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AVERAGING SCHEME FOR CONTROLLED SYSTEMS WITH MAXIMUM OF CONTROL FUNCTION

Кічмаренко О. Д., Сапожнікова К. Ю. Схема усереднення керованої системи з максимумом за керуванням. В даній статті розглядається керована система з максимумом за керуванням. Для дослідження системи використовується метод усереднення. Розроблено алгоритм співвідношення керувань початкової та усередненої систем. Наводиться обґрунтування методу усереднення для керованої системи з максимумом за керуванням.
Ключові слова: керовані системи з максимумом, асимптотичні методи, усереднення керованих систем, алгоритм відповідності керувань.

Кичмаренко О. Д., Сапожникова Е. Ю. Схема усреднения управляемой системой с максимумом по управлению. В данной статье рассматривается управляемая система с максимумом по управлению. Для исследования системы применяется метод усреднения. Разработан алгоритм соответствия управлений начальной и усредненной систем. Представлено обоснование метода усреднения для управляемой системы с максимумом по управлению.
Ключевые слова: управляемые системы с максимумом, асимптотические методы, усреднение управляемых систем, алгоритм соответствия управлений.

Kichmarenko O. D., Sapozhnikova K. Yu. Averaging scheme for controlled systems with maximum of control function. In this paper controlled system with maximum of control function is considered. Averaging method is used for system researching. Algorithm of correspondence between control functions of getting and averaged systems is presented. Justification of the averaging method for controlled system with maximum of control function is established.
Key words: controlled systems with maximum, asymptotic methods, averaging method for controlled systems, algorithm of correspondence between control functions.

Introduction. Nowadays interest in differential equations with maximum increases exponentially. For instance, it is easy to notice in the theory of automatic regulation [9]. Of course systems that describe real processes sometimes are not easy to research. That is why it is important to find methods for getting more simple systems that could be studied easier. It can be averaging method for controlled system that N.N.Moiseev [4] offered in XX century. Application of the averaging method for differential equations with maximum and controlled systems with maximum of state has been studied extensively by O.D.Kichmarenko and V.A.Plotnikov [2], [3], [5]- [7], [10]. In this paper we present conditions for applying averaging method for controlled systems with maximum of control function.

Main Results

1.Problem statement of averaging method for controlled system with maximum of control function. We consider motion of automatic regulation system that is described by system of differential equation with maximum of control function and small parameter

$$\dot{x}(t) = \varepsilon\left(f(t, x(t)) + A(x(t))\varphi(t, u(t), \max_{s\in[g(t),\gamma(t)]} u(s))\right), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is a phase vector, $f : [0, L\varepsilon^{-1}] \times \mathbb{R}^n \to \mathbb{R}^n$ is 2π-periodic vector function on t; $\varepsilon > 0$ — small parameter; $t \in [0, L\varepsilon^{-1}]$ is a time of system exists; $A - n \times m$— matrix, $a_{ij} : \mathbb{R}^n \to \mathbb{R}^1$; $\varphi : [0, L\varepsilon^{-1}] \times U \times U \to \mathbb{R}^m$, — 2π—periodic vector function on t; $g(t), \gamma(t)$ - are known functions and $0 \leqslant g(t) \leqslant \gamma(t) \leqslant t$; u —is a control function, $u \in \mathfrak{U}$, $u(t) \in U \subset comp(\mathbb{R}^r)$; $\max\limits_{s\in[g(t),\gamma(t)]} u(s) = (\max\limits_{s\in[g(t),\gamma(t)]} (u_1(s)), ..., \max\limits_{s\in[g(t),\gamma(t)]} (u_r(s)))$.

Piecewise continuous functions are considered as admissible control functions and absolutely continuous functions as solution for problem (1).

Note that for different ε we get different $u(t)$ i.e. $u(t) = u(t, \varepsilon)$.

Let us consider the correspondence averaged problem

$$\dot{y}(t) = \varepsilon\left[\overline{f}(y(t)) + A(y(t))v(\tau)\right], \quad y(0) = x_0, \tag{2}$$

for problem (1). Where

$$\overline{f}(y(t)) = \frac{1}{2\pi}\int_0^{2\pi} f(t, x(t))dt, \tag{3}$$

$v(t)$ is the control function such that the following condition is holds:

$$\int_{2\pi}^{2\pi(i+1)} v(t)dt = \int_{2\pi}^{2\pi(i+1)} \varphi(t, u(t), \max_{s\in[g(t),\gamma(t)]} u(s))dt, \tag{4}$$

i.e.

$$V = \left\{\frac{1}{2\pi}\int_0^{2\pi} \varphi(t, u(t), \max_{s\in[g(t),\gamma(t)]} u(s))dt, u(t) \in U\right\}.$$

Here V is Auman integral [1], $v \in \mathfrak{V}$ is a set of admissible control functions of averaged problem (2). Furthermore, note that for different ε we obtain different $v(t)$ i.e. $v(t) = v(t, \varepsilon)$.

Average system is more simple and does not include maximum.

2.Algorithm of correspondence between control functions of getting and averaged systems.

Establish the correspondence between control functions $u(t) \in U$ and $v(t) \in V$.

i. For admissible control $v \in \mathfrak{V}$ find the correspondence admissible control $u \in \mathfrak{U}$ in the following way:

   (a) calculate points $v_i = \frac{1}{2\pi}\int\limits_{2\pi i}^{2\pi(i+1)} v(t)dt$,

(b) assign control $u(t)=\{u_i(t), 2\pi i \leqslant t \leqslant 2\pi(i+1), i=1,2,..\}$, where $u_i(t)$ can be obtained from the conditions:

$$\arg\min_{u(t)\in U}\left\|\frac{1}{2\pi}\int\limits_{2\pi i}^{2\pi(i+1)}\varphi(t,u(t),\max_{s\in[g(t),\gamma(t)]}u(s))dt - v_i\right\|. \quad (5)$$

ii. For admissible control $u\in\mathfrak{U}$ find the correspondence admissible control $v\in\mathfrak{V}$ in the following way:

(a) denote values $u_i(t)=\{u(t), 2\pi i \leqslant t \leqslant 2\pi(i+1), i=1,2,..\}$ calculate points $w_i=\frac{1}{2\pi}\int\limits_{2\pi i}^{2\pi(i+1)}\varphi(t,u_i(t),\max\limits_{s\in[g(t),\gamma(t)]}u_i(s))dt$;

(b) assign control $v(\varepsilon t) = \{v_i(t), 2\pi i \leqslant t \leqslant 2\pi(i+1), i=1,2,..\}$, where v_i can be obtained from condition: $\arg\min\limits_{v\in V}\|w_i - v\|$.

Remark. *Control functions $u(t)$ in it.1(b) and $v(t)$ in it.2(b) can be determined ambiguously.*

3. Justification of averaging method for controlled system with maximum of control function.

Theorem. *Suppose that in domain $Q=\{t\geqslant 0, x,y\in D\subset\mathbb{R}^n, u(t)\in U\subset comp(\mathbb{R}^r)\}$ the following conditions hold:*

1) $f(t,x)$ is continuous function on t, 2π— periodic and bounded by constant K_1,

$$\|f(t,x)\|\leqslant K_1,$$

satisfy the Lipschitz condition with respect to x and with constant λ_1:

$$\|f(t,x')-f(t,x'')\|\leqslant\lambda_1\|x'-x''\|;$$

2) $A(x)$ is bounded by constant K_2 matrix s.t.:

$$\|A(x)\|\leqslant K_2,$$

where $\|A(x)\|$ is Euclidian norm, and satisfy Lipschitz condition with constant λ_2:

$$\|A(x')-A(x'')\|\leqslant\lambda_2\|x'-x''\|;$$

3) $\varphi(t,u,w)$ is 2π — periodic function and continuous with respect to t,u,w;

4) for any admissible control $v(t)\in\mathfrak{V}$ the corresponding trajectory $y(t)$ of the averaged system (2), $y(0)=x_0\in D'$ is defined by $t\geqslant 0$, and with its ρ — neighborhood is in domain D.

Then for any $L>0$ there exists $\varepsilon^0>0$, $C>0$ such that for any $\varepsilon\in(0,\varepsilon^0]$ and $t\in[0,L\varepsilon^{-1}]$ following statements hold:

1) for any admissible control $u \in \mathfrak{U}$ of system (1) exists control function $v \in \mathfrak{V}$ of system (2), such that:

$$\|x(t) - y(t)\| \leqslant C\varepsilon, \tag{6}$$

where $x(t), y(t)$ are solutions of systems (1) and (2) accordingly, $x(0) = y(0) \in D' \subset D$.

2) for any admissible control $v \in \mathfrak{V}$ of system (2) exists control function $u \in \mathfrak{U}$ of system (1), such that (6) is holds.

Proof.
Using the integral equations for (1) and (2), we can write:

$$\|x(t) - y(t)\| \leq \varepsilon \int_0^t \|f(\tau, x(\tau)) - f(\tau, y(\tau))\| \, d\tau + \varepsilon \left\| \int_0^t \left[f(\tau, y(\tau)) - \overline{f}(y(\tau)) \right] d\tau \right\| +$$

$$+\varepsilon \int_0^t \left\| \left[A(x(\tau)) - A(y(\tau)) \right] \varphi(\tau, u(\tau), \max_{s \in [g(\tau), \gamma(\tau)]} u(s)) \right\| d\tau +$$

$$+\varepsilon \left\| \int_0^t A(y(\tau)) \left[\varphi(\tau, u(\tau), \max_{s \in [g(\tau), \gamma(\tau)]} u(s)) - v(\tau) \right] \right\| d\tau \leq$$

$$\leqslant \varepsilon \left\{ [\lambda_1 + M\lambda_2] \int_0^t \delta(\tau) d\tau + \int_0^t \left\| f(\tau, y(\tau)) - \overline{f}(y(\tau)) \right\| d\tau + \right.$$

$$\left. + \int_0^t \left\| A(y(\tau)) \left[\varphi(\tau, u(\tau), \max_{s \in [g(\tau), \gamma(\tau)]} u(s) - v(\tau) \right] \right\| d\tau \right\}, \tag{7}$$

note that $\delta(t) = \|x(\tau) - y(\tau)\|$ is the uniform metric, i.e. $\varphi(\tau, u, w)$ is continuous function with respect to τ, u, w then

$$\|\varphi(\tau, u, w)\| \leqslant M = \max_{\tau, u, w} \|\varphi(\tau, u, w)\| .$$

Since previous inequality (7) is justly for any $t \in [0, \tau]$, we get

$$\delta(t) \leqslant \varepsilon \left\{ [\lambda_1 + M\lambda_2] \int_0^t \delta(\tau) d\tau + I_1 + I_2 \right\},$$

where

$$I_1 = \left\| \int_0^t f(\tau, y(\tau)) - \overline{f}(y(\tau)) d\tau \right\|,$$

$$I_2 = \int\limits_0^t \left\| A(y(\tau)) \left[\varphi(\tau, u(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} u(s) - v \right] \right\| d\tau.$$

Applying the Gronwall-Bellman lemma, we obtain:

$$\varepsilon \left\{ [\lambda_1 + M\lambda_2] \int\limits_0^t \delta(\tau) d\tau + I_1 + I_2 \right\} \leq \varepsilon \left(I_1 + I_2 \right) e^{\varepsilon[\lambda_1 + M\lambda_2]\tau} \leq$$

$$\leqslant \varepsilon \left(I_1 + I_2 \right) e^{L[\lambda_1 + M\lambda_2]}. \tag{8}$$

We denote $t_i = 2\pi$. Let $t \in [t_i, t_{i+1})$. Then, using the additive property of the integral, we get for I_1:

$$I_1 \leq \sum_{i=0}^{k-1} \left\{ \int\limits_{t_i}^{t_{i+1}} \left[\| f(\tau, y(\tau)) - f(\tau, y(t_i)) \| + \left\| \overline{f}(y(\tau)) - \overline{f}(y(t_i)) \right\| \right] d\tau + \right.$$

$$\left. + \left\| \int\limits_{t_i}^{t_{i+1}} f(\tau, y(t_i)) - \overline{f}(y(t_i)) d\tau \right\| \right\} + \int\limits_{t_k}^{t} \left\| \overline{f}(y(\tau)) - \overline{f}(y(t)) \right\| d\tau \leq$$

$$\leqslant \sum_{i=0}^{k-1} \left\{ 2\lambda_1 \int\limits_{t_i}^{t_{i+1}} \| y(\tau) - y(t_i) \| \, d\tau + \left\| \int\limits_{t_i}^{t_{i+1}} f(\tau, y(t_i)) - \overline{f}(y(t_i)) d\tau \right\| \right\} +$$

$$+ \int\limits_{t_k}^{t} \left\| f(y(\tau)) - \overline{f}(y(\tau)) \right\| d\tau \leq$$

$$\leq 2 \sum_{i=0}^{k-1} \left\{ \lambda_1 \int\limits_{t_i}^{t_{i+1}} \| y(\tau) - y(t_i) \| \, d\tau \right\} + 4\pi K_1 \leqslant 2\lambda_1 \varepsilon (K_1 + K_2 M) \sum_{i=0}^{k-1} \int\limits_{t_i}^{t_{i+1}} (\tau - 2\pi i) \, d\tau + 4\pi K_1 =$$

$$= 2\pi\lambda_1 L(K_1 + K_2 M) + 4\pi K_1.$$

Similarly to the way of previous estimation we can write for I_2:

$$I_2 \leq \sum_{i=0}^{k-1} \int\limits_{t_i}^{t_{i+1}} \left\| (A(y(\tau)) - A(y(t_i))) \left(\varphi(\tau, u(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} u(s)) - v(\tau) \right) \right\| d\tau +$$

$$+ \sum_{i=0}^{k-1} \int\limits_{t_i}^{t_{i+1}} \left\| A(y(t_i)) \left(\varphi(\tau, u(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} u(s)) - v(\tau) \right) \right\| d\tau +$$

$$+\int\limits_{t_k}^{t}\left\|A(y(\tau))\left(\varphi(\tau,u(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}u(s))-v(\tau)\right)\right\|d\tau\leq$$

$$\leq\sum_{i=0}^{k-1}\left\{\lambda_2\int\limits_{t_i}^{t_{i+1}}\|y(\tau)-y(t_i)\|\left\|\varphi(\tau,u(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}u(s))-v(\tau)\right\|d\tau\right\}+$$

$$+\int\limits_{t_k}^{t}\left\|A(y(\tau))\left(\varphi(\tau,u(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}u(s))-v(\tau)\right)\right\|d\tau.$$

Then, using Cauchy formula, 2) assumption of the theorem, estimation for function φ and (4) we get:

$$I_2\leqslant 2\lambda_2M(K_1+K_2M)\pi L+4\pi K_2M.$$

So, from estimations for I_1, I_2 and using inequality (8), we obtain:

$$\|x(t)-y(t)\|\leqslant\varepsilon C,$$

where

$$C=2\pi\left[L\left(K_1+K_2M\right)\{\lambda_1+\lambda_2M+2\}\right].$$

Theorem is proved.

Conclusion. Thus, algorithm of correspondence between control function $u(t)$ of getting system and control function $v(t)$ of averaged system is established. Proved theorem substantiates applying the averaging method for controlled system with maximum of control function.

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