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COMPLETE AVERAGING SCHEME FOR THE DIFFERENTIAL EQUATION WITH MAXIMA

Кічмаренко О. Д. Схема повного усереднення диференціальних рівнянь з максимумом. В статті представлено обгрунтування методу усереднення для динамічних рівнянь з малим параметром та з максимумом. Отримано оцінки близькості розв'язків початкової та усередненої систем.
Ключові слова: метод усереднення, диференціальні рівняння з максимумом.

Кичмаренко О. Д. Схема полного усреднения дифференциальных уравнений с максимумом. В статье представлено обоснование метода усреднения для динамических уравнений с малым параметром и с максимумом. Получены оценки близости решений исходной и усредненной систем.
Ключевые слова: метод усреднения, дифференциальные уравнения с максимумом.

Kichmarenko O. D. Complete averaging scheme for the differential equation with maxima. In paper the averaging method for dynamical equations with small parameter and with maxima is justificated. An estimation for proximity of solutions of given and averaged system of equations was obtained.
Key words: averaging method, differential equations with maxima.

Introduction.

Differential equations with "maxima" are a special type of differential equations that contain the maximum of the unknown function over a previous interval. Such equations adequately model real world processes whose present state significantly depends on the maximum value of the state on a past time interval [17, 11, 1].

The qualitative theory and develops some approximate methods for differential equations with "maxima" in the works [2, 13, 4, 3] are present.

In averaging theory of differential equations are fundamental work N. M. Krylov, N. N. Bogolyubov [9, 10]. Later, this approach was developed N. N. Bogolyubov and his progeny [5, 16].

Justification of the averaging method for differential equations with delay in the papers [6, 7] is present. And justification of the averaging method for differential equations with "maxima" in the next works [12, 14, 15, 19] is present.

Let us consider the most general case of averaging problem statement:

$$\dot{x}(t) = \varepsilon f(t, x(t), \max_{s \in [g(t), \gamma(t)]} x(s)). \tag{1}$$

Here $x \in \mathbb{R}^n$ is a phase vector, ε is a small parameter, $f : [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is n dimensional vector function, $t \geq 0$, $g(t)$ and $\gamma(t)$ are known functions, $0 \leq g(t) \leq$ $\leq \gamma(t) \leq t$ and

$$\max_{s \in [g(t), \gamma(t)]} x(s) = \left(\max_{s \in [g(t), \gamma(t)]} x_1(s), \cdots, \max_{s \in [g(t), \gamma(t)]} x_n(s) \right).$$

Here $||x|| = \max_i | x_i|$, $x(0) = x_0$.

Note that if $g(t) = \gamma(t) = t - h$, then (1) is a differential equation with constant delay and if $g(t) = \gamma(t)$, then (1) is a differential equation with variable delay. Application of the averaging method for differential equations with variable delay has been studied in works [8]. The case when $g(t) = t - h, \gamma(t) = t$ considered in works [2, 19].

Main Results.

1. Complete averaging scheme for the differential equation with maxima

Let us consider the following averaged equation

$$\dot{y}(t) = \varepsilon f^0\left(y(t), \max_{s\in[g(t),\gamma(t)]} y(s)\right), \quad y(0) = x_0 \tag{2}$$

for the equation (1). Here

$$f^0(x,y) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t,x,y)dt. \tag{3}$$

Theorem 1. *In $Q = \{t > 0, x, y \in D \subset \mathbb{R}^n\}$ the following conditions hold:*

1) *$f(t,x,y)$ is a continuous function on t and*

$$||f(t,x,y)|| \le M, \tag{4}$$

$$||f(t,x,y) - f(t,\bar{x},\bar{y})|| \le \lambda[||x - \bar{x}|| + ||y - \bar{y}||], \tag{5}$$

2) *$g(t)$ and $\gamma(t)$ are evenly continuous functions and $0 \le g(t) \le \gamma(t) \le t$;*

3) *the limit (3) exists evenly with respect to x, y;*

4) *the solution of the equation (2) at $\varepsilon \in (0, \varepsilon_1]$, $t \ge 0$, $y(0) \in D' \subset D$ together with its $\rho-$neighbourhood belongs to D.*

Then for any $\eta > 0$, $L > 0$ there exists $\varepsilon^0(\eta, L) \in (0, \varepsilon_1]$ such that for all $t \in$ $\in [0, L\varepsilon^{-1}]$ the following estimate holds:

$$||x(t) - y(t)|| \le \eta, \tag{6}$$

where $x(t)$, $y(t)$ are solutions of systems (1) and (2) accordingly, $x(0) = y(0) \in D'$.

Proof. Using the integral equations for (1) and (2)

$$x(t) = x_0 + \varepsilon\int_0^t f(\tau, x(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} x(s))d\tau,$$

$$y(t) = x_0 + \varepsilon\int_0^t f^0(y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s))d\tau,$$

we can write

$$\|x(t)-y(t)\| \le$$

$$\le \varepsilon \int_0^t \left\| f(\tau, x(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} x(s)) - f(\tau, y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) \right\| d\tau + \tag{7}$$

$$+ \left\| \varepsilon \int_0^t \left[f(\tau, y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) - f^0(y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) \right] d\tau \right\| = I + I^0.$$

Note that:

$$\delta(t) = \max_{s\in[0,t]} \|x(s)-y(s)\|$$

is the uniform metric. Then, using this notation and (7), we get:

$$I \le \varepsilon\lambda \int_0^t \left[\|x(\tau)-y(\tau)\| + \left\| \max_{s\in[g(\tau),\gamma(\tau)]} x(s) - \max_{s\in[g(\tau),\gamma(\tau)]} y(s) \right\| \right] ds \le$$
$$\le 2\varepsilon\lambda \int_0^t \delta(\tau) d\tau. \tag{8}$$

We consider $t_i = i\Delta$, $i = 0,1,...,m$, $m\Delta = L\varepsilon^{-1}$ and $[0, L\varepsilon^{-1}] = \bigcup_{i=0}^{m-1} [t_i, t_{i+1}]$. Let $t \in [t_k, t_{k+1})$. Then, using the additive property of the integral, we get:

$$I^0 \le \sum_{i=0}^{k-1} \varepsilon \left\| \int_{t_i}^{t_{i+1}} \left[f(\tau, y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) - f^0(y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) \right] d\tau \right\| +$$

$$+\varepsilon \left\| \int_{t_k}^{t} \left[f(\tau, y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) - f^0(y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) \right] d\tau \right\| = \tag{9}$$

$$= \sum_{i=0}^{k-1} I_i + I_k.$$

Let us estimate I_k and I_i for all i, using the triangle inequality, we obtain:

$$I_i \le \varepsilon \left\| \int_{t_i}^{t_{i+1}} \left[f(\tau, y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s)) - f^0(y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s)) \right] d\tau \right\| +$$

$$+\varepsilon \int_{t_i}^{t_{i+1}} \left\| f(\tau, y(\tau), \max_{s\in[g(\tau),\gamma(\tau)]} y(s)) - f(\tau, y(t_i), \max_{s\in[g(t_i),\gamma(t_i)]} y(s)) \right\| d\tau + \tag{10}$$

$$+\varepsilon\int\limits_{t_i}^{t_{i+1}}\left\|f^0(y(t_i),\max_{s\in[g(t_i),\gamma(t_i)]}y(s))-f^0(y(\tau),\max_{s\in[g(\tau),\gamma(\tau)]}y(s))\right\|d\tau=J_i^0+J_i+J_i^{00}.$$

From (2) and estimate (10) and 1), 2) assumptions of the theorem 1, we get:

$$J_i\le\varepsilon\lambda\int\limits_{t_i}^{t_{i+1}}\left[\|y(\tau)-y(t_i)\|+\left\|\max_{s\in[g(\tau),\gamma(\tau)]}y(s)-\max_{s\in[g(t_i),\gamma(t_i)]}y(s)\right\|\right]d\tau\le$$

$$\le\varepsilon\lambda\int\limits_{t_i}^{t_{i+1}}\left[\varepsilon\int\limits_{t_i}^{\tau}\left\|f^0(y(x),\max_{s\in[g(x),\gamma(x)]}y(s))\right\|dx+\varepsilon M\ max\{\omega(\gamma,\Delta),\omega(g,\Delta)\}\right]d\tau\le$$

$$\le\varepsilon^2\lambda M\Delta\left(\frac{\Delta}{2}+\ max\{\omega(\gamma,\Delta),\omega(g,\Delta)\}\right),\tag{11}$$

where $\omega(\alpha,\Delta)$ is a continuity modulus [18] of the function $\alpha(t)$ on an interval $[0,\infty)$ and $\omega(\alpha,\Delta)=\sup\limits_{|t''-t'|\le\Delta}|\ \alpha(t'')-\alpha(t')|$.

Using the properties of the continuity modulus of the paper [18], we get:

$$\sum_{i=0}^{k-1}J_i\le\varepsilon\lambda ML\left(\frac{L}{2\varepsilon m}+\ max\left\{\omega\left(\gamma,\frac{L}{\varepsilon m}\right),\omega\left(g,\frac{L}{\varepsilon m}\right)\right\}\right)\le$$

$$\le\frac{\lambda ML}{m}\left(\frac{L}{2}+\ max\left\{\omega\left(\gamma,L\right),\omega\left(g,L\right)\right\}\right)+\varepsilon\lambda ML\ max\left\{\omega\left(\gamma,L\right),\omega\left(g,L\right)\right\}.\tag{12}$$

Similarly to the way the estimate (12) was obtained, it can be proved that

$$\sum_{i=0}^{k-1}J_i^{00}\le\lambda ML\left(\frac{L}{2m}+\left(\frac{1}{m}+\varepsilon\right)\ max\left\{\omega\left(\gamma,L\right),\omega\left(g,L\right)\right\}\right).\tag{13}$$

From assumption 3) of the theorem 1 it follows there exists a decreasing function $\Theta(t)\underset{t\to\infty}{\longrightarrow}0$ such that

$$\left\|\varepsilon\int\limits_0^{t_i}\left[f(\tau,y(t_i),\max_{s\in[g(t_i),\gamma(t_i)]}y(s))-f^0(y(t_i),\max_{s\in[g(t_i),\gamma(t_i)]}y(s))\right]d\tau\right\|\le$$

$$\le\varepsilon t_i\Theta(t_i)\le\tau_i\Theta(\frac{\tau_i}{\varepsilon}).$$

Therefore, for any η_1 exists $\varepsilon_0(\eta_1)>0$ such that for any $\varepsilon\le\varepsilon_0(\eta_1)$ the following inequality holds:

$$J_i^0\le\varepsilon\left\|\int\limits_0^{t_{i+1}}f(\tau,y(t_i),\max_{s\in[g(t_i),\gamma(t_i)]}y(s))-f^0(y(t_i),\max_{s\in[g(t_i),\gamma(t_i)]}y(s))d\tau\right\|+$$

$$+\varepsilon\left\|\int\limits_0^{t_i}f(\tau,y(t_i),\max_{s\in[g(t_i),\gamma(t_i)]}y(s))-f^0(y(t_i),\max_{s\in[g(t_i),\gamma(t_i)]}y(s))d\tau\right\|\le2\eta_1.\tag{14}$$

From (4) it follow that

$$I_k \le \frac{2ML}{m}. \tag{15}$$

Using estimates (11)-(15), we get

$$\begin{gathered} I^0 \le \frac{2ML}{m}\left(\lambda\frac{L}{2} + \lambda\ max\left\{\omega\left(\gamma, L\right), \omega\left(g, L\right)\right\} + 1\right) + \\ +2\varepsilon\lambda ML\ max\left\{\omega\left(\gamma, L\right), \omega\left(g, L\right)\right\} + 2m\eta_1(\varepsilon) = \nu(m, \varepsilon). \end{gathered} \tag{16}$$

So, for $t \in [0, \tau]$ we can write

$$\|x(t) - y(t)\| \le 2\varepsilon\lambda\int\limits_0^t \delta(s)ds + \nu(m, \varepsilon) \le 2\varepsilon\lambda\int\limits_0^\tau \delta(s)ds + \nu(m, \varepsilon). \tag{17}$$

Then, using the definition of $\delta(\tau)$, we get

$$\delta(\tau) = \max_{0 \le s \le \tau}\|x(s) - y(s)\| \le 2\varepsilon\lambda\int\limits_0^\tau \delta(s)ds + \nu(m, \varepsilon). \tag{18}$$

Applying the Gronwall-Bellman lemma, we get

$$\delta(t) \le \nu(m, \varepsilon)e^{2\varepsilon\lambda t} \le \nu(m, \varepsilon)e^{2\varepsilon\lambda L} < \eta.$$

Note that by appropriate choice of sufficiently large m and sufficiently small ε, the value $\nu(m, \varepsilon)$ can be made as small as possible.

Theorem 1 is proved.

Remark 1. *In the statement of the problem of averaging it is possible to decline the limitation of nonnegativeness of function $g(t)$. Let us consider the equation (1) with function $g(t)$ such, that $g(t) \le \gamma(t) \le t$. If $t_* = \inf\limits_{t \ge 0} g(t)$ $(t_* > -\infty)$ exists, then the assignation of initial problem will contain prehistory $x(s, \varepsilon) = \Phi(s, \varepsilon)$ at $t_* \le s \le 0$. If the initial function $\Phi(s, \varepsilon) \in D$ is continuous and satisfies conditions: $|\Phi(s, \varepsilon)| \le M$, $h(\Phi(s', \varepsilon), \Phi(s'', \varepsilon)) \le \varepsilon\mu|s' - s''|$, so it is possible to prove the existence of an full average, whereas the averaging problem (2) will contain the same initial function $y(s, \varepsilon) = \Phi(s, \varepsilon)$ with respect to $t_* \le s \le 0$.*

2. Partially averaging scheme for the differential equation with maxima

Now, we consider the following partially averaged equation

$$\dot{y} = \varepsilon F\left(t, y(t), \max_{s \in [g(t), \gamma(t)]} y(s)\right), \quad y(0) = x_0 \tag{19}$$

for the equation (1). Here

$$F(t, y, z) = \left\{F^i(y, z) = \frac{1}{T}\int\limits_{iT}^{(i+1)T} f(t, y, z)dt,\ t \in [i\,T, (i+1)T),\ i = 0, 1, \cdots\right\}, \tag{20}$$

T is a constant. The step-averaging scheme of (19) for the system (1) will be used.

This scheme the estimation of proximity of solutions given and averaged systems are more exact.

Theorem 2. *Let the conditions* 1), 2) *of the theorem 1 be fulfilled and also:*

3) solution of the equation (19) at $\varepsilon \in (0, \varepsilon_1]$, $t \geq 0$ *and* $y(0) \in D'' \subset D$ *together with its* $\rho-$*neighbourhood belongs to* D.

Then for any $L > 0$ *there exist such* $C > 0$ *and* $\varepsilon^0(L) \in (0, \varepsilon_1]$ *that the following estimate is fulfilled:*

$$\|x(t) - y(t)\| \leq C\ \varepsilon, \tag{21}$$

where $x(t)$, $y(t)$ *are solutions of systems (1) and (19) accordingly,* $x(0) = y(0) \in D''$.

Proof.

The proof of theorem 2 is similarly to proof of the theorem 1, only in the estimate (11) $\Delta = t_{i+1} - t_i = T$ and Δ is not depend on ε.

So, from (11) it follows that

$$\begin{aligned} &\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \left\| f(\tau, y(\tau), \max_{s \in [g(\tau), \gamma(\tau)]} y(s)) - F(y(\tau), \max_{s \in [g(\tau), \gamma(\tau)]} y(s)) \right\| d\tau \leq \\ &\qquad \leq \varepsilon \lambda M \left(\frac{\Delta}{2} + \ max\{\omega(\gamma, \Delta), \omega(g, \Delta)\} \right), \end{aligned} \tag{22}$$

Similarly to the way the estimate (22) was obtained, it can be proved that

$$\begin{aligned} &\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \left\| f^0(y(\tau), \max_{s \in [g(\tau), \gamma(\tau)]} y(s)) - F(y(t_i), \max_{s \in [g(t_i), \gamma(t_i)]} y(s)) \right\| d\tau \leq \\ &\qquad \leq \lambda M L \left(\frac{L}{2m} + \left(\frac{1}{m} + \varepsilon \right)\ max \left\{ \omega\left(\gamma, L\right), \omega\left(g, L\right) \right\} \right). \end{aligned} \tag{23}$$

Using (20), we get

$$\int_{t_i}^{t_{i+1}} \left[f(\tau, y(t_i), \max_{s \in [g(t_i), \gamma(t_i)]} y(s)) - F(y(t_i), \max_{s \in [g(t_i), \gamma(t_i)]} y(s)) \right] d\tau = 0,$$

$$\begin{aligned} &\varepsilon \left\| \int_{t_k}^{t} \left[f(\tau, y(\tau), \max_{s \in [g(\tau), \gamma(\tau)]} y(s)) - F(y(\tau), \max_{s \in [g(\tau), \gamma(\tau)]} y(s)) \right] d\tau \right\| \leq \\ &\qquad \leq \varepsilon 2MT. \end{aligned} \tag{24}$$

From (23), (24) the inequality (21) holds. Theorem 2 is proved.

Conclusion.

So, in present paper the complete and partially scheme of averaging method for dynamical equations with small parameter and with maxima when right part of differential equations contains the maximum of the unknown function over a previous interval with variable boundaries is justificated. An estimation for proximity of solutions of given and averaged system of equations was obtained.

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