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ON EXPONENTIAL SUMS CONTAINING THE DIVISOR FUNCTION OF THE GAUSSIAN INTEGERS

В роботі знайдено асимптотичні формули тригонометричних сум спеціального вигляду, які містять функцію дільників цілих гаусових чисел. Ці результати узагальнюють дослідження М. Ютіли.

В работе найдены асимптотические формулы тригонометрических сумм специального вида, содержащих функцию делителей целых гауссовых чисел. Эти результаты обобщают исследования М. Ютилы.

Asymptotic formulas are found for the trigonometric sums of the special form containing the divisor function of Gaussian integers. These results generalize the investigations of M.Jutila.

1. Introduction and preliminary results. Let $\tau(\alpha)$ denote the divisor function of Gaussian integers, $\tau(\alpha)=\sum\limits_{\delta\backslash\alpha,\ \delta\in Z[i]} 1$.

The object of this paper is to consider sums of the form

$$\sum_{\alpha\in G}\tau(\alpha)g(\alpha), \tag{1.1}$$

where $g(\alpha)$ is a weight function.

G. Voronoy [1], [2] developed a method of summation for the values of an arithmetical function defined for $n\in N$. M.Jutila [3], [4] and T.Meurman [5] generalized Voronoy's method and considered the exponential sums of type $\sum_{n\leq\eta} f(n)g(n)$, where $f(n)$ is the divisor function $\tau(n)$ or the Fourier coefficient $a(n)$ of a modular form and $g(n)=\exp(2\pi i\, hn/q)$, $(h,q)=1$.

We investigate similar sums in the ring of the Gaussian integers in the case

$$g(\alpha)=\exp\left(2\pi i\operatorname{Re}\left((\alpha\beta)\gamma^{-1}\right)\right) \text{ or } g(\alpha)=K(\gamma;1,\alpha)=\sum_{\substack{\delta(\operatorname{mod}\gamma)\\ \delta\delta'\equiv 1(\operatorname{mod}\gamma)}}\exp\left(2\pi i\operatorname{Re}\left((\delta+\alpha\delta')\gamma^{-1}\right)\right).$$

The sum $\sum_{N(\alpha)\leq x}\exp\left(2\pi i\operatorname{Re}\left((\alpha\beta)\gamma^{-1}\right)\right)$ can be used for a study of the summatory function

$$\sum_{\alpha\equiv\delta \operatorname{mod}\gamma,\ N(\alpha)\leq x}\tau(\alpha), \tag{1.2}$$

and the sum $\sum_{N(\alpha)\leq x}\tau(\alpha)K(\gamma;1,\alpha)$ appears in the representation of the error term of the asymptotic formula for (1.2).

In the present paper, standard notation is used. $\Gamma(s)$ is the gamma-function; E is Euler's constant; ε is a positive constant, not necessarily always the same, which can be taken arbitrarily small;

$f(x)=O(g(x))$ and $f(x)\ll g(x)$ both mean $|f(x)|<Cg(x)$ for some $C>0$ and $x\geq x_0$, $e(x)=\exp(2\pi ix)$; $\bar\delta$ denote that $\delta\cdot\bar\delta\equiv 1\pmod\beta$ for corresponding β;

$Q(i)$, $Z[i]$ denote the field of Gaussian numbers and the ring of Gaussian integers; $N(\alpha)$ is a norm of α, $N(\alpha)=|\alpha|^2$. For $\alpha,\beta\in Z[i]$ (α,β) denotes the greatest common divisor of α and β.

$$\xi_m(s;\alpha,\beta)=\sum_{\substack{w\in Z[i]\\ w\neq-\alpha}} e\left(2m\pi^{-1}\arg(w+\alpha)+\operatorname{Re}(\beta w)\right)\cdot N(w+\alpha)^{-s}.$$

Lemma1. If $m\neq 0$, then $\xi_m(s;\alpha,\beta)$ is an entire function. If $m = 0$, and β is not a Gaussian integer, then the function $\xi_m(s;\alpha,\beta)$ is entire; if $m = 0$ and β is a Gaussian integer, then $\xi_m(s;\alpha,\beta)$ holomorphic except at $s = 1$, where it has a simple pole with the residue π.

The functional equation

$$\pi^{-s}\Gamma\left(\tfrac{1}{2}|m|+s\right)\xi_m(s;\alpha,-\beta)=\pi^{-(1-s)}\Gamma\left(\tfrac{1}{2}|m|+1-s\right)\xi_m(1-s;\beta,\alpha)e(-\operatorname{Re}(\alpha\beta))$$

holds in both cases.

For a proof in the case $\xi_m(s;0,0)$, see [6]. The proof in the general case is similar.

Let $\alpha_0,\gamma\in Z[i]$, $(\alpha_0,\gamma)=1$. For $\operatorname{Re} s>1$, we consider the series

$$F_m\left(s;\alpha_0\gamma^{-1}\right)=\sum_{\substack{w\in Z[i]\\ N(w)\neq 0}}\tau(w)N(w)^{-s}e\left(2m\pi^{-1}\arg w+\operatorname{Re}\left(\alpha_0 w\gamma^{-1}\right)\right).$$

Lemma2. The function $F_m\left(s;\alpha_0\gamma^{-1}\right)$ is entire for $m\neq 0$. For $m = 0$, it is holomorphic except at $s = 1$, where it has a second order pole with an expansion

$$F_0\left(s;\alpha_0\gamma^{-1}\right)=\pi^2N(\gamma)^{-1}(s-1)^{-2}+\pi N(\gamma)^{-1}\left(2E\pi+8L'(1,\chi_4)-2\pi\log N(\gamma)\right)(s-1)^{-1}+A_0+\cdots$$

Moreover, the functional equation

$$\pi^{-2s}\Gamma^2\left(\tfrac{1}{2}|m|+s\right)N(\gamma)^s F_m\left(s;\alpha_0\gamma^{-1}\right)=\pi^{-2(1-s)}\Gamma^2\left(\tfrac{1}{2}|m|+1-s\right)N(\gamma)^{1-s}F_m\left(1-s,-\bar{\alpha}_0\gamma^{-1}\right)$$

holds in both cases. This lemma follows from Lemma 1.

Corollary. Let $0<\varepsilon_1,\varepsilon_2<1$. Then, in a rectangle $-\varepsilon_1\leq\operatorname{Re}s\leq 1+\varepsilon_2$, we have

$$(s-1)^2F_m\left(s;\alpha_0\gamma^{-1}\right)\ll\varepsilon_2^{-\frac{\sigma+\varepsilon_1}{1+\varepsilon_1+\varepsilon_2}}\left(1+m^2+t^2\right)^{\frac{1+\varepsilon_2-\sigma}{1+\varepsilon_1+\varepsilon_2}}\left(1+t^2\right);\quad F_m\left(0;\alpha_0\gamma^{-1}\right)\ll 1+m^2.$$

2. An exponential sum containing the divisor function. Now we consider the sum

$$D\left(x;\alpha_0\gamma^{-1}\right)=\sum\nolimits_{0<N(\alpha)\leq x}\tau(\alpha)e\left(\operatorname{Re}\left(\alpha_0\alpha\gamma^{-1}\right)\right).$$

By Perron's formula, we get

$$D\left(x;\alpha_0\gamma^{-1}\right)=(2\pi i)^{-1}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT}F_0\left(s;\alpha_0\gamma^{-1}\right)x^s s^{-1}ds+O\left(x^{1+\varepsilon}T^{-1}\varepsilon^{-2}\right), \tag{2.1}$$

To transform the integral in (2.1), we move the segment of integration to $\operatorname{Re}(s)=-\varepsilon_1$. By the residue theorem, Lemma 2, and Corollary, we get that

$$D\left(x;\alpha_0\gamma^{-1}\right)=\pi^2x\log xN(\gamma)^{-1}+2\pi x\left(E\pi+4L'(1,\chi_4)-\tfrac{1}{2}\pi-\pi\log N(\gamma)\right)N(\gamma)^{-1}+$$

$$+(2\pi i)^{-1}\int_{-\varepsilon_1-iT}^{-\varepsilon_1+iT}F_0\left(s;\alpha_0\gamma^{-1}\right)x^s s^{-1}ds+O\left(x^{1+\varepsilon}T^{-1}\varepsilon^{-2}\right).$$

By Stirling's formula for $-1<\operatorname{Re}s\leq 1/2$, $|\operatorname{Im}s|\to\infty$ we have

$$\Gamma^2(1-s)\Gamma^{-2}(s)s^{-1}=(2\pi)^{-1/2}\Gamma(3/2-4s)e^{2\pi i\left(s-\frac{1}{2}\right)}\left(1+O\left(|s|^{-1}\right)\right).$$

Therefore from the functional equation for $F_0\left(s;\alpha_0\gamma^{-1}\right)$, we get

$$(2\pi i)^{-1}\int_{-\varepsilon_1-iT}^{-\varepsilon_1+iT} F_0\left(s;\alpha_0\gamma^{-1}\right)x^s s^{-1}ds =$$

$$= \frac{\pi^{2/3}N(\gamma)e^{(-1/8)}}{4\sqrt{2}}\left(\sum_{\omega}\tau(\omega)y^{3/8}N(\omega)^{-1}I(\omega)+O\left(T^{1+\varepsilon}y^{-\varepsilon_1}N(\omega)^{-1}\right)\right) \tag{2.2}$$

where $y=\pi^4 xN(\omega)N^{-2}(\gamma)$, $I(\omega)=(2\pi i)^{-1}\int_{3/2+4\varepsilon_1-iT}^{3/2+4\varepsilon_1+iT}\Gamma(s)e^{-\pi is/2}y^{-s/4}ds$

Now we have

$$I(\omega)=\frac{1}{2\pi i}\left(\int_{1/2-i\infty}^{1/2+i\infty} - \int_{1/2-i\infty}^{1/2-4iT} - \int_{1/2-i4T}^{3/2+4\varepsilon_1-i4T} - \int_{3/2+4\varepsilon_1+iT}^{1/2+i4T} - \int_{1/2-i4T}^{1/2+i\infty}\right) = A_1 - A_2 - A_3 - A_4 - A_5 .$$

Since $\int_0^{\infty}e^{-iy}y^{s-1}dy=\Gamma(s)e^{-\frac{\pi is}{2}}$, $(\mathrm{Re}(s)>0)$, then, by Mellin's transformation,

$$A_1=(2\pi i)^{-1}\int_{1/2-i\infty}^{1/2+i\infty}\Gamma(s)e^{-\pi is/2}\left(y^{1/4}\right)^{-s}ds=e^{-\pi iy/4}, \tag{2.3}$$

The Integrals A_2, A_3, A_4, A_5, can be estimated by use of Lemmas 1.4.3 and 1.4.5 in [7]:

$$|A_2|,|A_5| \ll y^{-1/8}\min\left(\left(\log XN(\omega)^{-1}\right)^{-1},T^{1/2}\right), \tag{2.4}$$

$$|A_3|,|A_4| \ll T^{1+4\varepsilon_1}N(\gamma)^{3/4+\varepsilon_1}\cdot(xN(\omega))^{-3/8-\varepsilon_1}, \tag{2.5}$$

Therefore, for $N(\omega)\le X=(4/\pi)^4T^4N^2(\gamma)x^{-1}$ we get

$$I(\omega)=e^{-\pi iy^{1/4}}+O\left(y^{-1/8}\min\left(\left(\log\frac{X}{N(\omega)}\right)^{-1},T^{1/2}\right)\right)+O\left(T^{1+4\varepsilon_1}\left(\frac{N(\gamma)^2}{xN(\omega)}\right)^{3/8+\varepsilon_1}\right) \tag{2.6}$$

Moreover, we have that

$$I(\omega)\ll y^{-3/8-\varepsilon_1}T^{1+4\varepsilon_1}\min\left(\left(\log XN(\omega)^{-1}\right)^{-1},T^{1/2}\right)\log T \tag{2.7}$$

if $N(\omega)>X$. We put $\varepsilon_1=(\log x)^{-1}$ and obtain

$$D_0\left(x;\alpha_0\gamma^{-1}\right)=\pi^2x\log xN(\gamma)^{-1}+2\pi x\left(E\pi+4L'(1,\chi_4)-\tfrac{1}{2}\pi-\pi\log N(\gamma)\right)N(\gamma)^{-1}+$$

$$+\pi^3 2^{-5/2}x^{3/8}N(\gamma)^{1/4}\sum_{N(\omega)\le X}\tau(\omega)e\left(\frac{1}{2}\left(xN(\omega)N(\gamma)^{-2}\right)^{1/4}+\mathrm{Re}\left(\alpha_0\omega\gamma^{-1}\right)\right)\cdot N(\omega)+ \tag{2.8}$$

$$+O(TN(\gamma)\log x)+O\left(x\log^2 xT^{-1}\right).$$

Thus, we obtain an analogue of a truncated form of the Voronoy identity for $\Delta\left(x;\alpha_0\gamma^{-1}\right)$, where

$$\Delta\left(x;\alpha_0\gamma^{-1}\right)=D_0\left(x;\alpha_0\gamma^{-1}\right)-\pi^2x\log xN(\gamma)^{-1}-2\pi x\left(E\pi+4L'(1,\chi_4)-\tfrac{1}{2}\pi-\pi\log N(\gamma)\right)N(\gamma)^{-1}.$$

From (2.8), we derive the following result:

Theorem 1. *For* $x\to\infty$, $\Delta\left(x;\alpha_0\gamma^{-1}\right)=O\left(x^{3/5}N(\gamma)^{2/5}\log^2 x\right)$.

Also, for $m\neq 0$,

$$D_m\left(x;\alpha_0\gamma^{-1}\right)=\sum_{N(\alpha)\le x}\tau(\alpha)e\left(2m\pi^{-1}\arg\alpha+\mathrm{Re}\left(\alpha_0\alpha\gamma^{-1}\right)\right)\ll N(\gamma)^{1+\varepsilon}m^{2+\varepsilon}+x^{3/5}N(\gamma)^{2/5}\log^2 x.$$

And then the standard method yields the following

Theorem2. *For* $\varphi_2-\varphi_1\ge x^{-2/5-\varepsilon}N(\gamma)^{7/5}$

$$\sum_{\substack{\varphi_1<\arg\alpha\le\varphi_2\\ N(\alpha)\le x}}\tau(\alpha)e\left(\mathrm{Re}\left(\alpha_0\alpha\gamma^{-1}\right)\right)=(\varphi_2-\varphi_1)(2\pi)^{-1}\times$$

$$\times\left(\pi^2 x\log xN(\gamma)^{-1}+2\pi x\left(E\pi+4L'(1,\chi_4)-\frac{1}{2}\pi-\pi\log N(\gamma)\right)N(\gamma)^{-1}\right)+O\left(x^{\frac{3}{5}}N(\gamma)^{\frac{2}{5}}\log^2 x\right).$$

3. The divisor function weighting by Kloosterman sum. We consider Kloosterman's sum in $Z[i]$, $\mathrm{K}(\gamma;\alpha,\beta)=\sum_{\substack{\delta(\mathrm{mod}\,\gamma)\\ \delta\delta'\equiv1(\mathrm{mod}\,\gamma)}}e\left(\mathrm{Re}\left((\alpha\delta+\beta\delta')\gamma^{-1}\right)\right).$

If $(\alpha,\gamma)=1$ (or $(\beta,\gamma)=1$) then $\mathrm{K}(\gamma;\alpha,\beta)=\mathrm{K}(\gamma;1,\alpha\beta)=\mathrm{K}(\gamma;\alpha\beta,1)$. (3.1)

If $\gamma=\gamma_1\cdot\gamma_2,(\gamma_1,\gamma_2)=1$ then $\mathrm{K}(\gamma;\alpha,\beta)=\mathrm{K}(\gamma_1;\alpha,\beta_1)\cdot\mathrm{K}(\gamma_2;\alpha,\beta_2)$, (3.2)

where β_1,β_2 satisfy the congruence condition $\beta\equiv\beta_1\gamma_2^{\,2}+\beta_2\gamma_1^{\,2}(\mathrm{mod}\,\gamma)$. (3.3)

Let us now establish an asymptotic formula for the sum $\mathrm{K}(x,\gamma)=\sum_{N(\omega)\le x}\tau(\omega)\cdot\mathrm{K}(\gamma;1,\omega)$.

We first prove the following statement:

Lemma. *Let* $\gamma=\gamma_1\cdot\gamma_2,(\gamma_1,\gamma_2)=1$ *where* γ_1 *be squarefree and* γ_2 *be squarefull numbers. Let* $(w,\gamma)=\delta_1\delta_2$ *where* $\delta_1|\gamma_1,\delta_2|\gamma_2$. *Then for* $N((\delta_1,\delta_2))>1$we have

$$\mathrm{K}(\gamma;1,\omega)=\begin{cases}\mu(\delta_1)\mathrm{K}\left(\gamma\delta_1^{-1};1,\omega\delta_1^{\,2}\right) & \text{if } N(\delta_2)=1\\ 0, & \text{if } N(\delta_2)>1\end{cases}.\qquad(3.4)$$

Proof. By (3.2), it suffice to show that

$\mathrm{K}(\gamma_1;1,\omega)=\mu(\delta_1)\mathrm{K}\left(\gamma\delta_1^{-1};1,\omega\delta_1^{\,2}\right)$ if $(\omega,\gamma_1)=\delta_1$, $N(\delta_1)>1$

and $\mathrm{K}(\gamma;1,\omega)=0$, if $(\omega,\gamma_2)=\delta_2$, $N(\delta_2)>1$.

Let $(w,\gamma_2)=\delta_2,N(\delta_2)>1$. Then by (3.2) it suffice to consider the case where $\gamma_2=\rho^m,m>1,\delta_2=\rho^l,l\ge1,\rho$ is a Gaussian prime.

Let $k=[m/2]$ and let $\alpha=\alpha_0\left(1+\gamma^{m-k}\beta\right)$, where α_0 runs through the reduced system of classes $\left(\mathrm{mod}\,\rho^{m-k}\right)\beta$ runs $\left(\mathrm{mod}\,\rho^k\right)$.

If $\alpha\cdot\alpha'\equiv1\left(\rho^m\right)$, then $\alpha'=\bar{\alpha}_0\left(1-\rho^{m-k}\beta\right),\alpha_0\bar{\alpha}_0\equiv1\left(\mathrm{mod}\,\rho^m\right)$.
Hence,

$$K\left(\rho^m;1,w\right)=\sum_{\substack{\alpha\left(\mathrm{mod}\,\rho^m\right)\\ \alpha\alpha'\equiv1\left(\mathrm{mod}\,\rho^m\right)}}e\left(\mathrm{Re}\frac{\alpha+w\alpha'}{\rho^m}\right)=\sum_{\substack{\alpha_0\left(\rho^{m-k}\right)\\ (\alpha_0,\rho)=1}}\sum_{\beta\left(\rho^k\right)}e\left(\mathrm{Re}\frac{\alpha_0\left(1+\rho^{m-k}\beta\right)+\bar{\alpha}_0\left(1-\rho^{m-k}\beta\right)}{\rho^m}\right)=$$

$$=\sum_{\substack{\alpha_0\left(\rho^{m-k}\right)\\ (\alpha_0,\rho)=1}}e\left(\mathrm{Re}\left(\left(\alpha_0+\bar{\alpha}_0w\right)\rho^{-m}\right)\right)\sum_{\beta\left(\rho^k\right)}e\left(\mathrm{Re}\left(\left(\alpha_0-\alpha_0w\right)\rho^{-k}\right)\right)=0.$$

Thus we have proved that $K(\gamma_2;1,w)=0$ if $(w,\gamma_2)\neq 1$. Let now $(w,\gamma)=(w,\gamma_1)=\delta_1$, $N(\delta_1)>1$. We denote $\gamma_1=\rho_1\cdots\rho_r$, $\delta_1=\rho_1\cdots\rho_l$, $1\le l\le r$, $\rho_1,\cdots,\rho_r$ are Gaussian prime. Then

$$K(\gamma;1,w)=K\left(\delta_1;1,w\left(\gamma\delta_1^{-1}\right)^2\right)\cdot K\left(\gamma\delta_1^{-1};1,w\delta_1^2\right).$$

Moreover, for $w=\rho w_1$ we have $K(\rho;1,w)=\sum\limits_{\substack{t(\bmod \rho)\\(t,\rho)=1}} e(\operatorname{Re}(t/\rho))=-1=\mu(\rho)$.

Hence we obtain

$$K(\gamma;1,\omega)=\prod_{j=1}^{l}K\left(\rho_j;1,\omega\left(\gamma\bar{\rho}_j\delta_1^{-1}\right)^2\right)\cdot K\left(\gamma\delta_1^{-1};1,\omega\bar{\delta}_1^{2}\right)=\mu(\delta_1)K\left(\gamma\delta_1^{-1};1,\omega\bar{\delta}_1^{2}\right).$$

Corollary. Let γ be a Gaussian integer, $\gamma=\gamma_1\gamma_2$, where γ_1 is squarefree and γ_2 squarefull, $(\gamma_1,\gamma_2)=1$. Then the following relation holds for $\operatorname{Re} s>1$:

$$\sum_{\substack{w\in Z[i]\\ w\neq 0}}\tau(w)K(\gamma;1,w)N(w)^{-s}=\sum_{\delta|\gamma_1}\mu(\delta)\sum_{\substack{w\\(w,\gamma)=\delta}}\tau(w)N(w)^{-s}K\left(\frac{\gamma}{\delta};1,w\bar{\delta}^2\right), \tag{3.5}$$

Let us now prove the following statement:

Theorem 3. *Let* $\gamma\in Z[i]$, $N(\gamma)>1$. *Then*

$$\sum_{N(w)\le x}\tau(w)K(\gamma;1,w)=\frac{\pi^2\mu(\gamma)C_0(\gamma)\tau(\gamma)x\log x}{N(\gamma)}+\frac{\mu(\gamma)C_1(\gamma)x}{N(\gamma)}+O\left(x^{\frac{3}{5}}N(\gamma)^{\frac{7}{10}}\tau^3(\gamma)\log x\right),$$

where $C_0(\gamma)=\prod_{\delta|\gamma_1}\left(2-\phi(\gamma)N^{-2}(\gamma)\right)$, $1\le C_0(\gamma)\le 2^{\nu(\gamma_1)}$,

$$C_1(\gamma)=\pi^2\left(2E-1+8L'(1,\chi_4)/\pi\right)C_0(\gamma)-2\tau(\gamma)\sum_{\delta|\gamma_1}\frac{\mu(\delta)\phi(\delta)}{\tau(\delta)N(\delta)^2}\log N(\delta)$$

Proof. Let $\delta|\gamma_1$. Then $(\delta,\gamma\delta^{-1})=1$. We have

$$\sum_{\substack{\alpha_1,\alpha_2\left(\bmod \frac{\gamma}{\delta}\right)\\ \alpha_1\alpha_2\equiv 1(\gamma/\delta)}}\ \sum_{\substack{\delta_1,\delta_2\\ \delta_1\delta_2=\delta}}\ \sum_{\substack{w_1\in Z[i]\\(w_1,\gamma/\delta)=1}} e\left(\operatorname{Re}\left(\frac{\alpha_1 w_1\delta_1\bar{\delta}}{2\gamma/\delta}\right)\right)\left(N(w_1)N(\delta_1)\right)^{-s}$$

$$\sum_{\substack{w_2\in Z[i]\\(w_2,\gamma/\delta)=1}} e\left(\operatorname{Re}\left(\frac{\alpha_2 w_2\delta_2\bar{\delta}}{2\gamma/\delta}\right)\right)\left(N(w_2)N(\delta_2)\right)^{-s}=\sum_{\substack{w\\(w,\gamma)=\delta}}N(w)^{-s}\sum_{\delta_1\delta_2=\delta}\ \sum_{\substack{w_1,w_2\\ \delta_1w_1\cdot\delta_2w_2=w}}$$

$$\sum_{\substack{\alpha_1,\alpha_2(\bmod \gamma/\delta)\\ \alpha_1\alpha_2\equiv 1(\bmod \gamma/\delta)}} e\left(\operatorname{Re}\left(\left(\alpha_1\delta_1w_1\bar{\delta}+\alpha_2\delta_2w_2\bar{\delta}\right)(\gamma/\delta)^{-1}\right)\right)=\sum_{\substack{w\\(w,\gamma)=\delta}}\tau(w)N(w)^{-s}K\left(\gamma\delta^{-1};1,w\bar{\delta}^2\right) \tag{3.6}$$

On the other hand, the left-hand side (3.6) is

$$\sum_{\substack{\alpha_1,\alpha_2\left(\bmod \frac{\gamma}{\delta}\right)\\ \alpha_1\alpha_2\equiv 1(\gamma/\delta)}}\ \sum_{\substack{\delta_1,\delta_2\\ \delta_1\delta_2=\delta}}\ \sum_{w_1} e\left(\operatorname{Re}\frac{\alpha_1w_1\delta_1\bar{\delta}}{\gamma/\delta}\right)N(w_1)^{-s}$$

$$\sum_{\beta_1|(w_1,\gamma/\delta)}\mu(\beta_1)\sum_{w_2} e\left(\operatorname{Re}\frac{\alpha_2w_2\delta_2\bar{\delta}}{\gamma/\delta}\right)N(w_2)^{-s}\sum_{\beta_2|(w_2,\gamma/\delta)}\mu(\beta_2)=$$

$$= \sum_{\beta_1,\beta_2|\frac{\gamma}{\delta}} \mu(\beta_1)\mu(\beta_2)N(\beta_1\beta_2)^{-s} \sum_{\alpha_1\alpha_2\equiv 1\left(\bmod \frac{\gamma}{\delta}\right)} \sum_{\delta_1\delta_2=\delta} \xi_0\left(s;0,\frac{\alpha_1\delta_1\beta_1\bar{\delta}}{\gamma/\delta}\right)\xi_0\left(s;0,\frac{\alpha_2\delta_2\beta_2\bar{\delta}}{\gamma/\delta}\right) \quad (3.7)$$

Thus, for $\mathrm{Re}\, s>1$: $F(s)=\sum_w \tau(w)K(\gamma;1,w)N(w)^{-s}=\sum_{\delta|\gamma}\mu(\delta)F(s;\gamma,\delta)N(\delta)^{-s}$, (3.8)

where

$$F(s;\gamma,\delta)=\sum_{\beta_1,\beta_2|\gamma/\delta}\frac{\mu(\beta_1)\mu(\beta_2)}{N(\beta_1\beta_2)^s}\sum_{\alpha_1\alpha_2\equiv 1(\gamma/\delta)}\sum_{\delta_1\delta_2=\delta}\xi_0\left(s;0,\frac{\alpha_1\delta_1\beta_1\bar{\delta}}{\gamma/\delta}\right)\xi_0\left(s;0,\frac{\alpha_2\delta_2\beta_2\bar{\delta}}{\gamma/\delta}\right).$$

By using estimates for $\xi_m(s;\alpha,\beta)$, we get immediately that

$$F(\sigma+it)<<\sigma_0^{-1}N(\gamma)^{\frac{4+9\sigma_0+4\sigma_0^2-\sigma(3+4\sigma_0)}{1+2\sigma_0}}|t|^{\frac{(2+4\sigma_0)(1+\sigma_0-\sigma)}{1+2\sigma_0}}\cdot\tau^2(\gamma)$$

for $-\sigma_0\le \mathrm{Re}\, s\le 1+\sigma_0$, $|\mathrm{Im}\, s|=|t|\ge 1$, $\left(0<\sigma_0\le\frac{1}{2}\right)$.

Therefore by Perron's formula

$$\sum_{N(w)\le x}\tau(w)K(\gamma;1,w)=\underset{s=1}{res}\left(F(s)x^s s^{-1}\right)+\underset{s=0}{res}\left(F(s)x^s s^{-1}\right)+(2\pi i)^{-1}\int_{-\sigma_0-iT}^{-\sigma_0+iT}F(s)x^s s^{-1}ds+$$

$$+O\left(x^{1+\sigma_0}\left(T\sigma_0^2\right)^{-1}\right)+O\left(xN(\gamma)^{1/2}\tau(\gamma)\log xT^{-1}\right)+O\left(\tau^2(\gamma)N(\gamma)^{1+\sigma_0}\cdot\sigma_0^{-1}T^{1+4\sigma_0}\right). \quad (3.9)$$

Now in the same way as in theorem 1, we get

$$\sum_{N(w)\le x}\tau(w)K(\gamma;1,w)=\sum_{s=0,1}res\left\{F(s)\cdot x^s s^{-1}\right\}+\sum_{\delta|\gamma}\sum_{\beta_1,\beta_2|\frac{\gamma}{\delta}}\mu(\beta_1)\mu(\beta_2)\sum_{\delta_1\delta_2=\delta}$$

$$\sum_{\substack{N(w)\le X\\ w\equiv\beta_1\beta_2\delta^4(\gamma)}}\tau(w)e\left(-\sqrt{xN(w)\left(N^2(\gamma)N(\beta_1\beta_2)\right)^{-1}}+3/8\right)x^{3/8}N(\gamma)^{5/4}N(\beta_1\beta_2)^{-3/8}N(w)^{-5/3}+$$

$$+O\left(TN(\gamma)\tau^3(\gamma)\log x\right)+O\left(x^{3/8}N(\gamma)^{1/4}T\tau^3(\gamma)\log x\right),$$

where $X=T^4(4\pi)^{-1}N^2(\gamma)N(\beta_1\beta_2)x^{-1}$. We put $T=x^{2/5}N(\gamma)^{1/5}$ and that completes the proof of the theorem.

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