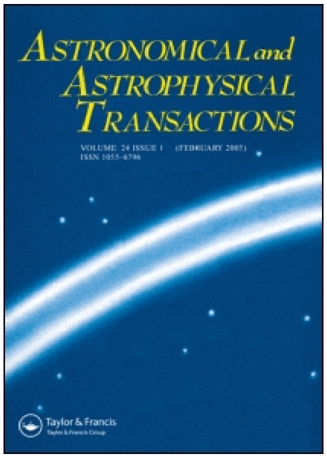

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A nonlinear multidimensional gravitational model $R + R^{-1}$ with form fields and stabilized extra dimensions

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We studied multidimensional gravitational models with scalar curvature nonlinearity of the type R^{-1} and with form fields (fluxes) as a matter source. It is assumed that the higher-dimensional space–time undergoes Freund–Rubin-like spontaneous compactification to a warped product manifold. It is shown that for certain parameter regions the model allows for freezing stabilization of the internal space near the positive minimum of the effective potential, which plays the role of the positive effective cosmological constant. However, the parameters of model should be fine tuned to obtain the observable dark energy.

Keywords: Multidimensional cosmology; Accelerating expansion; Extra-dimension stabilization

1. Introduction

There are two great challenges in modern theoretical physics and cosmology. The first great puzzle consists of the 'dark side' of our Universe. Recent observations of the luminosity distances of type Ia supernovae, angular temperature fluctuations of the cosmic microwave background on degree scales, and measurements of the power spectrum of galaxy clustering indicate that our Universe is spatially flat with about 24% of its critical energy in non-relativistic cold dark matter and about 72% in a smooth component having a large negative pressure (dark energy). The latter results in the accelerating expansion of our Universe which began approximately at a red shift $z \approx 1$ and continues until the present time. On the other hand, there is also the possibility that the late-time accelerating expansion of our Universe is caused by modification of gravity on a Galactic scale. For example, it was proposed [1] to add a R^{-1} term in the Einstein–Hilbert action to modify general relativity. It is clear that such modification may affect the dynamics of the Universe at late times of its evolution and on large scales where the scalar curvature becomes small. In fact, it was shown [1] that this term can give an accelerating solution of the field equation without the need to introduce dark energy.

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The second great challenge is the possible multidimensionality of our Universe which naturally follows from theories unifying different fundamental interactions with gravity, such as the string or M-theory. So, there is a great temptation to explain the dark matter and the accelerating expansion of our Universe with the help of extra dimensions. However, it is well known that the dynamic behaviour of internal spaces usually results in variations in the effective four-dimensional (4D) fundamental ‘constants’ (e.g. gravitational constant, fine-structure constant, etc.) [2, 3]. There are strong experimental bounds on such variations [4]. So, one of the main problem of higher-dimensional models lies in stable compactification of the internal spaces. Scale factors of the internal spaces play the role of scalar fields moving in our 4D space–time. Their dynamics are defined by an effective potential in dimensionally reduced theory. Thus, the internal spaces are stabilized in the case of a minimum of this potential [5]. Small excitations around this minimum look in our Universe like massive scalar fields (gravitational excitons [5]) with Planck scale suppression of their interaction with usual matter. Therefore, they may play the role of dark matter. Additionally, if the minimum of the effective potential is positive, it contributes to the positive cosmological constant providing acceleration of the Universe.

In the present paper, we consider a nonlinear gravitational multidimensional cosmological model with action of the type $R + R^{-1}$ with form fields as a matter source. We also include a bare cosmological term as an additional parameter of the theory. It is assumed that the corresponding higher-dimensional space–time manifold undergoes spontaneous compactification to a manifold with a warped product structure of the external and internal spaces. Each of the spaces has its own scale factor. A model without form fields and a bare cosmological constant was considered in [6] where freezing stabilization of the internal space was achieved owing to the negative minimum of the effective potential. Thus, such a model is of an asymptotically anti-de Sitter type without accelerating behaviour of our Universe. It is well known that inclusion of the usual matter can increase the potential to positive values [7]. One of the main tasks of our present investigations is to observe such an increase. Indeed, we demonstrate that for certain parameter regions the late-time acceleration scenario in our model becomes reachable. However, the parameters of the model should be fine tuned to obtain the observable dark energy. It is also worth noting that the effective potential in our reduced model has a branch point. It gives the very interesting possibility of investigating transitions from one branch to another by analogy with catastrophe theory or by similarity to phase transitions in statistical theory.

2. General set-up

We consider a $D = (D_0 + D')$-dimensional nonlinear gravitational theory with the action functional

$$S = \frac{1}{2\kappa_D^2} \int_M d^D x \ (|\bar{g}|)^{1/2} f(\bar{R}) - \frac{1}{2} \int_M d^D x \ (|\bar{g}|)^{1/2} \sum_{i=1}^{n} \frac{1}{d_i!} \left(F^{(i)}\right)^2, \tag{1}$$

where $f(\bar{R})$ is an arbitrary smooth function of scalar curvature $\bar{R} = R[\bar{g}]$ constructed from the D-dimensional metric $\bar{g}_{ab}$ $(a, b = 1, \ldots, D)$. D' is the number of extra dimensions. κ_D^2 denotes the D-dimensional gravitational constant. In action (1), a form field (flux) F has a block-orthogonal structure consisting of n blocks. Each of these blocks is described by its own antisymmetric tensor field $F^{(i)} (i = 1, \ldots, n)$ of rank d_i (d_i-form field strength). Additionally, we assume that $\sum_{i=1}^{n} d_i = D'$ holds for the sum of the ranks.

Following [6–9], we can show that the nonlinear gravitational theory (1) is equivalent to a linear theory $R = R[g]$ with the conformally transformed metric

$$g_{ab} = \Omega^2 \bar{g}_{ab} = \left[f'(\bar{R})\right]^{2/(D-2)} \bar{g}_{ab} \tag{2}$$

and an additional minimal scalar field $\phi = ln[f'(\bar{R})]/A$ with a self-interaction potential $U(\phi)$ given by

$$U(\phi) = \frac{1}{2}\mathrm{e}^{-B\phi}\left[\bar{R}(\phi)\mathrm{e}^{A\phi} - f\left(\bar{R}(\phi)\right)\right], \tag{3}$$

where

$$A = \left(\frac{D-2}{D-1}\right)^{1/2}, \qquad B = \frac{D}{((D-2)(D-1))^{1/2}}. \tag{4}$$

Furthermore, we assume that the multidimensional spacetime manifold undergoes spontaneous compactification

$$M \longrightarrow M = M_0 \times M_1 \times \cdots \times M_n \tag{5}$$

in accordance with the block-orthogonal structure of the field strength F, and that the form fields $F^{(i)}$, each nested in its own d_i-dimensional factor space $M_i (i = 1, \ldots, n)$, respect a generalized Freund–Rubin *Ansatz* [10]. Here, $(D_0 = 4)$-dimensional space–time M_0 is treated as our external Universe with metric $g^{(0)}(x)$.

This allows us to perform a dimensional reduction of our model along the lines used in [5–8, 11, 12]. The factor spaces M_i are then Einstein spaces with metrics $g^{(i)} \equiv \mathrm{e}^{2\beta^i(x)}\gamma^{(i)}$ which depend only through the warp factors $a_i(x) := \mathrm{e}^{\beta^i(x)}$ on the coordinates x of the external space–time M_0. For the corresponding scalar curvatures, $R[\gamma^{(i)}] = \lambda^i d_i \equiv r_i$ holds (in the case of the constant-curvature spaces $\lambda^i = k_i(d_i - 1), k_i = 0 \pm 1$). The warped product of Einstein spaces leads to a scalar curvature $\bar{R}$ which depends only on the coordinate x of the D_0-dimensional external space–time M_0: $\bar{R}[\bar{g}] = \bar{R}(x)$. This implies that the nonlinearity field ϕ is also a function of only x: $\phi = \phi(x)$. Additionally, it can be easily seen [6] that the generalized Freund–Rubin *Ansatz* results in the following expression for the form fields $(F^{(i)})^2 = f_i^2/a_i^{2d_i}$, where $f^i =$ constant.

In general, the model will allow for several stable scale factor configurations (minima in the landscape over the space of the volume moduli). We choose one of these (which we expect to correspond to the current evolution stage of our observable Universe), denote the corresponding scale factors as β_0^i and work further with the deviations $\hat{\beta}^i(x) = \beta^i(x) - \beta_0^i$.

Without loss of generality, we consider in the present section a model with only one d_1-dimensional internal space. (The difference between a general model with $n > 1$ internal spaces and the particular model with $n = 1$ consists of an additional diagonalization of the geometrical moduli excitations). After dimensional reduction and subsequent conformal transformation to the Einstein frame (along the lines used in [8]), the action functional (1) is

$$S = \frac{1}{2\kappa_0^2}\int_{M_0} d^{D_0}x \left(|\tilde{g}^{(0)}|\right)^{1/2}\left\{R\left[\tilde{g}^{(0)}\right] - \tilde{g}^{(0)}\partial_\mu\varphi\partial_\nu\varphi - \tilde{g}^{(0)\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2U_{\mathrm{eff}}(\varphi, \phi)\right\}, \tag{6}$$

where $\varphi := -[d_1(D-2)/(D_0-2)]^{1/2}\hat{\beta}^1$ and $\kappa_0^2 := \kappa_D^2/V_{d_1}$ denotes the D_0-dimensional (4D) gravitational constant. $V_{d_1} \propto \exp(d_1\beta_0^1)$ is the volume of the internal space at the present time.

Stable compactification of the internal space M_1 is ensured when its scale factor φ is frozen at the minimum of the effective potential

$$U_{\mathrm{eff}} = \mathrm{e}^{b\varphi}\left[-\frac{1}{2}R_1\mathrm{e}^{a\varphi} + U(\phi) + h\mathrm{e}^{c\phi}\mathrm{e}^{ad_1\varphi}\right], \tag{7}$$

where $R_1 := r_1\mathrm{e}^{-2\beta_0^1}$ and $h := \kappa_D^2 f_1^2\mathrm{e}^{-2d_1\beta_0^1} > 0$; for brevity we introduce the notation

$$a := 2\left(\frac{D_0-2}{d_1(D-2)}\right)^{1/2}, \quad b := 2\left(\frac{d_1}{(D-2)(D_0-2)}\right)^{1/2}, \quad c := \frac{2d_1 - D}{[(D-1)(D-2)]^{1/2}}. \tag{8}$$

3. The model and results

In this section we analyse the conditions of the compactification for a model with

$$f(\bar{R}) = \bar{R} - \frac{\mu}{\bar{R}} - 2\Lambda_D. \tag{9}$$

Then from the relation $f'(\bar{R}) = \mathrm{e}^{A\phi}$ we obtain

$$\bar{R} = q\left(\frac{|\mu|}{s(\mathrm{e}^{A\phi}-1)}\right)^{1/2}, \quad q = \pm 1, \quad s = \mathrm{sgn}(\mu). \tag{10}$$

Thus, the ranges of variation in ϕ are $\phi \in (-\infty, 0)$ for $\mu < 0 (s = -1)$ and $\phi \in (0, +\infty)$ for $\mu > 0 (s = +1)$.

It is worth noting that the limit $\phi \to \pm 0$ ($f' \to 1$) corresponds to the transition to a linear theory: $f(\bar{R}) \to \bar{R} - 2\Lambda_D$ and $R \to \bar{R}$. On the other hand, equation (10) shows that this point is a singularity $\bar{R}$, $R \to \pm\infty$ for the model (9).

For our model (9), the potential $U(\phi)$ (equation (3)) is

$$U(\phi) = \frac{1}{2}\mathrm{e}^{-B\phi}\left[2qs(|\mu|)^{1/2}\left(s\,\mathrm{e}^{A\phi} - s\right)^{1/2} + 2\Lambda_D\right]. \tag{11}$$

It is well known (see, for example, [7, 8, 11]) that, in order to ensure stabilization and asymptotical freezing of the internal space M_1, the effective potential (7) should have a minimum with respect to both scalar fields φ and ϕ. It should be remembered that the minimum position is chosen with respect to φ at $\varphi = 0$. Additionally, the eigenvalues of the mass matrix of the coupled (φ, ϕ)-field system, *i.e.* the Hessian of the effective potential at the minimum position,

$$J := \left.\begin{pmatrix} \partial^2_{\varphi\varphi}U_{\mathrm{eff}} & \partial^2_{\varphi\phi}U_{\mathrm{eff}} \\ \partial^2_{\phi\varphi}U_{\mathrm{eff}} & \partial^2_{\phi\phi}U_{\mathrm{eff}} \end{pmatrix}\right|_{\mathrm{extr}}, \tag{12}$$

should be positive definite (this condition ensures the positiveness of the mass squared of scalar field excitations). According to the Silvester criterion this is equivalent to the condition

$$J_{11} > 0, \quad J_{22} > 0, \quad \det(J) > 0. \tag{13}$$

It is convenient in further considerations to introduce the following notation:

$$\phi_0 := \phi|_{\mathrm{extr}}, \ X := \left(s\,\mathrm{e}^{A\phi_0} - s\right)^{1/2} > 0 \to X_{(s=-1)} < 1. \tag{14}$$

Then we can rewrite the potentials $U(\phi)$ and $U_{\text{eff}}(\varphi, \phi)$ and the derivatives of U_{eff} at an extremum (possible minimum) position ($\varphi = 0, \phi_0$) as follows:

$$U_0 \equiv U|_{\text{extr}} = \frac{1}{2}\left(1 + sX^2\right)^{-B/A}\left[2qs(|\mu|)^{1/2}X + 2\Lambda_D\right], \tag{15}$$

$$U_{\text{eff}}|_{\text{extr}r} = -\frac{1}{2}R_1 + U_0(X) + h\left(1 + sX^2\right)^{c/A}, \tag{16}$$

$$\partial_\varphi U_{\text{eff}}|_{\text{extr}} = -\frac{a+b}{2}R_1 + bU_0(X) + (ad_1 + b)h\left(1 + sX^2\right)^{c/A} = 0, \tag{17}$$

$$\partial_\phi U_{\text{eff}}|_{\text{extr}} = ch\left(1 + sX^2\right)^{c/A} - BU_0(X) + \frac{qA(|\mu|)^{1/2}}{2X}\left(1 + sX^2\right)^{((A-B)/A)} = 0, \tag{18}$$

$$\partial^2_{\varphi\varphi} U_{\text{eff}}|_{\text{extr}} = -\frac{(a+b)^2}{2}R_1 + b^2U_0(X) + (ad_1 + b)^2\, h\left(1 + sX^2\right)^{c/A}, \tag{19}$$

$$\partial^2_{\varphi\phi} U_{\text{eff}}|_{\text{extr}} = chad_1\left(1 + sX^2\right)^{c/A}, \tag{20}$$

$$\begin{aligned}\partial^2_{\phi\phi} U_{\text{eff}}|_{\text{extr}} &= ch(c - A + 2B)\left(1 + sX^2\right)^{c/A} + B(A - B)U_0(X)\\ &\quad - \frac{qs(|\mu|)^{1/2}A^2}{4X^3}\left(1 + sX^2\right)^{((2A-B)/A)}.\end{aligned} \tag{21}$$

The most natural strategy for extracting detailed information about the location of the stability region of parameters in which compactification is possible would consist in solving equation (18) for X with subsequent back substitution of the roots found into the inequalities (13) and equation (17). To obtain the main features of the model under consideration, it is sufficient to investigate two particular non-trivial situations. Both of these cases are easy to handle analytically.

3.1 *Zero effective cosmological constant:* $\Lambda_{\text{eff}} = 0$

The condition $\Lambda_{\text{eff}} = U_{\text{eff}}|_{\text{extr}} = 0$, results in the relations

$$R_1 = 2d_1\, h\left(1 + sX^2\right)^{c/A} = \frac{2d_1}{d_1 - 1}U_0(X), \quad d_1 \geqslant 2, \tag{22}$$

which enable us to obtain (from equation (18)) a quadratic equation for X with the following solutions:

$$X_{(p)} = qs\frac{d_1}{2(d_1+1)}\left[-z + p\left(z^2 + 4s\frac{d_1^2 - 1}{d_1^2}\right)^{1/2}\right], \quad z \equiv \frac{2\Lambda_D}{(|\mu|)^{1/2}}, \quad p = \pm 1, \tag{23}$$

where, for $s = -1$, $|z| \geqslant z_0 \equiv 2(d_1^2 - 1)^{1/2}/d_1 < 2$.

Simple analysis shows that a zero minimum of the effective potential occurs only if $\mu < 0$ ($s = -1$), $p = +1, q = +1$ and $z \in (z_0, +\infty)$. $z = z_0$ is the exceptional value because the minimum degenerates for $d_1 = 4$.

3.2 *Decoupling of excitations:* $d_1 = D_0$

It can be easily seen from equation (8) that, in the case $d_1 = D_0$, the parameter $c = 0$ leads to the condition $\partial^2_{\varphi\phi} U_{\text{eff}}|_{\text{extr}} = 0$ (see equation (20)). Thus the Hessian (12) is diagonalized.

This means that the excitations of the fields φ and ϕ near the extremum position are decoupled from each other.

In spite of the fact that we does not use the condition $\Lambda_{\rm eff} = 0$, in the case $d_1 = D_0$ we obtain exactly the same equation for X as in the previous subsection. Thus its solution has the form (23). However, the parameters now satisfy the following conditions:

$$R_1 = 4\left[\frac{1}{3}U_0(X) + h\right], \quad \Lambda_{\rm eff}(X) = \frac{1}{3}U_0(X) - h > 0, \quad J_{11} = \frac{2}{3}\left[9h - U_0(X)\right] > 0 \quad (24)$$

and from the positivity of $\Lambda_{\rm eff}$ and J_{11} we obtain

$$h > \frac{1}{16}R_1 > \frac{1}{9}U_0(X) > \frac{1}{3}h > 0. \quad (25)$$

In the case when $d_1 = D_0$, these relations naturally coincide with the similar relations in [7] because here we do not use the explicit form of the potential $U(\phi)$. However, the expressions for J_{22} are different.

Similar to the previous case, the analysis shows that a positive minimum of the effective potential occurs only if $\mu < 0$ ($s = -1$), $p = +1$, $q = +1$ and $z \in (z_0, +\infty)$.

The conditions (24) and (25) clearly demonstrate the typical problem of stable compactification in multidimensional cosmological models. Here, a positive minimum occurs if the parameters are positive and the same order of magnitude: $\Lambda_{\rm eff} \approx R_1 \approx U(X) \approx h > 0$. On the other hand, in Kaluza–Klein models the size of the extra dimensions at the present time should be $b_{(0)1} \lesssim 10^{-17}$ cm $\equiv 1$ TeV^{-1}. In this case, $R_1 \gtrsim b_{(0)1}^{-2} \approx 10^{34}$ cm^{-2}. Thus, for an effective cosmological constant we obtain a value which is many orders of magnitude greater than the dark-energy value of about 10^{-57} cm^{-2} observable at the present time The necessary small value of the effective cosmological constant can be achieved only if the parameters R_1, $U(X)$ and h are extremely fine tuned to each other. We see two possibilities to avoid this problem. Firstly, the inclusion of different form fields or fluxes may result in a large number of minima (landscape) [13–16] with a sufficiently large probability of finding oneself in a dark-energy minimum. Secondly, we can avoid the restriction $R_1 \approx b_{(0)1}^{-2} \approx 10^{34}$ cm^{-2} if the internal space is Ricci flat: $R_1 = 0$. For example, the internal factor space M_1 can be an orbifold with branes at fixed points (see the corresponding discussion in [17]).

Another very interesting feature of the model under consideration is the multivalued form of the effective potential. As can easily be seen from equations (7) and (11), for each choice of μ, the potential $U(\phi)$ (and consequently $U_{\rm eff}$) has two branches ($q = \pm 1$) which join smoothly with each other at $\phi = 0$. It gives the very interesting possibility of investigating transitions from one branch to another by analogy with catastrophe theory or in a similar way to phase transitions in statistical theory. However, as we mentioned above, the point $\phi = 0$ corresponds to the singularity $\bar{R}$, $R \to \pm\infty$. Thus, the analogue of the second-order smooth phase transition through the point $\phi = 0$ is impossible in our model. However, there is still the possibility for the analogue of the first-order transition via quantum jumps from one branch to another.

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