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TORSION PROBLEM FOR AN ELASTIC TWICE-TRUNCATED CONE

The problem of an elastic twice-truncated cone wave field estimation is investigated in case of steady state torsional oscillations. The G. Ya. Popov integral transform with regard to an angular coordinate is applied. Thus reducing the original problem to one-dimensional boundary value problem in the transform's domain. The Green's function is build for one-dimensional boundary value problem. With it's help the solution of one-dimensional problem is constructed in an explicit form. The G. Ya. Popov inverse transformation helped to derive the solution in original domain in form of an infinite sum. With it's help dependence of the eigenfrequencies from the cone's geometric parameters is investigated. Stress field was found with the use of asymptotic procedure. Comparison plots are build for different opening angles.
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Key words: twice-truncated cone, steady state torsional oscillations, G. Ya. Popov integral transform, eigenfriquencies, wave field, Green's function.
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Introduction. An important problem widely common in engineering practice is to determine the dynamic stress state of a cone under the impact of a non-stationary load. A particularly important point is the ability to calculate the eigenfrequencies required to evaluate the dynamic stability of the constructions. It is possible to do with help of the initial boundary value problems apparatus of mathematical physics. The solving of the initial boundary value problems for cone-shaped elastic bodies is not a new problem, however, there are many unresolved issues. In [1], the three-dimensional Green's function for a transverse-isotropic thermoelastic cone with a stable heat source on the vertex is derived. In [2] the problem of equilibrium for a semi-infinite transverse-isotropic three-dimensional elastic cone under the antisymmetric force at its vertex, is considered. In [3], the solution of the problem for a half-infinite elastic cone is constructed under the concentrated force applied at it's vertex at a certain distance from it. In [4] the problem of deriving the displacement and the stress fields of a semi-infinite isotropic elastic cone under mixed boundary conditions on the surface of the cone is considered. In [5] one found a solution in quadratures for the linear problem of the dynamic theory of elasticity with respect to the deformation of an infinite elastic homogeneous isotropic space caused by the rotation of an absolutely rigid circular cone with an uneven lateral surface. In [6], problems were solved on the stress-strain state of an elastic cone of variable thickness in a three-dimensional formulation using both analytical methods and numerical approaches. In [7] an axisymmetric problem of tensile of a cone under the action of concentrated load is considered in view of large deformations. The exact solution of the torsion problem of an elastic conically-layered cone is obtained in [8]. In [9] the

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solution for semi-infinite elastic cone with help of Papkovycha-Neybera functions is derived and analyzed stress distribution in the length of their attenuation. In [10] a numerical method for constructing eigenfunctions for arbitrary conical bodies with smooth and non-smooth lateral surfaces is considered. In [11] the method of distribution of variables yields an effective solution of various boundary thermoelasticity problems for a hollow endless cone. The dynamic problem of instability of a surface of a finite circular anisotropic cone with an arbitrary aperture in the form of a hexagonal single crystal is considered in [12]. In [13], a study was made of the stress intensity factor near a spherical crack inside a semi-infinite cone under a compression load applied to the vertex of a cone. The question of steady-state oscillations of an elastic infinite cone under the action of a concentrated oscillating force added at the vertex of a cone is solved in [14]. The problem of torsion of an elastic cone, weakened by a spherical crack under the action of the shock moment, added to the vertex, is considered in [15].

Problem statement for finite cones adds some complexity when studying the similar problems. For example, in [16] the influence of wave propagation in an elastic truncated cone is experimentally investigated. The problem of determining the non-stationary wave field of an elastic truncated cone, taking into account its own weight, is investigated in [17]. In [18] the stressed state of the inhomogeneous thin truncated hollow cone is investigated. In [19] obtained an exact solution of the problem of torsion of a truncated hollow cone. In [20] an exact solution of the torsion problem of an elastic truncated layered cone is found in static formulation.

Much less often, similar problems are considered for a twice-truncated elastic cones. This is explained by the mathematical difficulties caused by the geometry of the problem. A thick-walled twice-truncated cone from two-dimensional, functionally graded materials exposed to the combined load is considered at [21]. In [22] the stress state of a twice-truncated cone resting on a rigid base of the lateral surface under a uniform load applied at a larger base is investigated. An axisymmetric problem for a twice-truncated anisotropic cone is solved in [23] with the help of the straight lines method for three-dimensional elasticity equations. The general solution for axisymmetric boundary value prob-lems for a twice-truncated cone is derived in [24]. More complicated, an axially mixed problem for a twice-truncated hollow cone under its own weight was con-sidered in [25]. Significantly less problems are confirmed with an investigation of the dynamic field of conical bodies. An axisymmetric dynamic problem for a twice truncated dynamic cone first was considered at [26], but a lot of unresolved questions connected with eigenvalues investigation still remain.

Main Results

1. Statement of the problem. The twice truncated elastic cone is considered in the spherical coordinate system $a < r < b, -\psi \le \theta \le \psi, -\pi \le \varphi < \pi$.

The problem is stated in case of steady-state oscillations, thus for all mechanical characteristic representation $\bar{f}(r,\theta,\varphi,t) = e^{i\omega t} f(r,\theta,\varphi)$ take place, where ω is the steady state frequency, factor $e^{i\omega t}$ will be omitted in next formulas.

The cone surface $a < r < b, \theta = \psi, -\pi \le \varphi < \pi$ is loaded

$$\tau_{\theta\varphi}\big|_{\theta=\psi} = F(r), \tag{1}$$

where $F(r) = 1/r^2$ is given load.

The upper spherical face of the cone $r = b, -\psi \leq \theta \leq \psi, -\pi \leq \varphi < \pi$ is fixed

$$w|_{r=b} = 0, \tag{2}$$

where $w(r,\theta) = u_\varphi(r,\theta)$ the only nonzero displacement in this problem statement.

The bottom spherical face of the cone $r = a, -\psi \leq \theta \leq \psi, -\pi \leq \varphi < \pi$ is free from stress

$$\tau_{r\varphi}|_{r=a} = 0. \tag{3}$$

One need find the displacement to satisfy the boundary conditions (1)-(3) and the torsion equation

$$(r^2 w')' + \frac{(\sin\theta w^\bullet)^\bullet}{\sin\theta} - \frac{w}{\sin^2\theta} = -r^2 q^2 w, \tag{4}$$

where $w^\bullet = \frac{\partial w(r,\theta)}{\partial\theta}$, $w' = \frac{\partial w(r,\theta)}{\partial r}$, $q = \frac{\omega}{c}$ is the wave number and $c = \sqrt{\frac{G}{\rho}}$ is the shear wave speed, ρ is density and G is the shear modulus.

Fig. 1. Geometry of the problem

2. Deriving the basis solutions of a one dimension boundary problem. The integral G. Ya. Popov transform [27] is applied to problem (1)-(4)

$$w_k(r) = \int_0^\psi \sin\theta P^1_{\nu_k}(\cos\theta) w(r,\theta) d\theta \tag{5}$$

with inverse transform formula

$$w(r,\theta) = \sum_{k=0}^{\infty} \frac{P^1_{\nu_k}(\cos\theta) w_k(r)}{\left\| P^1_{\nu_k}(\cos\theta) \right\|^2}, \tag{6}$$

where $P^1_{\nu_k}(\cos\theta)$ ia associated Legendre's function of the first kind, ν_k are the roots of the transcendental equation

$$\left.\frac{\partial P^1_{\nu_k}(\cos\theta)}{\partial\theta}\right|_{\theta=\psi} - ctg\omega \, P^1_{\nu_k}(\cos\psi) = 0 \tag{7}$$

Thus, a one-dimensional boundary value problem is received in the transform's domain:

$$\begin{aligned} &\left(r^2 w'_k\right)' - \nu_k\left(\nu_k+1\right) w_k - r^2 q^2 w_k = -rF(r)\sin\psi P^1_{\nu_k}(\cos\theta) \\ &w_k\big|_{r=b} = 0 \\ &\tau_{kr\varphi}\big|_{r=a} = \left(w'_k - r^{-1}w_k\right)\big|_{r=a} = 0 \end{aligned} \tag{8}$$

It's basis solution system $\{\Psi_{0k}(r), \Psi_{1k}(r)\}$ [28] has next form

$$\begin{aligned} &\Psi_{0k}(r) = \left(\tfrac{a}{r}\right)^{\frac{1}{2}} W_{\tilde{\nu}_k}(qr, qb)\,\Delta_k^{-1} \\ &\Psi_{1k}(r) = \left(\tfrac{b}{r}\right)^{\frac{1}{2}} \left(-\tfrac{3}{2}a^{-1}W_{\tilde{\nu}_k}(qa, qr)\,\Delta_k^{-1} + W^1_{\tilde{\nu}_k}(qa, qr)\right)\Delta_k^{-1}\ , \\ &\Delta_k = -\tfrac{3}{2}a^{-1}W_{\tilde{\nu}_k}(qa, qb) + W^1_{\tilde{\nu}_k}(qa, qb) \end{aligned} \tag{9}$$

where

$$\begin{aligned} &W_k(x,y) = J_k(x)\,Y_k(y) - J_k(y)\,Y_k(x) \\ &W^1_k(x,y) = J'_k(x)\,Y_k(y) - J_k(y)\,Y'_k(x) \end{aligned}\ , \tag{10}$$

where $J_k(x)$ and $Y_k(x)$ are Bessel's functions of the first and second kind respectively.

The solution of boundary value problem (8) is constructed in form

$$w_k(r) = \int_a^b R(\xi)\,G_k(r,\xi)\,d\xi, \tag{11}$$

where $R(\xi)$ is a right part of differential equation in (8) and $G_k(r,\xi)$ is Green's function [28].

3. Green's function deriving. Green's function is constructed in next form

$$G_k(r,\xi) = \begin{cases} a_0(\xi)\,\Psi_{0k}(r) + a_1(\xi)\Psi_{1k}(r),\, a<r<\xi \\ b_0(\xi)\,\Psi_{0k}(r) + b_1(\xi)\Psi_{1k}(r),\, \xi<r<b \end{cases} \tag{12}$$

where $a_i(\xi)$, $b_i(\xi)$ $i=0,1$ are unknown constants found from four defining properties of Green's function.

$$\begin{aligned} &a_0(\xi) \equiv 0,\ \ b_1(\xi) \equiv 0 \\ &a_1(\xi) = \xi^{-2}\tfrac{\Psi_{0k}(\xi)}{\delta(\xi)} \\ &b_0(\xi) = \xi^{-2}\tfrac{\Psi_{1k}(\xi)}{\delta(\xi)} \\ &\delta(\xi) = \Psi'_{0k}(\xi)\,\Psi_{0k}(\xi) - \Psi_{0k}(\xi)\,\Psi'_{1k}(\xi) \end{aligned} \tag{13}$$

Thus, Green's function become

$$G_k(r,\xi) = \xi^{-2}\begin{cases} \frac{\Psi_{0k}(\xi)\Psi_{1k}(r)}{\delta(\xi)},\, a<r<\xi \\ \frac{\Psi_{1k}(\xi)\Psi_{0k}(r)}{\delta(\xi)},\, \xi<r<b \end{cases}\ . \tag{14}$$

Considering basis solutions (9) one can obtain next form of green function

$$G_k\left(r,\xi\right)=-\frac{\pi q}{2\xi r}\begin{cases}\frac{F_k(\xi,r)}{\Delta_k}, a<r<\xi\\ \frac{F_k(r,\xi)}{\Delta_k}, \xi<r<b\end{cases}, \tag{15}$$

where Δ_k is known from (9) and

$$F_k(\xi,r)=W_{\tilde{\nu}_k}\left(q\xi,qb\right)\times\left(-\frac{3}{2}a^{-1}W_{\tilde{\nu}_k}\left(qa,qr\right)+W^1_{\tilde{\nu}_k}\left(qa,qr\right)\right) \tag{16}$$

4. The final calculation formulas construction. G. Ya. Popov inverse integral transform is applied to (11). Thus, solution can be written in next form

$$w_k\left(r,\theta\right)=\sin\psi\frac{\pi q}{2\sqrt{r}}\sum_{k=0}^{\infty}\frac{P^1_{\nu_k}(\cos\theta)P^1_{\nu_k}\left(\cos\psi\right)}{\left\|P^1_{\nu_k}(\cos\theta)\right\|^2}\int\limits_a^b\xi^{\frac{3}{2}}\bar{G}\left(r,\xi\right)d\xi, \tag{17}$$

where

$$\bar{G}_k\left(r,\xi\right)=\begin{cases}\frac{F_k(\xi,r)}{\Delta_k}, a<r<\xi\\ \frac{F_k(r,\xi)}{\Delta_k}, \xi<r<b\end{cases}. \tag{18}$$

Next is shown asymptotic procedure which is used to find behavior of sum in (17).

Let $f_k(k)$ be the function for which $F_k(x)$ it's asymptotic form when k is big enough. An infinite sum can be represented in form of two addends $\sum_{k=0}^{\infty}f_k\left(x\right)=\sum_{k=0}^{N}f_k\left(x\right)+\sum_{k=N+1}^{\infty}f_k\left(x\right)$, where N is big enough. In second addend function can be replaced with it's asymptotic representation. Thus sum can be rewritten in next form

$$\sum_{k=0}^{\infty}f_k\left(x\right)\approx\sum_{k=1}^{\infty}F_k\left(x\right)+\sum_{k=1}^{N}\left(f_k\left(x\right)-F_k\left(x\right)\right)+f_0\left(x\right) \tag{19}$$

After applying this procedure to displacement one can differentiate them to receive unknown stress.

5. Numeric results discussion. From the point of view of mechanical applications, one of the most important goals is to find the eigenfrequencies. The solving of the transcendental equation is required

$$D\left(\omega\right)=\prod_{k=0}^{N}\left(\left(\frac{c\tilde{\nu}_k}{a}-\frac{3\omega}{2a}\right)W_{\tilde{\nu}_k}\left(\omega\frac{a}{c},\omega\frac{b}{c}\right)-\omega\widetilde{W}^1_{\tilde{\nu}_k}\left(\omega\frac{a}{c},\omega\frac{b}{c}\right)\right), \tag{20}$$

where

$$\widetilde{W}^1_k\left(x,y\right)=J_{k+1}\left(x\right)Y_k\left(y\right)-J_k\left(y\right)Y_{k+1}\left(x\right). \tag{21}$$

The following input parameters were selected for the calculation: $N=5$, $G=45.5\cdot10^10\,g/cm/s^2$, $\rho=8.92\,g/cm^3$, $a=10\,cm$, $b=3a$, $c=2.26\cdot10^5\,cm/s$.

For three different cone angles $\psi=15°,45°,75°$ the first five eigenfrequencies are shown in Table 1.

ψ	Ω_i				
15°	1.534381498	3.287793796	5.213091256	7.179694419	9.160953572
45°	1.534381498	3.287793796	4.633579360	5.213091256	6.196233245
75°	1.534381498	3.204860521	3.287793796	4.578468556	4.589225785

Table 1. Eigenfrequencies dependence from cone's opening angle

Here $\Omega_i = \dfrac{2\omega_i l}{\pi c}$ and $l = b - a$.

In Table 2 one can see how changing the cone size influences the values of first eigenfrequencies ($\psi = 45°$).

b	1.5a	2a	3a	10a
Ω_1	1.202871561	1.321210018	1.534381498	2.139666391

Table 2. Eigenfrequencies dependence from cone's linear size

In Fig. 2 one can see values $\tau_\psi = \frac{\tau_{r\varphi}(b,\theta)}{G}$ where $0 \leq \theta \leq \psi$ and $w_\psi = w(r, \psi)$ where $a < r < b$ for the two cone angles $\psi = 45°$ and $\psi = 75°$.

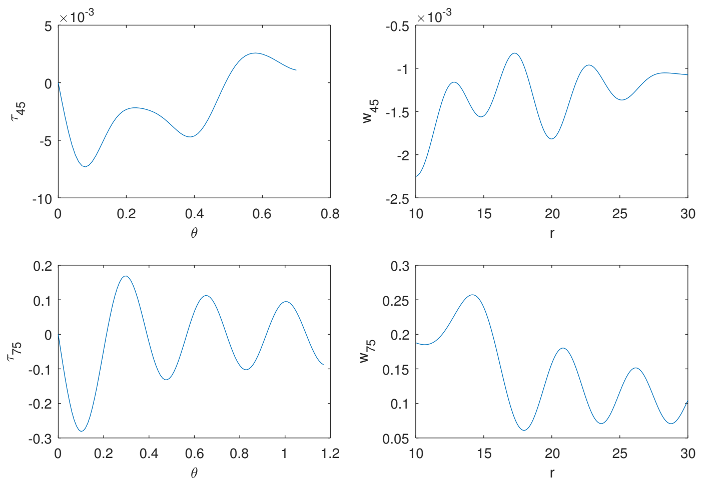

Fig. 2. The values of stress and displacement for opening angles 45° and 75°.

Conclusion. The explicit formulae for stress and displacement fields of an elastic twice truncated cone under dynamic torsion are derived in this paper. The dependencies of the eigenfrequencies on cones geometric parameters was stated. Comparison of eigenfrequencies was made to the results in [26]. In case when load is applied on the lateral surface, the first eigenfrequencies are lower than ones when load is applied through an overlay to bottom face. The proposed approach can be used in case when an elastic twice-truncated cone is weakened by a spherical crack.

Мисов К. Д.
Проблема кручення еластичного двічи-усіченного конуса

Резюме

Досліджено задачу визначення хвильового поля пружного двічі-зрізаного конуса у випадку встановлених коливань. Застосовується інтегральне перетворення Г. Я. Попова, відносно кутової координати. Таким чином, вихідна задача зводиться до одновимірної крайової задачі в області трансформант. Функція Гріна побудована для одновимірної крайової задачі. З її допомогою розв'язок одновимірної проблеми побудовано точно. Обернене перетворення Г. Я. Попова допомогло отримати розв'язок в оригінальному просторі у формі нескінченної суми. З його допомогою досліджена залежність власних частот від геометричних показників конуса. Поле напружень було знайдено за допомогою асимптотичної процедури. Графіки порівняння побудовані для різних кутів отвору.
Ключові слова: двічи-зрізаний конус, встановленні коливання, інтегральне перетворення Г. Я. Попова, власні частоти, хвильове поле, функція Гріна .

Мысов К. Д.
Проблема кручения эластичного дважды-усеченного конуса

Резюме

Исследована задача определения волнового поля упругого дважды-усеченного конуса в случае установившихся колебаний. Применяется интегральное преобразование Г. Я. Попова относительно угловой координаты. Таким образом исходная задача сводиться к одномерной краевой задаче в области трансформант. Функция Грина построена для одномерной краевой задаче. С ее помощью решение одномерной краевой задачи построено точно. Обратное преобразование Г. Я. Попова помогло получить решение в пространстве оригиналов в форме бесконечной суммы. С его помощью исследована зависимость собственных частот от геометрических показателей конуса. Поле напряжений было построено с помощью асимптотической процедуры. Графики сравнений построены для разных углов раствора.
Ключевые слова: дважды-усеченный конус, установившиеся колебания, интегральное преобразование Г. Я. Попова, собственные частоты, волновое поле, функция Грина .

References

1. Yong-Tie Xu, Li-Li Zhang. (2012). *3D Green's function for a transversely isotropic thermoelastic cone. Elsevier. Applied Mathematical Modelling*, Vol. 36, P. 5891–5900.
2. William, T. Chen. (1965). *Stresses in a transversely isotropic elastic cone under an asymmetric force at its vertex. ZAMP*, Vol. 16, P. 337–344.
3. Bhargava, R. D., Gupta S. C. (1968). *Stresses in an elastic cone due to an axial force at a point on the axis. Acta Mechanica*, Vol. 6, P. 255–274.
4. Roger, D. low, Harry, J. Weiss. (1962). *On a mixed boundary value problem for an infinite elastic cone. ZAMP*, Vol. 13, P.232–242.
5. Sabodash, P. F. (1989). *Elastic transverse waves initiated in an infinite medium by rotation of a circular cone. Soviet Applied Mechanics*, Vol. 25, P. 797–801.
6. Vasilenko, A. T., Pankratova N. D. (1987). *Solution of problems on the elastic equilibrium of an anisotropic inhomogeneous cone. Soviet Applied Mechanics*, Vol. 23, P. 1141–1147.

7. Gao, Y. C., Chen, S. H. (2000). *Asymptotic analysis and finite element calculation of a rubber cone under tension. Acta Mechanica*, Vol. 141, P. 149–159.
8. Popov, G., Vaisfel'd, N. (2014). *The torsion of the conical layered elastic cone. Acta Mechanica*, Vol. 225, P.67–76.
9. Thompson, T. R. (1970). *End effects in a truncated semi-infinite cone. Quart. J. Mech. Appl. Math.*, Vol. 23, P. 185–196.
10. Korepanova, T. O., Matveenko, V. P., Sevodina, N. V. (2013). *Numerical analysis of stress singularity at singular point of three-dimensional elastic bodies. Acta mechanica*, Vol. 224, P. 2045–2063.
11. Khomosuridze, N. G. (2003). *The thermoelastic equilibrium of the conical bodies. Appl. Maths Mechs*, Vol. 67, P. 111–120.
12. Rossikhin, Y. A., Shitikova, M. V. (2004). *The influence of the initial stresses on the dynamic instability of an anisotropic cone. Kluwer academic publisher*, Vol. 163, P.257–270.
13. Popov, G., Vaysfel'd, N. (2009). *The stress concentration in the neighborhood of the spherical crack inside the infinite elastic cone. Modern Analysis and Applications*, Vol. 191, P. 173–186.
14. Popov, G., Vaysfel'd, N. (2011). *The steady-state oscillations of the elastic infinite cone loaded at vertex by a concentrated force. Acta Mechanica*, Vol. 221, P. 261–270.
15. Vaysfel'd, N. D. (2002). *Nonstationary problem of torsion for an elastic cone with spherical crack. Materials Science*, Vol. 38, P. 698–708.
16. Kenner, V. H., Goldsmith, W. (1968). *Elastic waves in truncated cone. Experimental Mechanics*, Vol. 8, P. 442–449.
17. Kebli, B., Popov, G. Ya., Vaysfel'd N. D. (2011). *Dynamics of a truncated elastic cone. International Applied Mechanics*, Vol. 46, P. 1284–1291.
18. Akhmedov, N. K., Mekhtiyev, M. F. (1993). *Analysis of a three-dimensional problem of the theory of elasticity for an inhomogeneous truncated hollow cone. Journal of Applied Mathematics and Mechanics*, Vol. 57, P. 871–877.
19. Popov, G. Ya. (2012). *Torsion of an infinite truncated hollow elastic cone. Doklady Physics*, Vol. 57, P. 492–496.
20. Popov, G. Ya., Vaisfel'd N. D. (2014). *Torsion of a truncated conically layered elastic cone. Journal of Mathematical Sciences*, Vol. 203, P. 135–148.
21. Kamran, A., Manouchehr, S., Mehdi, A. (2011). *Elastic solution of a two dimensional functionally graded thick truncated cone with finite length under hydrostatic combined loads. Acta Mechanica*, Vol. 217, P. 119–134.
22. Savchenko, V. I., Shokot'ko, S. G., Uspenskii, A. A. (1977) *Study of the stress-strain state of truncated elastic cones. Strength of Materials*, Vol. 9, P. 817–821.
23. Uspemskii, A. A. (1977). *State of stress of an anisotropic cone under an axisymmetric load. Soviet Applied Mechanics*, Vol. 13, P. 436–440.
24. Popov, G. Ya. (2005). *On the axisymmetrical problems of elasticity for the truncated hollow cone. Journal of Applied Mathematics and Mechanics*, Vol. 69, P. 417–426.
25. Vaisfel'd, N. D., Popov, G. Ya., Reut, A. V. (2014). *Axisymmetric problem of stressed state for a twice truncated cone. Journal of Mathematical Sciences*, Vol. 201, P. 229–244.
26. Mysov, K., Vaysfeld, N. (2018). *Steady state torsion of twice truncated elastic cone. Young Scientist* Vol. 10, P. 118–121.

27. Popov, G. Ya. (2003). *New transforms for the resolving equations in elastic theory and new integral transforms, with applications to boundary-value problems of mechanics. International Applied Mechanics*, Vol. 39(12), P. 1400–1424.

28. Popov, G. Ya., Abdimanapov, S.A., Efimov, V.V. (1999). *Green's functions and matrix of one-dimensional boundary value problems.* Almati: Rauan, 113 p.