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ON EXPONENTIAL SUMS INVOLVING THE DIVISOR FUNCTION OVER $\mathbb{Z}[i]$

We apply the van der Corput transform to investigate the sums of view $\sum r(n)g(n)e(f(n)))$, where $r(n)$ is the number of representations of n as the sum of two squares of integer numbers. Such sums have been studied by M. Jutila, O. Gunyavy, M. Huxley and etc. Depending of differential properties of the functions $g(n)$ and $f(n)$ there have been obtained the different kinds of error terms in bounds of the considered sums. In the special case, O. Gunyavy improved the result of M. Jutila in the problem on estimate the exponential sum involving the divisor function $\tau(n)$. We obtain the asymptotic formula of the sum $\sum \tau(\alpha)e\left(\frac{a}{q}N(\alpha)\right)$ over the ring of Gaussian integers which is an analogue of the asymptotic formulas obtained by M. Jutila and O. Gunyavy.
MSC: 11K45.
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Introduction. In 1985 M. Jutila [4], [5] constructed an asymptotic formula for the divisor function $\tau(n)$ weighted by trigonometric unit

$$T(x;a,q)=\sum_{n\leqslant x}\tau(n)e^{2\pi i\frac{an}{q}}=\frac{x}{q}\left(\ln\frac{x}{q^2}+2\gamma-1\right)+R(x),$$

where $R(x)=O\left(x^{\frac{1}{3}+\varepsilon}q^{\frac{2}{3}}\right)$.

This formula is a nontrivial for $q\ll x^{\frac{2}{5}-\varepsilon}$, moreover, a constant in the symbol "O" doesn't depend at b, q, x.

Hereafter O. Gunyavy [1] improved an error term $R(x)=O\left(x^{\frac{1}{2}+\varepsilon}\right)$ and hence carry over a region of nontriviality of the formula for $T(x;a,q)$.

In the works [4], [1] the main method of investigation is founded on the van der Corput transform

$$\sum_{N\leqslant n\leqslant N'}^{\quad *} g(n)e^{2\pi i f(n)}=\sum_{f'(N)\leqslant n\leqslant f'(N')}\frac{1}{\sqrt{|f''(\varphi(n))|}}e^{2\pi i(f(\varphi(n))-n\varphi(n)+\frac{1}{8})}+R,$$

where f and g are real-valued three times continuously differentiable on the interval $[N,N']$, and $\varphi(n)$ is unique solution to $4x^2f'(x)=n$ in the interval $[N,N']$. A starred sum indicates that if a limit of summation is an integer, the corresponding summand is multiplied by $\frac{1}{2}$.

The main goal of this paper derive an analogue of the Gunyavy theorem for the weighted divisor function by trigonometric unit over the ring of Gaussian integers.

We prove the following theorems

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Theorem 1. *Let $1 \leqslant a \leqslant q$, $(a, q) = 1$, $q \ll x^{\frac{1}{2}-\varepsilon}$, and let $r(n)$ be a number of representations of n as sum of two square integers. Then*

$$A\left(x, \frac{a}{q}\right) := \sum_{n \leqslant x} r(n) e^{2\pi i \frac{an}{q}} = qA\left(\frac{x}{q^2}\right) + O\left(x^{\frac{1}{2}+\varepsilon}\right) = \frac{\pi x}{q} + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Theorem 2. *Let α_0, β be the gaussian integers, $(\alpha_0, \beta) = 1$, and $\tau(\alpha)$ be a divisor function over the ring of Gaussian numbers. Then for $N(\beta) \ll x^{\frac{1}{4}-\varepsilon}$ the following asymptotic formula*

$$\sum_{N(\alpha) \leqslant x} \tau(\alpha) e^{2\pi i N\left(\frac{\alpha_0 \alpha}{\beta}\right)} = C_1(\beta) \frac{x \log x}{N(\beta)} + C_2(\beta) \frac{x}{N(\beta)} + O\left(x^{\frac{3}{4}+\varepsilon}\right) + O\left(x^{\frac{1}{2}+\varepsilon} N(\beta)\right)$$

holds, where $C_i(\beta)$ be the computable constants, $N(\beta)^{-\varepsilon} \ll C_i(\beta) \ll N(\beta)^{\varepsilon}$, $i = 1, 2$.

Notation. We will frequently use the Landau and Vinogradov asymptotic notations. The big "O" notation $f(x) = O(g(x))$ (equivalently, $f(x) \ll g(x)$) means that there exists some constant C such that $|f(x)| \leqslant C|g(x)|$ on the domain in question. By $f(x) \asymp g(x)$, we shall mean that $g(x) \ll f(x) \ll g(x)$. A symbol $k \sim B$ under the sign of $\sum$ denotes that a summation variable k runs all integers from $[B, B']$, $B < B' \leqslant 2B$. We denote $e^{2\pi x}$ as $e(x)$.

Auxiliary arguments. In order to prove the main results we need the following preliminary lemmas.

Lemma 1 (Generalized van der Corput transform, see [3], Lemma 5.5.3)**.** *Suppose that $f(x)$ is real and four times continuously differentiable on $[a, b]$. Suppose that there are positive numbers M and T, with $M > b - a$, such that, for $x \in [a, b]$, we have*

$$f''(x) \asymp \frac{T}{M^2}, \; F^{(3)}(x) \ll \frac{T}{M^3}, \text{ and } f^{(4)}(x) \ll \frac{T}{M^4}.$$

Let $g(x)$ be a real function of bounded variation V on closed interval $[a, b]$. Then

$$\sum_{a \leqslant n \leqslant b} g(n) e(f(n)) = \sum_{f''(a) \leqslant n \leqslant f'(b)} \frac{g(\varphi(n)) e\left(f(\varphi(n)) - n\varphi(n) + \frac{1}{8}\right)}{\sqrt{f''(\varphi(n))}} +$$

$$+O\left((V + |g(a)|)\left(\frac{M}{\sqrt{T}} + \log\left(f'(b) - f'(a) + 2\right)\right)\right),$$

where $\varphi(n)$ is the unique solution in $[a, b]$ to $f'(x) = n$.

The implicit constants in the big O-term depends on the implicit constants in the relations between T, M and the derivatives of $f(x)$.

Lemma 2. *Consider a function $f : [N, N'] \to \mathbb{R}$ that is C^4, and a function $g : [N, N'] \to \mathbb{R}$ that is C^3 with the positive real numbers T, M, C, C_2, C_3, C_4, D such that*

$$\frac{CT}{M^2} \leqslant f''(x) \leqslant \frac{C_2 T}{M^3}, \; f^{(j)}(x) \leqslant C_j \frac{T}{M^j}, \; j = 3, 4;$$

$$f'(x_0) = 0 \text{ for some } x_0 \text{ in } [N, N'],\ x_0 - N \leqslant M;$$

$$0 < g^{(j)}(x) \ll \frac{U}{K^j},\ j = 0, 1, 2.$$

Then

$$\int_N^{N'} g(x)e(f(x))dx = g(x_0)\frac{e\left(f(x_0)+\frac{1}{8}\right)}{\sqrt{f''(x_0)}} +$$

$$+O\left(\frac{UM^4}{T^2(\min(x_0-N, N'-x_0))^3}\right) + O\left(\frac{UM}{T^{\frac{3}{2}}}\left(1+\frac{M}{K}\right)^2\right).$$

(For proof, see [7], Lemma 5.4).

Unfortunately, for many interesting cases, the above error is insufficient. Thus, it's often impose additional constraint of the function $g(n)$ and its derivatives.

The van der Corput transform has been studied in much more general problems for construction of asymptotic formulas for the sums of values of arithmetical functions weighted by a trigonometric units. For example, M. Jutila [4] [5] investigated sums of the form $\sum b(n)g(n)e(f(n))$ for certain multiplicative function $b(n)$ (see, also Gunyavy [1] and M. Huxley [3], Ch. 20).

The following lemmas of van der Corput are well-known (see, [7], Lemmas 5.6 and 5.7).

Lemma 3 (First derivative test)**.** *Let $f(x)$ be real and differentiable on the open interval (α, β) with $f'(x)$ monotone and $f'(x) \geqslant x > 0$ on $[\alpha, \beta]$. Let $g(x)$ be real, and let V be the total variation of $g(x)$ on the closed interval $[\alpha, \beta]$ plus maximal modules of $g(x)$ on $[\alpha, \beta]$. Then*

$$\left|\int_\alpha^\beta g(x)e(f(x))dx\right| \leqslant \frac{V}{\pi x}.$$

Lemma 4 (Second derivative test)**.** *Let $f(x)$ be real and twice differentiable on the open interval (α, β) with $f''(x) \geqslant \lambda > 0$ on (α, β). Let $g(x)$ be real, and let V be the total variation of $g(x)$ on the closed interval $[\alpha, \beta]$ plus maximum modules of $g(x)$ on $[\alpha, \beta]$. Then*

$$\left|\int_\alpha^\beta g(x)e(f(x))dx\right| \leqslant \frac{4V}{\sqrt{\pi\lambda}}.$$

Remark. *We can bound V in Lemmas 3 and 4 by*

$$|g(\alpha)| + |g(\beta)| + \int_\alpha^\beta |g'(y)|dy \ll U + 2M \cdot \frac{U}{M} \ll U,$$

using condition on derivatives of f and g.

Lemma 5 ([7], Lemma 5.9)**.** *Let $f \in C^3([\alpha, \beta])$ and $g \in C^2([\alpha, \beta])$, and define $h_m(x)$ by*

$$h_m(x) := \frac{(f'(x) - m)g'(x) - g(x)f''(x)}{(f'(x) - m)^3}.$$

Suppose that $f'(x) \neq m$ on an interval $[\alpha, \beta]$, and let

$$K_m(\alpha, \beta) := \sum |h_m(x)|,$$

where the sum ranges over all $x \in [\alpha, \beta]$, where $h'_m(x) = 0$.

Then we have

$$\int_\alpha^\beta g(x)e(f(x) - mx)dx = \left[\tfrac{g(x)}{2\pi i(f'(x)-m)}e(f(x) - mx)\right]_\alpha^\beta + \\ +O(K_m(\alpha, \beta)) + O(|h_m(\alpha)| + |h_m(\beta)|).$$

We always propose that the functions $f(x)$, $g(x)$ are quadruply (and, respectively, three times) continuously differentiable and satisfy the specified requirements on $[N, N']$, $N \leqslant N' \leqslant 2N$, such that

$$f^{(j)}(x) \asymp \frac{T}{M^j}, \ j = 2, 3, 4;$$
$$g^{(j)}(x) \asymp \frac{U}{M^j}, \ j = 0, 1, 2,$$

where $0 < T$, $M \ll N$ and

$$g(x) \ll g(N), \ \frac{1}{N} \ll |f'(N)| \ll |f'(x)| \ll |f'(N)|,$$
$$|f'(N)| \ll |f'(x) + 2xf''(x)| \ll f'(N), \ f^{(3)} \ll \frac{|f'(N)|}{N^2}.$$

These bounds on the derivatives of f and g allows us to use estimates on stationary phase integrals and, moreover, guarantee the uniqueness of the solution of equation $n = f'(x)$ or $n = 2xf'^2(x)$ if solutions of these equations there exist.

The following theorem you can consider as the special case of Lemma of van der Corput.

Theorem 3. *Let us the functions $f(x)$ and $g(x)$ satisfy the conditions above, and let $r(n)$ be the number of the representation of n by form $n = u^2 + v^2$, $u, v \in \mathbb{Z}$. Then we have*

$$\sum_{n\sim N} r(n)g(n)e^{2\pi i f(n)} =$$

$$= \sum_{n\sim N(f'(N))^2} r(n)g(\varphi(n))\sqrt{|\varphi'(n)|}e^{2\pi i(f(\varphi(n))-2\sqrt{n\varphi(n)})} +$$

$$+O\left(N^\varepsilon \left|\tfrac{g(N)}{f'(N)}\right|\right) + O(N^\varepsilon |g(N)|) +$$

$$+O\left(|g(N)| N^{\frac{1}{2}+\varepsilon} \min\left(\sqrt{|f'(N)|}, \ \tfrac{1}{\sqrt{|f'(N)|}}\right)\right).$$

where

$$\omega_0(f) = \begin{cases} -1 & if \quad f'(N)(f'(n) + 2Nf''(N)) > 0, \\ i & if \quad f'(N) > 0, \ f'(N) + 2Nf''(N) < 0, \\ -i & if \quad f'(N) < 0, \ f'(N) + 2Nf''(N) > 0, \end{cases}$$

$\varepsilon > 0$ is an arbitrary small, and constatnts in symbols "O" depend only on ε.

Proof. It is well known that for $x > 1$ we have the representation

$$\sum_{n \leqslant x} r(n) = \pi x + \sqrt{x} \sum_{n=1}^{\infty} \frac{r(n)}{\sqrt{n}} \Im_1(2\pi\sqrt{nx}), \tag{1}$$

where $\Im_1(z)$ is the Bessel function of first kind and order one.

For the Bessel function $\Im_\nu(z)$, $\nu = 0, 1, \dots$ there exist the asymptotic expanding

$$\Im_\nu(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{2\pi\nu}{4} - \frac{\pi}{4}\right) + O\left(|z|^{-\frac{3}{2}}\right), \text{ for } z \to \infty, \tag{2}$$

moreover, for $\nu \geqslant 1$,

$$\frac{d}{dz}\left(z^{\frac{\nu}{2}} \Im_\nu(2\sqrt{z})\right) = z^{\frac{\nu-1}{2}} \Im_{\nu-1}(2\sqrt{z})$$

The asymptotical series (1) is boundedly convergent and may be differentiate.Thus, by Abelian summation for any $X > 1$, we obtain

$$\begin{gathered} \sum_{N \leqslant n \leqslant N' \leqslant 2N} r(n) g(n) e(f(n)) = G(x) R_X(x) \Big|_N^{N'} + \\ + \sum_{n \leqslant X} r(n) \int\limits_N^{N'} G(x) \Im_0(2\pi\sqrt{nx}) dx + \pi \int\limits_N^{N'} G(x) dx - \int\limits_N^{N'} G'(x) R_X(x) dx, \end{gathered} \tag{3}$$

where

$$G(x) = g(x) e(f(x)), \;\; R_X(x) = \sum_{n > X} \sqrt{\frac{x}{n}} r(n) \Im_1(2\pi\sqrt{nx}).$$

First we consider the case $N^{-1} \ll f'(N) \ll 1$. We take up of every summand in right side of (2) in separately. We put $X_0 = 4N(f'(N))^2$, $X = X_0 + \sqrt{X_0 f'(N)}$.

Since $R_X(x) \ll x^\varepsilon \left(1 + \sqrt{\frac{x}{X}}\right)$, $\varepsilon > 0$, and $g'(x)$ is monotoneness on $[N, N']$, we have

$$\begin{gathered} G(x) R_X(x) \Big|_N^{N'} \ll N^\varepsilon g(N) \left(1 + \frac{1}{f'(N)}\right) \ll N^\varepsilon \frac{g(N)}{f'(N)}, \\ \int\limits_N^{N'} g'(x) e(f(x)) R_X(x) dx \ll N^\varepsilon \int\limits_N^{N'} |g'(x)| \frac{1}{f'(N)} dx \ll N^\varepsilon \frac{g(N)}{f'(N)}. \end{gathered} \tag{4}$$

$$\int\limits_N^{N'} g(x) f'(x) e(f(x)) R_X(x) dx \ll N^\varepsilon \left(1 + \sqrt{\frac{N}{X}}\right) \cdot |f'(N)| \cdot g(N) \frac{1}{f'(N)} \ll N^\varepsilon \frac{g(N)}{f'(N)},$$

$$\int\limits_N^{N'} g(x) e(f(x)) dx \ll \frac{g(N)}{f'(N)}. \tag{5}$$

The bounds (5) follow by "*First derivative test*".

Next, we denote

$$A_0(N,x) := \pi \sum_{n\leqslant X} r(n)g(x)e(f(x))\mathfrak{I}_0(2\pi\sqrt{nx}), \tag{6}$$

$$A_1(N,x) := 2\pi i \sum_{n>X} \frac{r(n)}{\sqrt{n}} g(x)f'(x)\sqrt{x}\mathfrak{I}_1(2\pi\sqrt{nx})e(f(x)). \tag{7}$$

Now we employ the asymptotic expanding of Bessel function for $x > 1$

$$\mathfrak{I}_0(2\pi\sqrt{nx}) = \frac{cos\left(2\pi\sqrt{nx} - \frac{\pi}{4}\right)}{\pi(nx)^{\frac{1}{4}}}\left(1 + O\left(\frac{1}{(nx)^{\frac{1}{2}}}\right)\right),$$

$$\mathfrak{I}_1(2\pi\sqrt{nx}) = -\frac{cos\left(2\pi\sqrt{nx} + \frac{\pi}{4}\right)}{\pi(nx)^{\frac{1}{4}}}\left(1 + O\left(\frac{1}{(nx)^{\frac{1}{2}}}\right)\right).$$

So, we may write

$$B_0(N) := \int_N^{N'} A_0(N,x)dx = \pi \sum_{n\leqslant X} r(n) \int_N^{N'} g(x)e(f(x))\mathfrak{I}_0(2\pi\sqrt{nx})dx = \tag{8}$$

$$= \frac{e^{\frac{\pi i}{4}}}{2}\pi \sum_{n\leqslant X} \frac{r(n)}{n^{\frac{1}{4}}} \int_N^{N'} \frac{g(x)}{x^{\frac{1}{4}}} e(f(x) - \sqrt{nx})dx +$$

$$+ \frac{e^{\frac{\pi i}{4}}}{2} \sum_{n\leqslant X} \frac{r(n)}{n^{\frac{1}{4}}} \int_N^{N'} \frac{g(x)}{x^{\frac{1}{4}}} e(f(x) + \sqrt{nx})dx := I_{01} + I_{02},$$

say.

Without loss of generality, we can suppose that $f'(x) > 0$. Consider the integral

$$\int_N^{N'} g(x)x^{-\frac{1}{4}} e(f(x) - \sqrt{nx})dx,$$

and notice that the function $\frac{d}{dx}(f(x) - \sqrt{nx})$ goes to zero in the point x_0, where x_0 is a solution of the equation $4f'^2(x) = n$. We denote $x_0 = \varphi(n)$ and call a stationary phase point for $f_n(x) = f(x) - \sqrt{nx}$. On the interval $[N, N']$ a stationary phase points of $f_n(x)$ exist only if $n \in [X_0 - \sqrt{X_0 f'(N)}, X_0 + \sqrt{X_0 f'(N)}]$.

The functions $\widetilde{f}_n(x) = f(x) + \sqrt{nx}$ for all n and also those $f_n(x)$, for which $n \notin [X_0 - \sqrt{X_0 f'(N)}, X_0 + \sqrt{X_0 f'(N)}]$ have not stationary phase points on $[N, N']$.

Hence, by "*First derivative test*", we have

$$I_{02} \ll N^{\varepsilon} X^{\frac{3}{4}} g(N) \cdot \frac{X^{\frac{1}{4}}}{N^{\frac{1}{2}}} \ll N^{\frac{1}{2}+\varepsilon}\sqrt{f'(N)}g(N). \tag{9}$$

For estimate of $B_0(N)$ remained to calculate two sums

$$\sum\nolimits_1 := \sum_{n\sim X_0} \frac{r(n)}{n^{\frac{1}{4}}} \int_N^{N'} \frac{g(x)}{x^{\frac{1}{4}}} e(f(x) - \sqrt{nx})dx,$$

where the sign $n \sim X_0$ denotes that $n \in [X_0 - \sqrt{X_0 f'(N)}, X_0 + \sqrt{X_0 f'(N)}]$, and

$$\sum\nolimits_2 := \sum_{n < X_0 - \sqrt{X_0 f'(N)}} \frac{r(n)}{n^{\frac{1}{4}}} \int\limits_N^{N'} \frac{g(x)}{x^{\frac{1}{4}}} e(f(x) - \sqrt{nx}) dx.$$

For the sum $\sum_2$ the "*Second derivative test*" gives

$$\int\limits_N^{N'} \frac{g(x)}{x^{\frac{1}{4}}} e(f(x)) dx \ll \frac{g(N)}{N^{\frac{1}{4}}} \cdot N^{\varepsilon} \left(f_n''(N)\right)^{-\frac{1}{4}}. \tag{10}$$

We make the some auxiliary calculations

$$f_n''(x) = f''(x) + \frac{1}{4} \frac{\sqrt{n}}{x^{\frac{3}{4}}} = \frac{1}{4x} \left(4x f''(x) + \sqrt{\frac{n}{x}}\right),$$

and hence, for $n \sim X_0$, we have

$$\begin{cases} f_n''(x) \asymp \frac{1}{N}(f'(N) + 4N f''(N)) \asymp \frac{f'(N)}{N} > 0, \\ f_n^{(3)}(x) = f^{(3)}(x) - \frac{3}{8} \frac{\sqrt{n}}{x^{\frac{5}{2}}} \ll \frac{f'(N)}{N^2}. \end{cases} \tag{11}$$

Next, by the relation $x = 4\varphi(x) f'^2(\varphi(x))$, we infer

$$1 = 4(\varphi'(x) f'^2(\varphi(x)) + 2\varphi(x) f'(\varphi(x)) f''(\varphi(x)) \varphi'(x)),$$

$$\varphi'(x) = \frac{1}{4f'(\varphi(x))(f'(\varphi(x)) + 2\varphi(x) f''(\varphi(x)))} \asymp \frac{1}{f'^2(N)}.$$

At last, for $n \sim 4N f'^2(N)$, we have

$$\varphi(n) - \varphi(X_0) \asymp \varphi'(n)(n - X_0) \asymp \frac{n - X_0}{f'^2(N)}.$$

Thus,

$$\begin{aligned} &\varphi(n) - N = \varphi(n) - \varphi(X_0) \ll \frac{1}{\sqrt{f_n''(N)}} \\ &\Leftrightarrow \frac{n, X_0}{f'^2(N)} \ll \frac{\sqrt{N}}{\sqrt{f'(N)}} \Leftrightarrow \\ &\Leftrightarrow n - X_0 \ll \sqrt{N} (f'(N))^{\frac{3}{2}} \Leftrightarrow \\ &\Leftrightarrow n - X_0 \ll \sqrt{X_0 f'(N)}. \end{aligned} \tag{12}$$

So, from (10), we obtain

$$\sum\nolimits_2 \ll \frac{g(N)}{N^{\frac{1}{4}} f'(N)} + g(N)\sqrt{N} \cdot \sqrt{f'(N)}.$$

To the integral in $\sum_1$ we apply Lemma 2. Then by (12), we have

$$\int_N^{N'} \frac{g(x)}{x^{\frac14}} e(f_n(x))dx = \omega(f)\frac{g(\varphi(n))e(f_n(\varphi(n)))}{|\varphi(n)|^{\frac14}\sqrt{|f_n''(\varphi(n))|}} + \\ + O\left(\frac{g(N)}{N^{\frac14}(\varphi(n)-N)f_n''(N)}\right) + O\left(\frac{g(N)}{N^{\frac14}Nf_n''(N)}\right), \tag{13}$$

where

$$\omega(f) = \begin{cases} e^{\frac{\pi i}{4}} & if \quad f_n''(\varphi(n)) > 0, \\ e^{-\frac{\pi i}{4}} & if \quad f_n''(\varphi(n)) < 0. \end{cases}$$

Moreover,

$$\begin{cases} \frac{g(N)}{N^{\frac14}(\varphi(n)-N)f''(N)} \ll \frac{g(N)N^{\frac12}f'(N)}{X_0-n}, \\ \frac{g(N)}{N^{\frac54}f_n''(N)} \ll \frac{g(N)}{N^{\frac14}f'(N)}. \end{cases} \tag{14}$$

From the definition $\varphi(n)$ and $f_n(x)$, we have

$$n^{\frac14}(\varphi(n))^{\frac14}\sqrt{|f_n''(\varphi(n))|} = \frac{n^{\frac14}(\varphi(n))^{\frac14}}{\sqrt{\varphi(n)}}|f'(\varphi(n)) + 2\varphi(n)f''(\varphi(n))|^{\frac12} = \\ = \sqrt{f'(\varphi(n))}|f'(\varphi(n)) + 2\varphi(n)f''(\varphi(n))|^{\frac12} = |\varphi'(n)|^{-\frac12}. \tag{15}$$

Hence, we obtain

$$B_0(N) = \frac{\pi}{2}\omega_0(f)\sum_{n\sim 4Nf'^2(N)} r(n)g(\varphi(n))\sqrt{|\varphi'(n)|}e(f_n(\varphi(n))) + \\ + O\left(N^{\varepsilon}\frac{g(N)}{|f'(N)|}\right) + \\ + O\left(N^{\varepsilon}g(N)\right) + \\ + O\left(g(N)N^{\frac12+\varepsilon}\min\left(\sqrt{|f'(N)|}, \frac{1}{|f'(N)|}\right)\right), \tag{16}$$

where

$$\omega_0(f) = \begin{cases} e^{\frac{\pi i}{4}}\omega(f) & if \quad f'(N) + 2Nf''(N) > 0, \\ e^{-\frac{\pi i}{4}}\omega(f) & if \quad f'(N) + 2Nf''(N) < 0. \end{cases}$$

Now we will calculate $A_1(N,X)$, where $X = X_0 + \sqrt{X_0 f'(N)}$, $X_0 = 4N(f'(N))^2$. We must note that on the interval of integration $[N, N']$, a subintegral function has not stationary phase points. So, by analogy with the case of $A_0(N,X)$, we obtain for $n \geqslant X$

$$\int_N^{N'} x^{\frac14}g(x)f'(x)e(f(x)+\sqrt{nx})dx \ll \frac{g(N)N^{\frac34}f'(N)}{\sqrt{n}},$$

and hence, the summation over $n > X$ gives the bound

$$O\left(N^{\varepsilon} g(N)\sqrt{Nf'(N)}\right).$$

Next, the integral

$$I(n) = \int\limits_{N}^{N'} x^{\frac{1}{4}} g(x) f'(x) e(f(x) - \sqrt{nx}) dx$$

we calculate with help of "*First derivative test*"

$$I(n) \ll N^{\frac{1}{4}} g(N) \frac{f'(N)}{f'_n(N)}.$$

But we have

$$f'_n(N) = f'(N) - \sqrt{\frac{n}{N}} = \frac{1}{\sqrt{N}}\left(\sqrt{N}f'(N) - \frac{\sqrt{n}}{2}\right) = \frac{1}{\sqrt{N}}\left(\sqrt{X_0} - \frac{1}{2}\sqrt{n}\right) =$$
$$= \frac{1}{\sqrt{N}}\left(\sqrt{X_0} - \frac{1}{2}\sqrt{n}\right) \asymp \sqrt{\frac{n}{N}}.$$

And then we derive the following estimates

$$\sum_{X \leqslant n \leqslant 2X} \frac{r(n)}{n^{\frac{3}{4}}} I(n) \ll \sum_{X \leqslant n \leqslant 2X} \frac{r(n)}{n^{\frac{3}{4}}} N^{\frac{1}{4}} g(N) f'(N) \frac{\sqrt{XN}}{n - X_0} \ll$$
$$\ll \frac{N^{\frac{3}{4}}}{X_0^{\frac{1}{4}}} g(N) f'(N) \sum_{X \leqslant n \leqslant 2X} \frac{r(n)}{n - X_0} \ll \frac{N^{\frac{3}{4}}}{X_0^{\frac{1}{4}}} g(N) f'(N) \left(\frac{1}{\sqrt{X_0 f'(N)}} + \log N\right) \ll$$
$$\ll \frac{g(N)}{f'(N)} + g(N)\sqrt{Nf'(N)};$$

$$\sum_{n > 2X} \frac{r(n)}{n^{\frac{3}{4}}} I(n) \ll \sum_{n > 2X} \frac{r(n)}{n^{\frac{1}{4}}} N^{\frac{1}{4}} g(N) f'(N) \sqrt{\frac{N}{n}} \ll$$
$$\ll N^{\frac{3}{4}} g(N) f'(N) \sum_{n > X} \frac{r(n)}{n^{\frac{1}{4}}} \ll g(N)\sqrt{Nf'(N)}.$$

So,

$$\sum_{n > X} \frac{r(n)}{\sqrt{n}} \int\limits_{N}^{N'} \sqrt{x} g(x) f'(x) e(f(x)) \mathfrak{I}_1(2\pi\sqrt{nx}) dx \ll$$
$$\ll \frac{g(N)}{f'(N)} + g(N)\sqrt{Nf'(N)}. \tag{17}$$

From (16), (17) Theorem 1 follows for $f'(N) \ll 1$ with $\omega_0(f) = e^{\frac{\pi i}{4}} \omega(f)$. In the case $f'(N) \gg 1$ we consider the expression

$$\overline{\omega}_{f_0} = \sum r(n) g(\varphi(n)) \sqrt{|\varphi'(n)|} e(-\widetilde{f}(\varphi(n))),$$

where a bar denotes the complex conjugate value, and

$$\widetilde{f}(n) = -f(\varphi(n)) + \varphi(n)f'(\varphi(n)).$$

Its clear that the following equations

$$\widetilde{f}'(x) = \frac{1}{f'(\varphi(x))} \ll 1,$$

$$\widetilde{f}'(x) + 2x\widetilde{f}''(x) = 4[f'(\varphi(x)) + 2\varphi(x)f''(\varphi(x))]^{-1},\ 4x\widetilde{f}'^2(x) = \varphi(x).$$

are true.

Hence, for $n \sim X_0$

$$g(\varphi(n))\sqrt{\varphi'(n)} \ll g(N)(f'(N))^{-1}.$$

So, we have

$$\omega_0(f) \sum_{n\sim X_0} r(n)g(\varphi(n))\sqrt{|\varphi'(n)|}e(\widetilde{f}(\varphi(n))) =$$

$$= \sum_{n\sim N} r(n)g(n)e(f(n)) + O\left(N^\varepsilon g(N)\right) + O\left(N^\varepsilon g(N)\frac{\sqrt{N}}{\sqrt{f'(N)}}\right).$$

At last, in the case $f'(N) < 0$ suffice it consider a complex conjugate sum.

Thus the proof of Theorem 3 is concluded.

Remark. *The proof of Theorem 3 in the idea sense is close to the method of estimation the exponential sums by using the van der Corput transform (and also to the method of exponential pairs).*

As the corollary of Theorem 3 there is the statement of Theorem 1 mentioned above.

Main Results

1. Proof of Theorem 1. In the interval $[1, x]$ the function $f(x) = \frac{ax}{q}$ satisfy all conditions of Theorem 3, moreover, the function $\varphi(y) = \frac{q^2}{a^2}y$ be the inversion for $y = xf'^2(x)$. All conditions of Theorem 3 for the functions $g(n) \equiv 1$ and $f(n) = \frac{a}{q}x$ are followed out. Then we have

$$\begin{aligned} A\left(x, \frac{a}{q}\right) &= \frac{q}{a}A\left(\frac{xa^2}{q^2}, -\frac{q}{a}\right) + \\ &\quad + O\left(x^\varepsilon \frac{q}{a}\right) + O\left(x^\varepsilon\right) + O\left(x^{\frac{1}{2}+\varepsilon}\min\left(\sqrt{\frac{q}{a}}, \sqrt{\frac{a}{q}}\right)\right) = \\ &= \frac{q}{a}A\left(\frac{xa^2}{q^2}, -\frac{q}{a}\right) + \\ &\quad + O\left(x^\varepsilon \frac{q}{a}\right) + O\left(x^{\frac{1}{2}+\varepsilon}\left(\frac{q}{a}\right)^{\frac{1}{2}}\right). \end{aligned} \tag{18}$$

Let us take a_1, $1 \leqslant a_1 < a$, $a_1 \equiv -q(mod\ a)$. Then

$$A\left(\frac{xa^2}{q^2}, -\frac{q}{a}\right) = A\left(\frac{xa^2}{q^2}, \frac{a_1}{a}\right),\ 1 \leqslant a_1 < a.$$

Use again the Theorem 3 to the right-hand sum. Then we obtain

$$A\left(x\frac{a^2}{q^2}, \frac{b}{a}\right) = \frac{a}{b}A\left(\frac{xa^2}{q^2}, \frac{b^2}{a^2}\right) + O\left(\left(x\frac{a^2}{q^2}\right)^{\varepsilon} \cdot \frac{a}{b}\right) + O\left(\left(\frac{xa^2}{q^2}\right)^{\varepsilon} \cdot \left(\frac{xa^2}{q^2} \cdot \frac{b}{a}\right)^{\frac{1}{2}}\right) =$$
$$= \frac{a}{b}A\left(\frac{xb^2}{q^2}, -\frac{a}{b}\right) + O\left(x^{\varepsilon}\frac{q}{a_1}\right) + O\left(x^{\frac{1}{2}+\varepsilon}\left(\frac{a_1}{a}\right)^{\frac{1}{2}}\right).$$

Hence, we have the following chain

$$A\left(x, \frac{a}{q}\right) \to \frac{q}{a}A\left(\frac{xa^2}{q^2}, \frac{a_1}{a}\right) \to \frac{q}{a_1}A\left(\frac{xa_1^2}{q^2}, \frac{a_2}{a_1}\right) \to \cdots$$
$$\cdots \to \frac{q}{a_{M-1}}A\left(\frac{xa_{M-1}^2}{q^2}, \frac{a_M}{a_{M-1}}\right) \to$$
$$\to qA\left(\frac{x^2}{q^2}, 1\right),\ \text{(i.e. } a_M = 1).$$

The number M is bounded above by the number of nearest to q the number of Fibonacci sequence f_n, i.e. $|f_M - q| < |f_n - q|$, where $n \neq M$, $n = 0, 1, 2, \ldots$.

It is easy to see that $M \ll \log q \ll x^{\varepsilon}$.

Thus from (18) we infer

$$A\left(x, \frac{a}{q}\right) = qA\left(\frac{x}{q^2}\right) + O\left(x^{\frac{1}{2}+\varepsilon}\right) = \frac{\pi x}{q} + O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

□

2. Proof of Theorem 2. Denote $N(\alpha_0) = a$, $N(\beta) = q$. Then we have

$$\sum_{N(\alpha)\leqslant x} \tau(\alpha)e^{2\pi i N\left(\frac{\alpha\alpha_0}{\beta}\right)} = \sum_{\substack{\alpha_1,\alpha_2\in\mathbb{Z}[i]\\ N(\alpha_1\alpha_2)\leqslant x}} e^{2\pi i\frac{aN(\alpha_1)N(\alpha_2)}{q}} = \sum_{mn\leqslant x} r(m)r(n)e^{2\pi i\frac{amn}{q}} =$$

$$= 2\sum_{d|q}\sum_{\substack{(m,q)=d\\ m\leqslant x^{\frac{1}{2}}}} r(m)\sum_{n\leqslant\frac{x}{m}} r(n)e^{2\pi i\frac{amn}{q}} - \sum_{d|q}\sum_{\substack{m\leqslant x^{\frac{1}{2}}\\ (m,q)=1}}\sum_{n\leqslant x^{\frac{1}{2}}} r(m)r(n)e^{2\pi i\frac{amn}{q}} =$$

$$= 2\sum\nolimits_1 - \sum\nolimits_2,$$

say. For $(m,q) = d$ let us suppose $m = m_1 d$, $q = q_1 d$, $(m_1, q_1) = 1$. Then, from the

Theorem 1, we get

$$\begin{aligned}
\textstyle\sum_1 &= \sum_{d|q} \sum_{m_1 \leqslant \frac{x^{\frac{1}{2}}}{d}} r(m_1 d) \sum_{n \leqslant \frac{x}{m_1 d}} r(n) e^{2\pi i \frac{a m_1 n}{q_1}} = \\
&= \sum_{d|q} \left[\frac{\pi x}{d q_1} \sum_{m_1 \leqslant \frac{x^{\frac{1}{2}}}{d}} \frac{r(m_1 d)}{m_1} + O\left(\left(\sum_{\substack{m_1 \leqslant \frac{x^{\frac{1}{2}}}{d} \\ (m_1, q_1)=1}} \left(\frac{x}{m_1 d}\right)^{\frac{1}{2}+\varepsilon} + q_1 \left(\frac{x}{m_1 d}\right)^{\varepsilon} \right) \right) \right] = \\
&= \sum_{d|q} \left\{ \left[\frac{\pi x}{q} \operatorname*{res}_{s=1} (\zeta(s) L(s,\chi_4)) \sum_{d_1|d} \chi_4(d_1) \prod_{p|\frac{d}{d_1}} \left(1 - \frac{\chi_4(d_1)}{p^s}\right) \frac{x^{s-1}}{s-1} \right] + \right. \\
&\left. + O\left(\left(\frac{x^{\frac{1}{2}}}{d} \right)^{\frac{1}{3}+\varepsilon} \right) + O\left(x^{\frac{3}{4}+\varepsilon} d^{-1-\varepsilon} \right) + O\left(q x^{\frac{1}{2}+\varepsilon} \right) \right\} = \\
&= A_1(q) \frac{x \log x}{q} + A_2(q) \frac{x}{q} + O\left(x^{\frac{1}{2}+\varepsilon} q \right) + O\left(x^{\frac{3}{4}+\varepsilon} \right),
\end{aligned} \tag{19}$$

where $A_1(q)$, $A_2(q)$ be the computable constants, $q^{-\varepsilon} \ll A_i(q) \ll q^{\varepsilon}$, $i = 1, 2$.
As before, we obtain

$$\begin{aligned}
\textstyle\sum_2 &= \sum_{d|q} \sum_{\substack{m_1 \leqslant \frac{x^{\frac{1}{2}}}{d} \\ (m_1, q_1)=1}} r(m) \left(\frac{\pi x^{\frac{1}{2}}}{q_1} + O\left(x^{\frac{1}{4}+\varepsilon} \right) + O\left(q_1 x^{\varepsilon} \right) \right) = \\
&= B(q) \frac{x}{q} + O\left(x^{\frac{3}{4}+\varepsilon} \right) + O\left(q_1 x^{\varepsilon} \right),
\end{aligned} \tag{20}$$

where $q^{-\varepsilon} \ll B(q) \ll q^{\varepsilon}$. Collecting out estimates(19)-(20) together, we obtain the desired result of theorem. □

Conclusion. The scheme of the proof of Theorem 3 may be applied for investigation the weighted exponential sums over the ring of Gaussian integers of the following view

$$\sum_{N(\alpha) \leqslant x} \tau(\alpha) g(N(\alpha)) e(N(\alpha)).$$

But now instead of representation the sum $\sum r(n)$ in view of the series on Bessel functions it is necessary to study the sums of view

$$\sum_{\alpha \in \mathbb{Z}[i]} \tau(\alpha) \mathfrak{Y}(\alpha) g(N(\alpha)) e(N(\alpha)),$$

where

$$\mathfrak{Y}(\alpha) = y^{\frac{3}{8}} \int\limits_{c-i\infty}^{c+i\infty} \Gamma(s) y^{-\frac{3}{4}} e\left(-\frac{s}{4}\right) ds, \ \ y = \pi^4 x N(\alpha).$$

Function $\mathfrak{Y}(\alpha)$ may be considered as an analogue of Bessel function $\mathfrak{I}_1(\alpha)$.

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Варбанецъ С.
Тригонометричні суми функції дільників над $\mathbb{Z}[i]$

Резюме

Ми застосовуємо перетворення Ван дер Корпута для дослідження сум виду $\sum r(n)g(n)\times$ $\times e(f(n))$, де $r(n)$ є число зображень n як суми двох квадратів цілих чисел. Такі суми вивчались М. Ютілою, О. Гунявим, М. Хакслі та ін. Спираючись на властивості диференціювання функцій $g(n)$ та $f(n)$, нами були отримані різні типи залишкових членів на границях розглянутих сум. В спеціальному випадку О. Гунявий покращив результат М. Ютіли в проблемі оцінювання тригонометричної суми від функції дільників $\tau(n)$. Ми отримуємо асимптотичну формулу для суми $\sum \tau(\alpha)e\left(\frac{a}{q}N(\alpha)\right)$ над кільцем цілих гаусових чисел, яка є аналогом асимптотичних формул, отриманих М. Ютілою та О. Гунявим.
Ключові слова: тригонометричні суми.

Варбанец С.
Тригонометрические суммы функции делителей над $\mathbb{Z}[i]$

Резюме

Мы применяем преобразование Ван дер Корпута для исследования сумм $\sum r(n)g(n)\times$ $\times e(f(n))$, где $r(n)$ есть число представлений n в виде суммы двух квадратов целых чисел. Такие суммы изучались М. Ютилой, О. Гунявым, М. Хаксли и др. Опираясь на свойства дифференцирования функций $g(n)$ и $f(n)$, нами были получены различные типы остаточных членов на границах рассматриваемых сумм. В специальном случае О. Гунявый улучшил результат М. Ютилы в проблеме оценки тригонометрической суммы от функции делителей $\tau(n)$. Мы получаем асимптотическую формулу для $\sum \tau(\alpha)e\left(\frac{a}{q}N(\alpha)\right)$ над кольцом целых гауссовых чисел, являющуюся аналогом асимптотических формул, полученных М. Ютилой и О. Гунявым.
Ключевые слова: тригонометрические суммы.