficients belong to the class $S_m(\varepsilon_0)$ relatively derivatives $\partial^r u(t, \varepsilon, \theta)/\partial \theta^r$ $(r = \overline{0, l+2})$, which are calculated by values of argument θ is equal $\theta + v(t, \varepsilon, \theta)$ and $\partial^r v(t, \varepsilon, \theta)/\partial \theta^r$ $(r = \overline{0, l+2})$. Given (1) we obtain, that $w(t, \varepsilon, \theta) \in F^{\theta}_{m,\infty}(\varepsilon_0)$.

10) The follows chains hold:

$$F^{\theta}_{0,\infty}(\varepsilon_0) \supset F^{\theta}_{1,\infty}(\varepsilon_0) \supset \ldots \supset F^{\theta}_{m,\infty}(\varepsilon_0), \quad S_0(\varepsilon_0) \supset S_1(\varepsilon_0) \supset \ldots \supset S_m(\varepsilon_0).$$

Definition 3. We say, that vector $f = \mathrm{colon}(f_1, ..., f_N)$ belong to the class $F^{\theta}_{m,\infty}(\varepsilon_0)$ (or $S_m(\varepsilon_0)$), if $f_j \in F^{\theta}_{m,\infty}(\varepsilon_0)$ (relatively $f_j \in S_m(\varepsilon_0)$) $(j = \overline{1, N})$.

Definition 4. We say, that matrix $(a_{jk})_{j,k=\overline{1,N}}$ belong to the class $F^{\theta}_{m,\infty}(\varepsilon_0)$ (or $S_m(\varepsilon_0)$), if $a_{j,k} \in F^{\theta}_{m,\infty}(\varepsilon_0)$ (relatively $a_{j,k} \in S_m(\varepsilon_0)$) $(j, k = \overline{1, N})$.

Consider the next system of the differential equations:

$$\begin{aligned} \tfrac{dx}{dt} &= (\Lambda(t,\varepsilon) + \mu P(t,\varepsilon,\theta))x + f(t,\varepsilon,\theta), \\ \tfrac{d\theta}{dt} &= \omega(t,\varepsilon) + \mu a(t,\varepsilon,\theta), \end{aligned} \tag{2}$$

where $(t,\varepsilon) \in G(\varepsilon_0)$, $x = \mathrm{col}(x_1, ..., x_N)$, $\Lambda = \mathrm{diag}(\lambda_1, ..., \lambda_N) \in S_m(\varepsilon_0)$, $P = (p_{jk})_{j,k=\overline{1,N}} \in F^{\theta}_{m,\infty}(\varepsilon_0)$, scalar real functions $\omega \in S_m(\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)} \omega > 0$, $a \in F^{\theta}_{m,\infty}(\varepsilon_0)$, $\mu \in (0, \mu_0) \subset \mathbf{R}^+$.

We study the problem determine of the conditions of existence of integral manifold $x(t,\varepsilon,\theta,\mu)$ of the system (2), belongs to the class $F^{\theta}_{m^*,\infty}(\varepsilon_0)$, where $m^* \leqslant m$.

Auxiliary arguments.

Lemma 1. *There exists $\mu_1 \in (0, \mu_0)$ such that for all $\mu \in (0, \mu_1)$ there exists the real reversible transformation*

$$\theta = \varphi + \mu v(t,\varepsilon,\varphi,\mu), \ v \in F^{\varphi}_{m,\infty}(\varepsilon_0), \tag{3}$$

such the system (2) reduces to the following form:

$$\begin{aligned} \tfrac{dx}{dt} &= (\Lambda(t,\varepsilon) + \mu Q(t,\varepsilon,\varphi,\mu))x + g(t,\varepsilon,\varphi,\mu), \\ \tfrac{d\varphi}{dt} &= \omega(t,\varepsilon) + \mu b(t,\varepsilon,\mu) + \mu\varepsilon\beta(t,\varepsilon,\varphi,\mu), \end{aligned} \tag{4}$$

where $Q = P(t,\varepsilon,\varphi + \mu v(t,\varepsilon,\varphi,\mu)) \in F^{\varphi}_{m,\infty}(\varepsilon_0)$, $g = f(t,\varepsilon,\varphi + \mu v(t,\varepsilon,\varphi,\mu)) \in F^{\varphi}_{m,\infty}(\varepsilon_0)$, $b \in S_m(\varepsilon_0)$, $\beta \in F^{\varphi}_{m-1,\infty}(\varepsilon_0)$.

Proof. We define the function v from equation:

$$\omega(t,\varepsilon)\frac{\partial v}{\partial\varphi} = a(t,\varepsilon,\varphi + \mu v) - b(t,\varepsilon,\mu) - \mu b(t,\varepsilon,\mu)\frac{\partial v}{\partial\varphi}, \tag{5}$$

where the function $b(t,\varepsilon,\mu)$ is to be determined.

We seek the solution $v \in F^{\varphi}_{m,\infty}(\varepsilon_0)$ of equation (5) and function $b \in S_m(\varepsilon_0)$ by the method of succesive approximations, defining the initial approximations v_0, b_0 from the equation:

$$\omega(t,\varepsilon)\frac{\partial v_0}{\partial\varphi} = a(t,\varepsilon,\varphi) - b_0(t,\varepsilon), \tag{6}$$

and the subsequent approximations v_s, b_s $(s = 1, 2, \ldots)$ defining from the equations:

$$\omega(t,\varepsilon)\frac{\partial v_s}{\partial\varphi} = a(t,\varepsilon,\varphi + \mu v_{s-1}) - \mu b_{s-1}(t,\varepsilon,\mu)\frac{\partial v_{s-1}}{\partial\varphi} - b_s(t,\varepsilon,\mu), \ s = 1, 2, \ldots . \tag{7}$$

We denote:

$$b_0(t,\varepsilon) = \Gamma_0[a(t,\varepsilon,\varphi)], \tag{8}$$

$$v_0(t,\varepsilon,\varphi) = \frac{1}{\omega(t,\varepsilon)}\ I[a(t,\varepsilon,\varphi)], \tag{9}$$

$$b_s(t,\varepsilon,\mu) = \Gamma_0\left[a(t,\varepsilon,\varphi+\mu v_{s-1}) - \mu b_{s-1}\frac{\partial v_{s-1}}{\partial\varphi}\right], \tag{10}$$

$$v_s(t,\varepsilon,\varphi,\mu) = \frac{1}{\omega(t,\varepsilon)}\ I\left[a(t,\varepsilon,\varphi+\mu v_{s-1}) - \mu b_{s-1}\frac{\partial v_{s-1}}{\partial\varphi}\right] =$$

$$= \frac{1}{\omega(t,\varepsilon)}\ I\left[a(t,\varepsilon,\varphi+\mu v_{s-1})\right] - \mu\frac{b_{s-1}(t,\varepsilon,\mu)}{\omega(t,\varepsilon)}\ v_{s-1}(t,\varepsilon,\varphi,\mu),\ s=1,2,\dots\ . \tag{11}$$

We show, that all the approximations $b_s(t,\varepsilon,\mu)$, defined by the formulas (8), (10), belongs to the class $S_m(\varepsilon_0)$, and all the approximations $v_s(t,\varepsilon,\varphi,\mu)$, defined by the formulas (9), (11), belongs to the class $F^{\varphi}_{m,\infty}(\varepsilon_0)$.

Obviously, that $b_0 \in S_m(\varepsilon_0)$, $v_0 \in F^{\varphi}_{m,\infty}(\varepsilon_0)$, $v_0 \in \mathbf{R}$, and $\Gamma_0[v_0] \equiv 0$. Then, using the property 9) of the functions of the class $F^{\varphi}_{m,\infty}(\varepsilon_0)$, we obtain, that $a(t,\varepsilon,\varphi+\mu v_0(t,\varepsilon,\varphi)) \in F^{\varphi}_{m,\infty}(\varepsilon_0)$, and $b_1(t,\varepsilon,\mu) \in S_m(\varepsilon_0)$, $v_1(t,\varepsilon,\varphi,\mu) \in F^{\varphi}_{m,\infty}(\varepsilon_0)$.

Suppose by the induction, that $b_s(t,\varepsilon,\mu) \in S_m(\varepsilon_0)$, $v_s(t,\varepsilon,\varphi,\mu) \in F^{\varphi}_{m,\infty}(\varepsilon_0)$ $(s=\overline{2,k})$ we, by virtue of the property 9), show, that then $b_{k+1}(t,\varepsilon,\mu) \in S_m(\varepsilon_0)$, $v_{k+1}(t,\varepsilon,\varphi,\mu) \in F^{\varphi}_{m,\infty}(\varepsilon_0)$.

We introduce the sets:

$$\Omega_1 = \{\ b \in S_m(\varepsilon_0):\ \|b\|_m \leqslant d\ \},\ \Omega_2 = \left\{\ v \in F^{\varphi}_{m,\infty}(\varepsilon_0):\ \|v\|_{m,\varphi} \leqslant d\ \right\},\ d > 0.$$

Using techniques contraction mapping principle [10] it is easy to show that for sufficiently small values μ fulfilled $b_s \in \Omega_1$, $v_s \in \Omega_2$ $(s = 0,1,2,\dots)$, and process (8)–(11) converges to the solution of class $F^{\varphi}_{m,\infty}(\varepsilon_0)$ of the equation (5), in which function $b \in S_m(\varepsilon_0)$.

Then for sufficiently small values μ the function β is determined from the equation:

$$\left(1+\mu\frac{\partial v(t,\varepsilon,\theta,\mu)}{\partial\theta}\right)\beta = -\frac{1}{\varepsilon}\frac{\partial v(t,\varepsilon,\theta,\mu)}{\partial t}.$$

Now we show, that the transformation (3) are reversible for sufficiently small values μ. Means for sufficiently small values μ there exists the reverse transformation

$$\varphi = \theta + q(t,\varepsilon,\theta,\mu), \tag{12}$$

where $q \in F^{\theta}_{m,\infty}(\varepsilon_0)$.

We substitute (12) in (3) and obtain the equation respectively q:

$$q = -\mu v(t,\varepsilon,\theta+q,\mu). \tag{13}$$

We seek the solution $q \in F^{\theta}_{m,\infty}(\varepsilon_0)$ of equation (13) by the method of succesive approximations, defining:

$$q_0 \equiv 0,\quad q_{s+1} = -\mu v(t,\varepsilon,\theta+q_s,\mu),\ s=0,1,2,\dots\ . \tag{14}$$

Using techniques contraction mapping principle [10] it is easy to show that for sufficiently small values μ the process (14) converges by the norm $\|\cdot\|_{m,\theta}$ to the solution of class $F^{\theta}_{m,\infty}(\varepsilon_0)$ of the equation (13).

Lemma 2. *There exists $\mu_2 \in (0, \mu_1)$ such that for all $\mu \in (0, \mu_2)$ there exists the chain of reversible transformations of kind:*

$$\varphi = \psi_1 + \mu\varepsilon w_1(t, \varepsilon, \psi_1, \mu), \tag{15}$$

$$\psi_1 = \psi_2 + \mu\varepsilon^2 w_2(t, \varepsilon, \psi_2, \mu), \tag{16}$$

$$\ldots$$

$$\psi_{m_1-2} = \psi_{m_1-1} + \mu\varepsilon^{m_1-1} w_{m_1-1}(t, \varepsilon, \psi_{m_1-1}, \mu), \tag{17}$$

where $m_1 < m$, $w_k \in F^{\psi_k}_{m-k,\infty}(\varepsilon_0)$ $(k = \overline{1, m_1 - 1})$, such that the system (4) reduces to the following form:

$$\begin{gathered} \tfrac{dx}{dt} = (\Lambda(t, \varepsilon) + \mu R_{m_1-1}(t, \varepsilon, \psi_{m_1-1}, \mu))x + h_{m_1-1}(t, \varepsilon, \psi_{m_1-1}, \mu), \\ \tfrac{d\psi_{m_1-1}}{dt} = \omega(t, \varepsilon) + \mu b(t, \varepsilon, \mu) + \mu \textstyle\sum_{l=1}^{m_1-1} \varepsilon^k \beta_k(t, \varepsilon, \mu) + \\ + \mu\varepsilon^{m_1} \widetilde{\beta}_{m_1-1}(t, \varepsilon, \psi_{m_1-1}, \mu), \end{gathered} \tag{18}$$

where $R_{m_1-1} \in F^{\psi_{m_1-1}}_{m-m_1+1}(\varepsilon_0)$, $h_{m_1-1} \in F^{\psi_{m_1-1}}_{m-m_1+1}(\varepsilon_0)$, $\beta_k \in S_{m-k}(\varepsilon_0)$, $\widetilde{\beta}_{m_1-1} \in F^{\psi_{m_1-1}}_{m-m_1,\infty}(\varepsilon_0)$ $(k = \overline{1, m_1 - 1})$.

Proof. Consider the transformation (15). The application of this is reduce the system (4) to the kind:

$$\begin{gathered} \tfrac{dx}{dt} = (\Lambda(t, \varepsilon) + \mu R_1(t, \varepsilon, \psi_1, \mu))x + h_1(t, \varepsilon, \psi_1, \mu), \\ \tfrac{d\psi_1}{dt} = \omega(t, \varepsilon) + \mu b(t, \varepsilon, \mu) + \mu\varepsilon\beta_1(t, \varepsilon, \mu) + \mu\varepsilon^2 \widetilde{\beta}_1(t, \varepsilon, \psi_1, \mu), \end{gathered} \tag{19}$$

where $\beta_1, \widetilde{\beta}_1$ must be defined.

We define the functions w_1, β_1 from the equation:

$$(\omega(t, \varepsilon) + \mu b(t, \varepsilon, \mu)) \frac{\partial w_1}{\partial \psi_1} + \beta_1(t, \varepsilon, \mu) = \beta(t, \varepsilon, \psi_1 + \mu\varepsilon w_1, \mu) - \mu\varepsilon\beta_1(t, \varepsilon, \mu) \frac{\partial w_1}{\partial \psi_1}. \tag{20}$$

Then the function $\widetilde{\beta}_1$ are defined from the equation:

$$\left(1 + \mu\varepsilon \frac{\partial w_1}{\partial \psi_1}\right) \widetilde{\beta}_1 = -\frac{1}{\varepsilon} \frac{\partial w_1}{\partial t}. \tag{21}$$

Similarly to the proof of Lemma 1 we show that for sufficiently small values μ the equation (20) has a solution $w_1(t, \varepsilon, \psi_1, \mu) \in F^{\psi_1}_{m-1,\infty}(\varepsilon_0)$. In this $\beta_1 \in S_{m-1}(\varepsilon_0)$. And for sufficiently small values μ the equation (21) has a solution $\widetilde{\beta}_1 \in F^{\psi_1}_{m-2,\infty}(\varepsilon_0)$.

Similarly are defined the functions $w_2, \ldots, w_{m_1-1}$, $\beta_2, \ldots, \beta_{m_1-1}$, $\widetilde{\beta}_2, \ldots, \widetilde{\beta}_{m_1-1}$.

Main Results

Theorem. *Let the elements $\lambda_j(t, \varepsilon)$ $(j = \overline{1, N})$ of matrix $\Lambda(t, \varepsilon)$ in system (2) are such that $\inf\limits_{G(\varepsilon_0)} |\mathrm{Re}\lambda_j(t, \varepsilon)| \geqslant \gamma > 0$ $(j = \overline{1, N})$. Then there exists $\mu^* \in (0, \mu_0)$ such that for all $\mu \in (0, \mu^*)$ the system (2) has the integral manifold $\widetilde{x}(t, \varepsilon, \theta, \mu) \in F^{\theta}_{m_1,\infty}(\varepsilon_0)$, where $2m_1 \leqslant m$ $(m_1 \in \mathbf{N} \cup \{0\})$.*

Proof. Based on Lemmas 1,2, we reduce the system (2) to the kind (18). We denote:

$$\psi = \psi_{m_1-1},$$

$$R(t,\varepsilon,\psi,\mu) = R_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu),\ h(t,\varepsilon,\psi,\mu) = h_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu),$$

$$\omega_1(t,\varepsilon,\mu) = \omega(t,\varepsilon_0) + \mu b(t,\varepsilon,\mu) + \sum_{l=1}^{m_1-1} \varepsilon_k \beta_k(t,\varepsilon,\mu) + \varepsilon^{m_1} \widetilde{\beta}_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu).$$

Using condition of Theorem and property 10) of the functions of class $F^{\theta}_{m,\infty}(\varepsilon_0)$, we can state, that $R(t,\varepsilon,\psi,\mu)$, $h(t,\varepsilon,\psi,\mu) \in F^{\psi}_{m_1,\infty}(\varepsilon_0)$, $\omega_1(t,\varepsilon,\mu) \in S_{m_1}(\varepsilon_0)$. Then we write the system (18) in form:

$$\begin{aligned} \tfrac{dx}{dt} &= (\Lambda(t,\varepsilon) + \mu R(t,\varepsilon,\psi,\mu))x + h(t,\varepsilon,\psi,\mu), \\ \tfrac{d\psi}{dt} &= \omega_1(t,\varepsilon,\mu). \end{aligned} \tag{22}$$

With the system (22) we consider the system:

$$\begin{aligned} \tfrac{dx_0}{dt} &= \Lambda(t,\varepsilon)x_0 + h(t,\varepsilon,\psi,\mu), \\ \tfrac{d\psi}{dt} &= \omega_1(t,\varepsilon,\mu). \end{aligned} \tag{23}$$

By results [6] and condition of Theorem, we can state, that the system (23) has the integral manifold $x_0(t,\varepsilon,\psi,\mu) \in F^{\psi}_{m_1,\infty}(\varepsilon_0)$. And there exists $K \in (0,+\infty)$ such that

$$\|x_0\|_{m_1,\psi} \leqslant K\|h\|_{m_1,\psi}. \tag{24}$$

We seek the integral manifold of system (22) by the method of succesive approximations, defining as an initial approximation x_0, and the subsequents approximations defining from the systems:

$$\begin{aligned} \tfrac{dx_{s+1}}{dt} &= \Lambda(t,\varepsilon)x_{s+1} + h(t,\varepsilon,\psi,\mu) + \mu R(t,\varepsilon,\psi,\mu)x_s,, \\ \tfrac{d\psi}{dt} &= \omega_1(t,\varepsilon,\mu),\ s = 0,1,2,\ldots\ . \end{aligned} \tag{25}$$

Taking in to account the unequality (24) and using the ordinary technicue of the contraction mapping principle [10], it is easy to show, that there exists $\mu_3 \in (0,\mu_0)$ such that for all $\mu \in (0,\mu_3)$ all approximations x_s belong to class $F^{\psi}_{m_1,\infty}(\varepsilon_0)$, and process (25) converges by the norm $\|\cdot\|_{m,\psi}$ to integral manifold $x(t,\varepsilon,\psi,\mu) \in F^{\psi}_{m_1,\infty}(\varepsilon_0)$ of the system (22).

By virtue of the revesibility of the transformations (3), (15) – (17), we can state the existence for sufficiently small values μ of the integral manifold $\widetilde{x}(t,\varepsilon,\theta,\mu) \in F^{\theta}_{m,\infty}(\varepsilon_0)$ of the system (2).

Conclusion. Thus, for the system (2) the existence of the integral manifold of the class $F^{\theta}_{m_1,\infty}(\varepsilon_0)$ $(2m_1 \leqslant m)$ we prove for sufficiently small values of μ.

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Про існування інтегрального многовиду спеціального вигляду однієї нелінійної диференціальної системи

Резюме

Для нелінійної диференціальної системи, коефіцієнти якої представлені у вигляді абсолютно та рівномірно збіжних рядів Фур'є з повільно змінними коефіцієнтами та частотою, отримано умови існування інтегрального многовиду аналогічної структури.
Ключові слова: диференціальний, повільно змінний, ряди Фур'е.

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О существовании интегрального многообразия специального вида одной нелинейной дифференциальной системы

Резюме

Для нелинейной дифференциальной системы, коэффициенты которой представимы в виде абсолютно и равномерно сходящихся рядов Фурье с медленно меняющимися коэффициентами и частотой, получены условия существования интегрального многообразия аналогичной структуры.
Ключевые слова: дифференциальный, медленно меняющийся, ряды Фурье.

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