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THE STUDY OF THE INERTIAL PROPERTIES OF THE EXFOLIATED HARD INCLUSION IS IN THE CONDITIONS OF SMOOTH CONTACT WITH A NON-STATIONARY WAVES

Мойсеєнок О. П. Дослідження інерціальних властивостей жорсткого відшарованого включення за умов гладкого контакту за нестаціонарної хвильової дії. Розв'язана задача про визначення напруженого стану в околі тонкого жорсткого включення у вигляді смуги кінцевої ширини у безмежному тілі (матриці) при проходженні плоских нестаціонарних хвиль. Вважається, що матриця знаходиться у стані плоскої деформації, а на обох сторонах включення реалізовано умови гладкого контакту. Метод розв'язання полягає у застосуванні інтегрального перетворення Лапласа за часом і у поданні зображень напружень та переміщень через розривний розв'язок рівнянь Ламе за умов плоскої деформації. В результаті початкову задачу зведено до системи сингулярних інтегральних рівнянь відносно зображень невідомих стрибків напружень та переміщень. Для обернення перетворення Лапласа застосовано числовий метод, який ґрунтується на заміні інтеграла Мелліна рядом Фур'є.
Ключові слова: жорстке відшароване включення, нестаціонарна плоска хвиля, розривний розв'язок, числове обернення перетворення Лапласа, КІН.

Мойсеенок А. П. Исследование инерциальных свойств жесткого отслоившегося включения при условиях гладкого контакта при нестационарном волновом воздействии. Решена задача об определении напряженного состояния вблизи тонкого жесткого включения в виде полосы конечной ширины в неограниченном упругом теле (матрице) при прохождении плоских нестационарных волн. Считается, что матрица находится в состоянии плоской деформации, а на обеих сторонах включения реализованы условия гладкого контакта. Метод решения состоит в применении интегрального преобразования Лапласа по времени и представлении изображений напряжений и перемещений через разрывное решение уравнений Ламе для случая плоской деформации. В результате исходная задача сведена к системе сингулярных интегральных уравнений относительно изображений неизвестных скачков напряжений и перемещений. Для обращения преобразования Лапласа применен численный метод, основанный на замене интеграла Меллина рядом Фурье.
Ключевые слова: жесткое отслоившееся включение, нестационарная плоская волна, разрывное решение, численное обращение преобразования Лапласа, КИН.

Moysyeyenok A. P. The study of the inertial properties of the exfoliated hard inclusion is in the conditions of smooth contact with a non-stationary waves. The problem about determining the stress state near the thin rigid inclusion in a strip of finite width in an infinite elastic body (matrix) when passing of plane nonstationary waves is solved. It is considered that the matrix is in the conditions of plane strain and on both sides of the inclusion conditions of the smooth contact are implemented. The method of solution consists in applying the integral Laplace transform in time and presenting images of stresses and displacements through the discontinuous solution of Lame's equations for the case of

plane strain. As a result, the initial problem is reduced to a system of singular integral equations with respect to unknown images jumps of stresses and displacements. To inverse the Laplace transform the numerical method based on the replacement of the Mellin integral by the Fourier series is applied.
Key words: hard exfoliated inclusion, non-stationary waves, the discontinuous solution, numerical Laplace transfomation, SIF.

Introduction. Dynamic problems of the theory of elasticity for solids with thin inclusions often are considered on the assumption that between the matrix and the inclusion the conditions of full coupling are fulfilled. The solution of such 2D and 3D problems for the case of harmonic vibrations of solids with inclusions can be found in [1], [2]. The concentration of stresses near the thin rigid inclusion which is fully coupled with the matrix at the non-stationary loading was studied in [3]. The dynamic problems, when between the matrix and the inclusion the smooth contact conditions are realized, are not considered. In [4] the problem of the interaction of plane harmonic waves with the rigid inclusion under conditions of smooth contact is solved . In this paper we consider the similar problem of the interaction with non-stationary waves.

Main Results. Let us consider an infinite elastic body (matrix), which is in the plane strain and containing the inclusion in the form of of the rigid plate width a and thickness $h << a$. This inclusion in the plane Oxy occupies an area of $|x| \le a$, $-\frac{h}{2} \le y \le \frac{h}{2}$. Let at the initial moment $t = 0$ the non-stationary plane longitudinal wave with a potential $\varphi_0(x, y, t)$ or shear wave with a potential $\psi_0(x, y, z)$ interacts with the inclusion. The displacements caused by these waves we will be denoted as $u_0(x, y, t)$, $v_0(x, y, t)$. Then the displacements and stresses in the matrix can be represented as the sum of two terms

$$u = u_0 + u_1, v = v_0 + v_1, \sigma_y = \sigma_y^0 + \sigma_y^1, \tau_{yx} = \tau_{yx}^0 + \tau_{yx}^1; \tag{1}$$

where u_1, v_1, σ_y^1, τ_{yx}^1 – the displacements and stresses in the matrix caused by the waves reflected from the inclusion. The displacements u_1, v_1 satisfy the Lame equations for plane strain and zero initial conditions at $t = 0$.

The boundary conditions of the external environment on the inclusion due to its small thickness we formulate concerning its median plane. We shall assume that on both sides of the inclusion the conditions of smooth contact with the matrix are fulfilled. Then on the inclusion the stress σ_y^1 and displacement u_1 have discontinuity, which jumps we denote:

$$\sigma_y^1(x, +0, t) - \sigma_y^1(x, -0, t) = \chi_1(x, t), \tag{2}$$

$$u_1(x, +0, t) - u_1(x, -0, t) = \chi_4(x, t), \chi_4(\pm a, t) = 0, -a \le x \le a. \tag{3}$$

Also, from the conditions of the smooth contact, since the rigidity of inclusion should be the following equations:

$$v_1(x, 0, t) = \alpha_1(t) + \gamma(t)\, x - v_0(x, 0, t), \tau_{yx}^1(x, \pm 0, t) = -\tau_{yx}^0(x, 0, t), \tag{4}$$

where $\alpha_1(t)$ – the unknown displacement along the axis Oy, and $\gamma(t)$ – the unknown angle of the rotation of the inclusion. They are found from the equations of motion for inclusion as a rigid body:

$$m\ddot{\alpha}_1(t) = q(t) + R(t), J\ddot{\gamma}(t) = m(t) + M(t),$$

$$R(t) = \int_{-a}^{a} \chi_1(x,t)\,dx, M(t) = \int_{-a}^{a} \chi_1(x,t)\,xdx. \quad (5)$$

In these equations m – mass, a J – the moment of inertia per unit length of inclusion; $R(t)$, $M(t)$ – the force and the moment of the reaction on the inclusion from the part of the matrix.

Obtaining the integral equations and solution of the problem. To solve the formulated initial boundary value problem we apply the time-integral Laplace transform. Then from (2) - (4) for images we obtain the equalities

$$S_y^1(x,+0,p) - S_y^1(x,-0,p) = \mathrm{X}_1(x,p), U_1(x,+0,p) - U_1(x,-0,p) = \mathrm{X}_4(x,p), \quad (6)$$

$$V_1(x,0,p) = A_1(p) + G(p)\,x - V_0(x,0,p), \mathrm{T}_y^1(x,\pm 0,p) = -\mathrm{T}_y^1(x,0,p). \quad (7)$$

The equations of motion (5) after the Laplace transform take the form

$$mp^2 A_1(p) = \bar{R}_y(p), Jp^2 G(p) = \bar{M}(p),$$

$$\bar{R}_y(p) = \int_{-a}^{a} \mathrm{X}_1(x,p)\,dx, \bar{M}(p) = \int_{-a}^{a} \mathrm{X}_1(x,p)\,xdx. \quad (8)$$

In the recent equalities V_k, U_k, S_y^k, T_{yx}^k, $k = 0,1$; X_1, X_4, A_1, G –images of the corresponding functions, p – parameter of the Laplace transform.

The Laplace image of displacements and stresses in the matrix can be represented as the discontinuous solution of Lame equations for images with jumps (6). For this purpose in the corresponding formulas from [1], where they are given for the harmonic oscillations, we should set $\kappa_k = i\frac{p}{c_k}$, $k = 1,2$. Here c_1, c_2 - velocities of longitudinal and transverse waves in the matrix. Then, for the displacements and stresses which are included in boundary conditions we obtain

$$V_1 = \frac{1}{\mu p_2^2}\int_{-a}^{a} \mathrm{X}_1\left(\left(p_1^2 - \frac{\partial^2}{\partial x^2}\right)K_1 + \frac{\partial^2}{\partial x^2}K_2\right)d\eta +$$

$$+\frac{1}{p_2^2}\int_{-a}^{a} \mathrm{X}_4'\left(\left(2\frac{\partial^2}{\partial x^2} - p_2^2\right)K_2 - 2\left(2\frac{\partial^2}{\partial x^2} - p_1^2\right)K_1\right)d\eta,$$

$$\mathrm{T}_{yx}^1 = \frac{1}{p_2^2}\int_{-a}^{a} \mathrm{X}_1\left(\frac{\partial}{\partial x}\left(2\frac{\partial^2}{\partial x^2} - p_2^2\right)K_2 - 2\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - p_1^2\right)K_1\right)d\eta +$$

$$+\frac{\mu}{p_2^2}\int_{-a}^{a} \mathrm{X}_4'\left(4\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} - p_2^2\right)K_2 - 4\frac{\partial}{\partial x}\left(2\frac{\partial^2}{\partial x^2} + p_1^2\right)K_1 - p_2^4 K_2^*\right)d\eta. \quad (9)$$

In these formulas

$$K_j^* = \frac{i}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\alpha(\eta - x) - \gamma_j|y|}}{2\alpha\gamma_j}d\alpha, K_j = -\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{i\alpha(\eta - x) - \gamma_j|y|}}{2\gamma_j}d\eta =$$

$$- = \frac{1}{2\pi} K_0 \left(p_j \sqrt{(\eta - x)^2 + y^2} \right),$$

$$\gamma_j = \sqrt{\alpha^2 + \frac{p^2}{c_j^2}}, p_j = \frac{p}{c_j}, j = 1, 2.$$

The presentation from the discontinuous solution makes it possible to find the stresses and displacements in the matrix with using the formulas (1), (9), if we known the jumps of displacements and stresses. To determine these jumps from the conditions on the inclusion (4) we obtain the integral equations. For this purpose the first equation (7) we differentiate and add to the result the condition of the equivalence of the original and differentiated equations.

$$V_1' (x, 0, p) = G (p) - V_0' (x, 0, p), V_1 (x, 0, p) = A_1 (p) - G (p) a - V_0 (-a, 0, p), \quad (10)$$

After the substitution of (9) in the second condition (7) and in the conditions of (10) we obtain the system of integral equations with the additional condition concerning the images of unknown jumps. This system after the isolation of the singular components of kernels has the form

$$\frac{1}{2\pi} \int_{-1}^{1} \Phi_1 (z, q) \left(-\frac{(1 + \xi^2)}{2 (z - \zeta)} + F_{11} (q (z - \zeta)) \right) dz +$$

$$+ \frac{1}{2\pi} \int_{-1}^{1} \Phi_2 (z, q) \left(-\frac{\xi^2}{z - \zeta} + F_{12} (q (z - \zeta)) \right) dz =$$

$$= g (q) - f_1 (\zeta),$$

$$\frac{1}{2\pi} \int_{-1}^{1} \Phi_1 (z, q) \left(-\frac{\xi^2}{z - \zeta} + F_{21} (q (z - \zeta)) \right) dz +$$

$$+ \frac{1}{2\pi} \int_{-1}^{1} \Phi_2 (z, q) \left(\frac{2 (1 - \xi^2)}{z - \zeta} + F_{22} (q (z - \zeta)) \right) dz = f_2 (\zeta),$$

$$\frac{1}{2\pi} \int_{-1}^{1} \Phi_1 (z, q) \left(\frac{1 + \xi^2}{2} \ln (z + 1) + R_1 (q (z + 1)) \right) dz +$$

$$+ \frac{1}{2\pi} \int_{-1}^{1} \Phi_2 (z, q) \left(\xi^2 \ln (z + 1) + R_2 (q (z + 1)) \right) dz =$$

$$= d_1 (q) - g (q) - f_3 (q),$$

$$\int_{-1}^{1} \Phi_2 (z, q) \, dz = 0. \quad (11)$$

The functions F_{11}, F_{12}, F_{21}, F_{22}, R_1, R_2 are limited and continuous for $-1 \leq z, \zeta \leq 1$.

When obtaining the system (11) the following notation were introduced :

$$\eta = az, x = a\zeta, p = \frac{c_2 q}{a}, \xi = \sqrt{\frac{c_2}{c_1}}, p_1 = \frac{p}{c_1},$$

$$p_2 = \frac{p}{c_2}, d_1(q) = \frac{c_2}{a^2} A_1\left(\frac{c_2 q}{a}\right),$$

$$g(q) = \frac{c_2}{a} G\left(\frac{c_2 q}{a}\right), \mathrm{X}_1\left(az, \frac{c_2 q}{a}\right) = \frac{a\mu}{c_2}\Phi_1(z,q), \mathrm{X}_4'\left(az, \frac{c_2 q}{a}\right) = \frac{a}{c_2}\Phi_2(z,q), \quad (12)$$

$$f_1(\zeta) = -\frac{c_2}{a} V_0'\left(a\zeta, 0, \frac{c_2 q}{a}\right),$$

$$f_2(\zeta) = \frac{c_2}{a\mu}\mathrm{T}_{yx}^0\left(a\zeta, 0, \frac{c_2 q}{a}\right), f_3(q) = \frac{c_2}{a^2} V_0\left(-a, 0, \frac{c_2 q}{a}\right).$$

The recent equation (11) follows from (12) and $\mathrm{X}_4(\pm a, p) = 0$. To the system (11) we also need to add two more equations to determine the images of unknown amplitude of the motion of the inclusion $d_1(q)$ and the angle of the rotation $g(q)$. We obtain these equations from (8) going to the notation (12):

$$d_1(q) = \frac{\bar{\rho}}{4\varepsilon q^2}\int_{-1}^{1}\Phi_1(z,q)\,dz, g(q) = \frac{3\bar{\rho}}{16\varepsilon q^2}\int_{-1}^{1}\Phi_1(z,q)\,zdz, \bar{\rho} = \frac{\rho_1}{\rho_0}. \quad (13)$$

ρ_1, ρ_0 – the density of matrix and inclusions. The approximate solution of the (11), (13) we shall find in the form [5]

$$\Phi_j(z,q) = \frac{\Psi_j(z,q)}{\sqrt{1-z^2}}, j = 1,2.$$

The functions $\Psi_j(z,q)$ we approximate by interpolating polynomials

$$\Psi_j(z_m,q) = \sum_{m=1}^{n}\Psi_{mj}\frac{T_n(z_m)}{T_n'(z_m)(z-z_m)}, \Psi_{mj} = \Psi_j(z_m), j = 1,2, \quad (14)$$

where $T_n(z)$ – Chebyshev polynomials of the 2nd kind, $z_m = \cos\frac{\pi(2m-1)}{2n}$, $m = 1,..,n$ – roots of these polynomials. To find the unknown values of the images Ψ_{mj}, $j = 1,2$ in the interpolation points of (11) we get a system of linear algebraic equations. For this we substitute there

$$\zeta = \zeta_k, \zeta_k = \cos\left(\frac{k\pi}{n}\right), k = 1,..,n-1,$$

the integrals with Cauchy kernel substitute the special quadrature formula, and the integrals with regular kernels - Gauss-Chebyshev quadrature formulas [5].For the integral with a logarithmic singularity we use the formula [6]

$$\int_{-1}^{1}\Phi_j(z,q)\ln(z+1)\,dz = \sum_{m=1}^{n} a_m\Psi_{mj}B_m, B_m =$$

$$= -\ln 2 - 2\sum_{j=1}^{n-1}(-1)^j\frac{\cos\frac{j\pi(2n-1)}{2n}}{j}, a_m = \frac{\pi}{n}.$$

As a result we get the system of 2n +2 linear algebraic equations

$$\frac{1}{2\pi}\sum_{m=1}^{n}a_m\Psi_1\left(z_m,q\right)\left(-\frac{\left(1+\xi^2\right)}{2\left(z_m-\zeta_k\right)}+F_{11}\left(q\left(z_m-\zeta_k\right)\right)-\frac{3\pi\bar{\rho}z_m}{8\varepsilon q^2}\right)+$$

$$+\frac{1}{2\pi}\sum_{m=1}^{n}a_m\Psi_2\left(z_m,q\right)\left(-\frac{\xi^2}{z_m-\zeta_k}+F_{12}\left(q\left(z_m-\zeta_k\right)\right)\right)=f_1\left(\zeta_k\right),$$

$$\frac{1}{2\pi}\sum_{m=1}^{n}a_m\Psi_1\left(z_m,q\right)\left(-\frac{\xi^2}{z_m-\zeta_k}+F_{21}\left(q\left(z_m-\zeta_k\right)\right)\right)+$$

$$+\frac{1}{2\pi}\sum_{m=1}^{n}a_m\Psi_2\left(z_m,q\right)\left(\frac{2\left(1-\xi^2\right)}{z_m-\zeta_k}+F_{22}\left(q\left(z_m-\zeta_k\right)\right)\right)=f_2\left(\zeta_k\right),$$

$$\frac{1}{2\pi}\sum_{m=1}^{n}a_m\Psi_1\left(z_m,q\right)\left(-\left(\frac{1+\xi^2}{2}\right)B_m+R_1\left(q\left(z_m+1\right)\right)-\frac{\pi\bar{\rho}}{2\varepsilon q^2}+\frac{3\pi\bar{\rho}z_m}{8\varepsilon q^2}\right)+$$

$$+\frac{1}{2\pi}\sum_{m=1}^{n}a_m\Psi_2\left(z_m,q\right)\left(-\xi^2B_m+R_2\left(q\left(z_m+1\right)\right)\right)=f_3\left(q\right),$$

$$\sum_{m=1}^{n}a_m\Psi_2\left(z_m,q\right)=0,$$

$$d_1\left(q\right)=\frac{\bar{\rho}}{4\varepsilon q^2}\sum_{m=1}^{n}a_m\Psi_1\left(z_m,q\right),g\left(q\right)=\frac{3\bar{\rho}}{16\varepsilon q^2}\sum_{m=1}^{n}a_m\Psi_1\left(z_m,q\right)z_m. \tag{15}$$

The most interest to the fracture mechanics represent the stress state in the matrix near the inclusion. We should use the asymptotic formulas for the stresses near the ends of the inclusion [7], [8]. These formulas for the rigid inclusion which is in the smooth contact with the matrix after notation which was introduced in (12) have the form

$$\begin{bmatrix}\sigma_y\\ \sigma_x\\ \tau_{xy}\end{bmatrix}=\frac{\mu\sqrt{a}}{\sqrt{2r}}k_1^{\pm}\left(\tau\right)\begin{pmatrix}-\sin\theta_1+\sin\theta_5\\ -7\sin\theta_1-\sin\theta_5\\ 3\cos\theta_1+\cos\theta_5\end{pmatrix}+$$

$$+\frac{\mu\sqrt{a}}{\sqrt{2r}}k_2^{\pm}\left(\tau\right)\begin{pmatrix}\left(2\kappa+3\right)\sin\theta_1+\sin\theta_5\\ -\left(2\kappa-5\right)\sin\theta_1+\sin\theta_5\\ \left(2\kappa-1\right)\cos\theta_1-\cos\theta_5\end{pmatrix}+O\left(1\right),\theta_p=\frac{p\theta}{2}. \tag{16}$$

In the formulas (16) r, θ are coordinates in the polar coordinate system, the centers of which coincide with the ends of the inclusion $x=\pm a$, $\tau=\frac{c_2}{a}t$ - dimensionless time. From (16) it is clear that the stress state in the matrix near the inclusion determined by the coefficients $k_1^{\pm}$ and $k_2^{\pm}$. These coefficients, following [7], [8], we will call the dimensionless stress intensity factors (SIF) for inclusion. These factors equal

$$k_1^{\pm}\left(\tau\right)=\mp\frac{\psi_2\left(\pm1,\tau\right)}{8\left(1-\nu\right)},k_2^{\pm}\left(\tau\right)=\pm\frac{\psi_1\left(\pm1,\tau\right)}{16\left(1-\nu\right)}. \tag{17}$$

The approximate values of the images (SIF) using (14), (17) can be expressed through the solution of (15). There are following formulas

$$K_1^{\pm}(q) = \mp\frac{\Psi_2(\pm 1, q)}{8(1-\nu)}, K_2^{\pm}(q) = \mp\frac{\Psi_2(\pm 1, q)}{16(1-\nu)},$$

$$\Psi_j(\pm 1) = \mp\frac{(\pm 1)^n}{n}\sum_{m=1}^{n}\Psi_{mj}(-1)^m\left(ctg\frac{\gamma_m}{2}\right)^{\pm 1}, \gamma_m = \frac{(2m-1)\pi}{2n}.$$

The numerical results. The numerical implementation of the proposed solution carried out for the case of the interaction with the inclusion of longitudinal or transverse waves with the front parallel to the inclusion. Here by the symmetry for the longitudinal waves we have $k_j^+(\tau) = -k_j^-(\tau) = k_j(\tau)$, and for transverse waves we have $k_j^+(\tau) = k_j^-(\tau) = k_j(\tau)$, $j = 1, 2$.

The originals of the dimensionless (SIF) $k_j^{\pm}(\tau)$ were restored numerically using a method based on the replacement of the Mellin integral by the Fourier series [9], as well as the modification of this method proposed in [10].

Suppose that the inclusion interacts with the flat longitudinal waves. During the numerical analysis, it was found that in this case $|k_2| >> |k_1|$ and therefore only depending on the time graphics $k_2(\tau)$, are shown in Fig.1(a) and Fig.1(b) The curve in Fig.1(a) shows the graph of $k_2(\tau)$ under the action on the inclusion of a impact wave with the potential

$$\varphi_0 = (c_1 t - y)^2 H(c_1 t - y),$$

$H(t)$ – Heaviside function.

There is a rapid growth of (SIF) $k_2(\tau)$, and then it decreases to a value of 0. On Fig.1(b) it is shown the similar graph for the case when the incident wave is harmonic with potential.

$$\varphi_0 = \frac{c_1}{\omega}\cos\left(\omega\left(t - \frac{y}{c_1}\right)\right)H(c_1 t - y).$$

It was assumed that $\omega_0 = \frac{a\omega}{c_2} = 3$. It can be seen that the $\tau > 2$ we have access to the steady state.

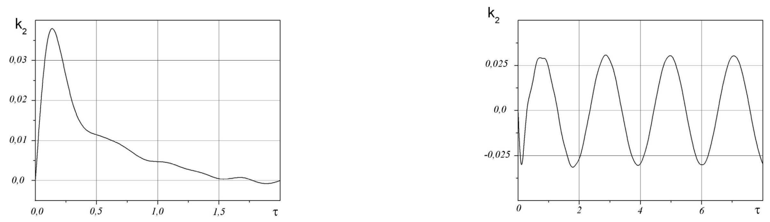
The calculation of (SIF) has also been performed for the interaction with the inclusion of transverse shear waves. The results of these calculations are shown in Fig.2(a) and Fig.2(b). As in this case, $|k_1| >> |k_2|$, then only studied the behavior of $k_1(\tau)$. On Fig.2(a) the variation of this coefficient under the action of the impact wave on the inclusion with the potential

$$\psi_0 = (c_2 t - y)^2 H(c_2 t - y).$$

is shown. Under this action $k_1(\tau) < 0$, and the absolute value of (SIF) increases to a maximum and then decreases to a constant value. The dependence of the $k_1(\tau)$ of time under the influence of transverse shear harmonic wave with the potential

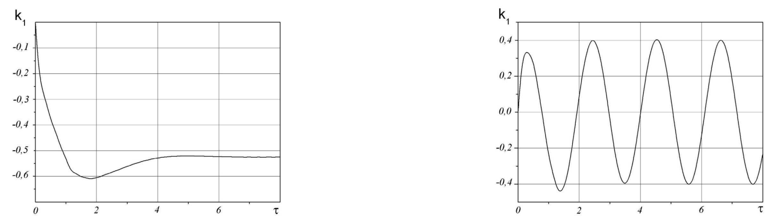
$$\psi_0 = \frac{c_2}{\omega}\cos\left(\omega\left(t - \frac{y}{c_2}\right)\right)H(c_2 t y), \omega_0 = \frac{a\omega}{c_2} = 3$$

is shown in Fig.2(b). It can be seen that at $\tau > 2$ (SIF) changes harmonically, and during the transition time the absolute values of (SIF) may slightly exceed the maximum values at the steady state.



a b

Fig. 1. The dependence of the dimensionless SIF on time during the action of the plane transverse impact wave on the inclusion (a) and of the plane transverse impact harmonic wave on the inclusion (b).



a b

Fig. 2. The dependence of the dimensionless SIF on time during the action of the plane longitudinal impact wave on the inclusion (a) and of the plane longitudinal harmonic impact wave on the inclusion (b).

Conclusion. In the state of plane strain the conditions of smooth contact significantly affect to the nature of the stress state near the the rigid inclusion [3] and the dependence of the (SIF) from time. It is established that at the impact wave action values of (SIF) are taking extreme values at the beginning of wave action. Under the sudden harmonic action, the value of (SIF) in the transition period is not significantly greater than the value of (SIF) in the steady state. Last fact allows when studying the stress state near the such inclusion, with harmonic action, to solve the stationary problem immediately.

1. **Popov V. G.** Sravnitelnyy analiz difractsionykh poley pri prokhozhdenii uprugikh voln cherez defecty razlichnoi prorody / V. G. Popov, A. E. Ulanovskiy // Izv. RAN. Mekhanika tverdogo tela. – 1995. – № 4. – P. 99–109.

2. **Kit G. S.** Analiz ustanovivshykhsya kolebaniy ploskogo absolyutno zhestkogo vklyucheniya v trekhmernom tele metodom granichnikh elementoov / G. S. Kit, V. V. Mikhaskiv, O. M. Khay // Prikl. mat. i mekhanika. – 2002. – T. 66, Vyp. 5. – P. 855–863.

3. **Moyseenok A. P.** Doslidzhennya napruzhenogo stany v okoli smugovogo vkluchennya za umov nestastionarnogo navantazhennya / A. P. Moyseenok, V. G. Popov // Pid zagalnoyu redakstieyu V. V. Panasyuka – Lviv: Fiz.-mekh. institut im. G. V. Karpenka NAN Ukrainy, 2004. – P. 143–148.

4. **Litvin O. V.** Vzaemodiya ploskikh pruzhnukh khvul z zhorstkym vkluchennyam za umovu gladkogo kontaktu / O. V. Litvin, V. G. Popov // Mashinoznavstvo, 2003. – №9. – P. 24–27.

5. **Belotserkovskiy S. M.** Chislennye metodu v singulyarnykh integralnukh uravneniyakh i ih primenenie v aerodinamike, teorii yprugosti, electrodinamike / S. M. Belotserkovskiy, I. K. Lifanov. – M.: Nayka, 1985. – 256 P.

6. **Nazarchyk Z. T.** Chislennoe issledovanie difraktsii voln na tsilindricheskikh strukturakh / Z. T. Nazarchyk. – K. : Naykova dymka, 1989. – 256 P.

7. **Grulitskiy D. V.** Uprugie napryazheniya v ploskosti s tonkostennymi vklyucheniyami / D. V. Grulitskiy, G. T. Sulim // Mat. metody i fiz.–mekh. polya. – 1975. – Vyp.1. – P. 41–48.

8. **Berezhnitskiy L. T.** Vzaimodeistvie zhestkikh lineinykh vklyucheniy i treshin v deformiruemom tele / L. T. Berezhnitskiy, N. G. Stashyuk. – K.: Naykova Dymka, 1983. – 288 P.

9. **Davies B.** Numerical inversion of the place Laplace transform: a survey and comparison of methods / B. Davies, B. Martin // J. of Comp. Phys. – 1979. – Vol. 33, №1. – P. 1–32.

10. **Jaemin A.** A flexible inverse place Laplace transform algorithm and its application / A. Jaemin, K. Sungkwon, K. YongHoon // Computing. – 2003. – Vol. 71, № 2. – P. 115–131.