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PARTIAL AVERAGING OF IMPULSIVE HYBRID SYSTEMS

Осадча О., Скрипник Н. Схема часткового усереднення для одного класу імпульсних змішаних систем. У статті розглядається обґрунтування схеми часткового усереднення для одного класу імпульсних змішаних систем, коли одне з рівнянь є імпульсним диференціальним рівнянням з похідною Хукухари, а друге – звичайним імпульсним диференціальним рівнянням.
Ключові слова: метод усереднення, змішана система, похідна Хукухари.

Осадчая О., Скрипник Н. Схема частичного усреднения для одного класса импульсных смешанных систем. В статье рассматривается обоснование схемы частичного усреднения для одного класса импульсных смешанных систем, когда одно из уравнений является импульсным дифференциальным уравнением с производной Хукухары, а второе – обыкновенным импульсным дифференциальным уравнением.
Ключевые слова: метод усреднения, смешанная система, производная Хукухары.

Osadcha O., Skripnik N. Partial averaging of impulsive hybrid systems. This paper contains the substantiation of the scheme of partial averaging for one class of impulsive hybrid systems where one equation is an impulsive differential equation with Hukuhara derivative and the second one is an impulsive ordinary differential equation.
Key words: averaging method, hybrid system, Hukuhara derivative.

Introduction.

In practice, there are often considered the so-called hybrid systems — systems that contain equations of different nature: for example, one of the equations is an equation in partial derivatives and the other one is an ordinary differential equation, or one of the equations is discrete and the other one is differential, etc. In this paper we consider the case of a hybrid system, where one of the equations is a differential equation with Hukuhara derivative and the second one is an ordinary differential equation. The interest in such systems follows from the fact, that some parameters of the model may be accurate, while the rest may contain the noise, errors and inaccuracies.

Main Definitions.

Development of the theory of multivalued mappings led to the question what should be understood as a derivative of a multivalued mapping. The main cause of difficulties for the inducting of such definition was the nonlinearity of the space $comp(R^n)$, which led to the absence of the concept of difference. There are several approaches to define the difference of two sets, one of them is the Hukuhara difference.

Let $conv(R^n)$ be the family of all nonempty compact convex subsets of R^n with the Hausdorff metric

$$h(A,B) = \max\{\max_{a\in A}\min_{b\in B}\|a-b\|,\ \max_{b\in B}\min_{a\in A}\|a-b\|\},$$

where $\|\cdot\|$ denotes the Euclidean norm in R^n.

Definition 1. [5] *Let $X, Y \in conv(R^n)$. The set $Z \in conv(R^n)$, where $X = Y + Z$, is called the Hukuhara difference of sets X and Y and is denoted by $X \overset{h}{-} Y$.*

Along with the inducted difference there appeared the concept of derivative.

Definition 2. [5] *A multuvalued mapping $X : I \to conv(R^n), I \subset R$, is called differentiable in the sense of Hukuhara at point $t \in I$ if there exists such $D_H X(t) \in \in conv(R^n)$ that the limits*

$$\lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}\left(X(t+\Delta t) \overset{h}{-} X(t)\right), \quad \lim_{\Delta t \downarrow 0} \frac{1}{\Delta t}\left(X(t) \overset{h}{-} X(t-\Delta t)\right)$$

exist and are equal to $D_H X(t)$. The set $D_H X(t)$ is called the Hukuhara derivative of the multivalued mapping $X : I \to conv(R^n)$ at point t.

In M. Hukuhara papers [5] along with the concept of derivative the concept of the integral of a multivalued mapping was inducted and the connection between those two concepts was found. In 1969 F. S. de Blasi and F. Iervolino were first to consider differential equation with the Hukuhara derivative [1, 2, 3, 4]. Its solution is a multivalued mapping. After that various existence and uniqueness theorems were proved, the stability of solutions of this type of equations was considered, integro-differential equations, impulsive differential equations, differential equations with fractional derivatives, control differential equations with Hukuhara derivative were considered, the possibility of applying some averaging schemes for such type of equations was investigated [9, 7, 12, 13, 8, 10, 11, 6].

Consider the hybrid system

$$\begin{cases} D_H X = F(t, X, y),\ X(t_0) = X_0, \\ \dot{y} = g(t, X, y),\ y(t_0) = y_0, \end{cases} \tag{1}$$

where $I = [t_0, T] \subset R$; $X : I \to conv(R^n)$ is a multivalued mapping; $y : I \to R^m$ is a vector function; $F : I \times conv(R^n) \times R^m \to conv(R^n)$ is a multivalued mapping; $g : I \times conv(R^n) \times R^m \to R^m$ is a vector function; $X_0 \in conv(R^n)$, $y_0 \in R^m$.

Consider a class S of pairs $(X(\cdot), y(\cdot))$, where $X(\cdot)$ is a continuously differentiable on I in the sense of Hukuhara multivalued mapping, $y(\cdot)$ is a continuously differentiable on I vector-function.

Definition 3. *A pair $(X(\cdot), y(\cdot)) \in S$ is called a solution of system (1), if for all $t \in I$ the following equalities fulfill $D_H X(t) = F(t, X(t), y(t))$, $\dot{y}(t) = g(t, X(t), y(t))$ and $X(t_0) = X_0$, $y(t_0) = y_0$.*

Theorem 1. *Let in the domain*

$$Q = \{(t, X, y) :\ t_0 \le t \le t_0 + a,\ h(X, X_0) \le b,\ \|y - y_0\| \le c\}$$

the multivalued mapping $F(t, X, y)$ and the vector function $g(t, X, y)$ are continuous and satisfy the Lipschitz condition in variables X and y, i. e. there exists such constant $\lambda > 0$ that

$$h\left(F(t, X_1, y_1), F(t, X_2, y_2)\right) \le \lambda\left[h(X_1, X_2) + \|y_1 - y_2\|\right],$$

$$\|g(t,X_1,y_1)-g(t,X_2,y_2)\| \le \lambda\left[h(X_1,X_2)+\|y_1-y_2\|\right].$$

Then system (1) has the unique solution defined on the interval $[t_0,t_0+d]$ *where* $d=\min\left(a,\dfrac{b}{M},\dfrac{c}{M}\right)$, *constant* M *satisfies the inequalities* $|F(t,X,y)|\le M$ *and* $\|g(t,X,y)\|\le M$ *in the domain* Q.

Main Results.

Consider impulsive hybrid system with a small parameter

$$\begin{aligned} &D_H X=\varepsilon F(t,X,y),\ t\ne\tau_i,\ X(0)=X_0,\\ &\dot y=\varepsilon g(t,X,y),\ t\ne\tau_i,\ y(0)=y_0, \end{aligned} \tag{2}$$

$$\Delta X|_{t=\tau_i}=\varepsilon I_i(X,y),\ \ \Delta y|_{t=\tau_i}=\varepsilon J_i(X,y), \tag{3}$$

where $t\in R_+$; the moments of impulses $\tau_{i+1}>\tau_i$; $X:R_+\to conv(R^n)$ is a multivalued mapping; $y:R_+\to R^m$ is a vector function; $F:R_+\times conv(R^n)\times R^m\to conv(R^n)$, $I_i:conv(R^n)\times R^m\to conv(R^n)$ are multivalued mappings; $g:R_+\times conv(R^n)\times R^m\to$ $\to R^m$, $J_i:conv(R^n)\times R^m\to R^m$ are vector functions; $X_0\in conv(R^n)$, $y_0\in R^m$.

Definition 4. *A pair* $(X(\cdot),y(\cdot))$ *is called a solution of system (2), (3) if it is a solution of system (2) on intervals between moments of impulses and satisfies impulse condition (3) in the points of impulses* τ_i.

It's easy to notice, that the existence and the uniqueness of a solution of system (2), (3) holds if the right sides of equations (2) satisfy Theorem 1 on intervals between moments of impulses.

Consider the following partially averaged system:

$$\begin{aligned} &D_H\bar X=\varepsilon\bar F(t,\bar X,\bar y),\ \ \bar X(0)=X_0,\\ &\dot{\bar y}=\varepsilon\bar g(t,\bar X,\bar y),\ \ \bar y(0)=y_0, \end{aligned} \tag{4}$$

$$\Delta X|_{t=\sigma_j}=\varepsilon\bar I_j(X,y),\ \Delta y|_{t=\sigma_j}=\varepsilon\bar J_j(X,y), \tag{5}$$

where

$$\begin{aligned} &\lim_{T\to\infty}\tfrac{1}{T}h\left(\int_0^T F(t,X,y)dt+\sum_{0\le\tau_i<T}I_i(X,y),\int_0^T\bar F(t,X,y)dt+\sum_{0\le\sigma_j<T}\bar I_j(X,y)\right)=0,\\ &\lim_{T\to\infty}\tfrac{1}{T}\left\|\int_0^T[g(t,X,y)-\bar g(t,X,y)]dt+\sum_{0\le\tau_i<T}J_i(X,y)-\sum_{0\le\sigma_j<T}\bar J_j(X,y)\right\|=0. \end{aligned} \tag{6}$$

The following theorem that proves the closeness of solutions of systems (2), (3) and (4), (5) holds.

Theorem 2. *Let in the domain* $Q=\{(t,X,y):t\ge0,\ X\in D_1,\ y\in D_2\}$ *the following conditions hold:*

1) the multivalued mappings $F(t,X,y),\bar F(t,X,y)$ *and vector functions* $g(t,X,y),\bar g(t,X,y)$ *are continuous in* t, *uniformly bounded with constant* M *and satisfy the Lipschitz condition in* X *and* y *with constant* λ;

2) the multivalued mappings $I_i(X,y), \bar{I}_j(X,y)$ and vector functions $J_i(X,y)$, $\bar{J}_j(X,y)$ are uniformly bounded and satisfy the Lipschitz condition in X and y with constant λ;

3) limits (6) exist uniformly with respect to $X \in D_1$ and $y \in D_2$;

4) there exists such constant $0 < d \le \infty$ that

$$\frac{1}{T} i(t,t+T) \le d, \ \frac{1}{T} j(t,t+T) \le d,$$

where $i(t,t+T)[j(t,t+T)]$ are the numbers of points of a sequence $\{\tau_i\}[\{\sigma_j\}]$ on the interval $[t,t+T)$.

5) the solution $\left(\bar{X}(t), \bar{y}(t)\right)$ of system (4), (5) with the initial condition $\bar{X}(0) = X_0 \in D_1^{'} \subset D_1$, $\bar{y}(0) = y_0 \in D_2^{'} \subset D_2$ is defined for all $t \ge 0$, $\varepsilon \in (0, \bar{\varepsilon}]$ and $\bar{X}(t)$ belongs with some ρ- neighborhood to the domain D_1, $\bar{y}(t)$ belongs with some ξ-neighborhood to the domain D_2.

Then for any $\eta > 0$ and $L > 0$ there exists such $\varepsilon_0(\eta, L) \in (0, \bar{\varepsilon}]$, that for $0 < \varepsilon \le \varepsilon_0$ and $0 \le t \le L\varepsilon^{-1}$ the following inequalities fulfill:

$$h\left(X(t), \bar{X}(t)\right) \le \eta, \|y(t) - \bar{y}(t)\| \le \eta,$$

where $(X(\cdot), y(\cdot))$ and $\left(\bar{X}(\cdot), \bar{y}(\cdot)\right)$ are the solutions of systems (2), (3) and (4), (5) with the initial conditions $X(0) = \bar{X}(0) \in D_1'$, $y(0) = \bar{y}(0) \in D_2'$.

Proof. From conditions 1)–3) of the theorem it follows that systems (2), (3) and (4), (5) have unique solutions that are defined for $t \ge 0$ if $X(t)$ and $y(t)$ (accordingly $\bar{X}(t)$ and $\bar{y}(t)$) belong to the domain $D_1 \times D_2$.

Replace systems (2), (3) and (4), (5) with the equivalent system of integral equations:

$$\begin{cases} X(t) = X_0 + \varepsilon \int\limits_0^t F(s, X(s), y(s))ds + \varepsilon \sum\limits_{0 \le \tau_i < t} I_i(X(\tau_i), y(\tau_i)), \\ y(t) = y_0 + \varepsilon \int\limits_0^t g(s, X(s), y(s))ds + \varepsilon \sum\limits_{0 \le \tau_i < t} J_i(X(\tau_i), y(\tau_i)), \end{cases} \tag{7}$$

$$\begin{cases} \bar{X}(t) = X_0 + \varepsilon \int\limits_0^t \bar{F}(s, \bar{X}(s), \bar{y}(s))ds + \varepsilon \sum\limits_{0 \le \sigma_j < t} \bar{I}_j(\bar{X}(\sigma_j), \bar{y}(\sigma_j)), \\ \bar{y}(t) = y_0 + \varepsilon \int\limits_0^t \bar{g}(s, \bar{X}(s), \bar{y}(s))ds + \varepsilon \sum\limits_{0 \le \sigma_j < t} \bar{J}_j(\bar{X}(\sigma_j), \bar{y}(\sigma_j)). \end{cases} \tag{8}$$

Then

$$
\begin{aligned}
& h\left(X(t),\bar X(t)\right) = \\
&= h\left(X_0+\varepsilon\int_0^t F(s,X(s),y(s))ds+\varepsilon\sum_{0\le\tau_i<t} I_i(X(\tau_i),y(\tau_i)),\right.\\
&\left. X_0+\varepsilon\int_0^t \bar F(s,\bar X(s),\bar y(s))ds+\varepsilon\sum_{0\le\sigma_j<t}\bar I_j(\bar X(\sigma_j),\bar y(\sigma_j))\right) = \\
&= \varepsilon h\left(\int_0^t F(s,X(s),y(s))ds+\sum_{0\le\tau_i<t} I_i(X(\tau_i),y(\tau_i)),\right.\\
&\left.\int_0^t \bar F(s,\bar X(s),\bar y(s))ds+\sum_{0\le\sigma_j<t}\bar I_j(\bar X(\sigma_j),\bar y(\sigma_j))\right) \le \\
&\le \varepsilon h\left(\int_0^t F(s,X(s),y(s))ds,\int_0^t F(s,\bar X(s),\bar y(s))ds\right)+\\
&+\varepsilon h\left(\sum_{0\le\tau_i<t} I_i(X(\tau_i),y(\tau_i)),\sum_{0\le\tau_i<t} I_i(\bar X(\tau_i),\bar y(\tau_i))\right)+\\
&+\varepsilon h\left(\int_0^t F(s,\bar X(s),\bar y(s))ds+\sum_{0\le\tau_i<t} I_i(\bar X(\tau_i),\bar y(\tau_i)),\right.\\
&\left.\int_0^t \bar F(s,\bar X(s),\bar y(s))ds+\varepsilon\sum_{0\le\sigma_j<t}\bar I_j(\bar X(\sigma_j),\bar y(\sigma_j))\right) \le \\
&\le \varepsilon\int_0^t h\left(F(s,X(s),y(s)),F(s,\bar X(s),\bar y(s))\right)ds+\\
&+\varepsilon\sum_{0\le\tau_i<t} h\left(I_i(X(\tau_i),y(\tau_i)),I_i(\bar X(\tau_i),\bar y(\tau_i))\right)+\\
&+\varepsilon h\left(\int_0^t F(s,\bar X(s),\bar y(s))ds+\sum_{0\le\tau_i<t} I_i(\bar X(\tau_i),\bar y(\tau_i)),\right.\\
&\left.\int_0^t \bar F(s,\bar X(s),\bar y(s))ds+\varepsilon\sum_{0\le\sigma_j<t}\bar I_j(\bar X(\sigma_j),\bar y(\sigma_j))\right) \le \\
&\le \varepsilon\lambda\int_0^t [h(X(s),\bar X(s))+|y(s)-\bar y(s)|]ds+\\
&+\varepsilon\lambda\sum_{0\le\tau_i<t}[h(X(\tau_i),\bar X(\tau_i))+|y(\tau_i)-\bar y(\tau_i)|]+\\
&+\varepsilon h\left(\int_0^t F(s,\bar X(s),\bar y(s))ds+\sum_{0\le\tau_i<t} I_i(\bar X(\tau_i),\bar y(\tau_i)),\right.\\
&\left.\int_0^t \bar F(s,\bar X(s),\bar y(s))ds+\varepsilon\sum_{0\le\sigma_j<t}\bar I_j(\bar X(\sigma_j),\bar y(\sigma_j))\right).
\end{aligned}
\tag{9}
$$

Similarly,

$$
\begin{aligned}
&\|y(t)-\bar y(t)\| \le \varepsilon\lambda\int_0^t \left[h(X(s),\bar X(s))+\|y(s)-\bar y(s)\|\right]ds+\\
&+\varepsilon\lambda\sum_{0\le\tau_i<t}\left[h(X(\tau_i),\bar X(\tau_i))+\|y(\tau_i)-\bar y(\tau_i)\|\right]+\\
&+\varepsilon\left\|\int_0^t g(s,\bar X(s),\bar y(s))ds+\sum_{0\le\tau_i<t} J_i(\bar X(\tau_i),\bar y(\tau_i))-\right.
\end{aligned}
$$

$$-\int_0^t \bar{g}(s,\bar{X}(s),\bar{y}(s))ds - \varepsilon \sum_{0\le\sigma_j<t} \bar{J}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j))\Bigg\|. \qquad (10)$$

Divide the interval $[0, L\varepsilon^{-1}]$ in m equal intervals by the points $t_p = \frac{pL}{\varepsilon m}$. Define by $(\bar{X}_p, \bar{y}_p) = (\bar{X}(t_p), \bar{y}(t_p))$ the solution of system (4), (5) in division points.

Let us estimate in the interval $[t_k, t_{k+1}]$, where $0 \le k \le m-1$ the expression

$$\begin{aligned}
&\varepsilon h\Bigg(\int_0^t F(s,\bar{X}(s),\bar{y}(s))ds + \sum_{0\le\tau_i<t} I_i(\bar{X}(\tau_i),\bar{y}(\tau_i)),\\
&\int_0^t \bar{F}(s,\bar{X}(s),\bar{y}(s))ds + \varepsilon\sum_{0\le\sigma_j<t}\bar{I}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j))\Bigg) =\\
&= \varepsilon h\Bigg(\sum_{p=0}^{k-1}\left[\int_{t_p}^{t_{p+1}} F(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_p\le\tau_i<t_{p+1}} I_i(\bar{X}(\tau_i),\bar{y}(\tau_i))\right] +\\
&\quad + \int_{t_k}^t F(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_k\le\tau_i<t} I_i(\bar{X}(\tau_i),\bar{y}(\tau_i)),\\
&\sum_{p=0}^{k-1}\left[\int_{t_p}^{t_{p+1}} \bar{F}(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_p\le\sigma_j<t_{p+1}} \bar{I}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j))\right] +\\
&\quad + \int_{t_k}^t \bar{F}(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_k\le\sigma_j<t} \bar{I}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j))\Bigg) \le\\
&\le \varepsilon h\Bigg(\sum_{p=0}^{k-1}\left[\int_{t_p}^{t_{p+1}} F(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_p\le\tau_i<t_{p+1}} I_i(\bar{X}(\tau_i),\bar{y}(\tau_i))\right],\\
&\sum_{p=0}^{k-1}\left[\int_{t_p}^{t_{p+1}} \bar{F}(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_p\le\sigma_j<t_{p+1}} \bar{I}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j))\right]\Bigg) +\\
&\quad +\varepsilon h\Bigg(\int_{t_k}^t F(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_k\le\tau_i<t} I_i(\bar{X}(\tau_i),\bar{y}(\tau_i)),\\
&\int_{t_k}^t \bar{F}(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_k\le\sigma_j<t} \bar{I}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j))\Bigg) \le
\end{aligned}$$

$$\begin{aligned}
&\le \varepsilon\sum_{p=0}^{k-1}\Bigg[h\Bigg(\int_{t_p}^{t_{p+1}} F(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_p\le\tau_i<t_{p+1}} I_i(\bar{X}(\tau_i),\bar{y}(\tau_i)), \int_{t_p}^{t_{p+1}} F(s,\bar{X}_p,\bar{y}_p)ds + \sum_{t_p\le\tau_i<t_{p+1}} I_i(\bar{X}_p,\bar{y}_p)\Bigg) +\\
&\quad +h\Bigg(\int_{t_p}^{t_{p+1}} F(s,\bar{X}_p,\bar{y}_p)ds + \sum_{t_p\le\tau_i<t_{p+1}} I_i(\bar{X}_p,\bar{y}_p), \int_{t_p}^{t_{p+1}} \bar{F}(s,\bar{X}_p,\bar{y}_p)ds + \sum_{t_p\le\sigma_j<t_{p+1}} \bar{I}_j(\bar{X}_p,\bar{y}_p)\Bigg) +\\
&\quad +h\Bigg(\int_{t_p}^{t_{p+1}} \bar{F}(s,\bar{X}_p,\bar{y}_p)ds + \sum_{t_p\le\sigma_j<t_{p+1}} \bar{I}_j(\bar{X}_p,\bar{y}_p), \int_{t_p}^{t_{p+1}} \bar{F}(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_p\le\sigma_j<t_{p+1}} \bar{I}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j))\Bigg)\Bigg] +\\
&\quad +\varepsilon\Bigg[h\Bigg(\int_{t_k}^t F(s,\bar{X}(s),\bar{y}(s))ds + \sum_{t_k\le\tau_i<t} I_i(\bar{X}(\tau_i),\bar{y}(\tau_i)), \int_{t_k}^t F(s,\bar{X}_k,\bar{y}_k)ds + \sum_{t_k\le\tau_i<t} I_i(\bar{X}_k,\bar{y}_k)\Bigg) +\\
&\quad +h\Bigg(\int_{t_k}^t F(s,\bar{X}_k,\bar{y}_k)ds + \sum_{t_k\le\tau_i<t} I_i(\bar{X}_k,\bar{y}_k), \int_{t_k}^t \bar{F}(s,\bar{X}_k,\bar{y}_k)ds + \sum_{t_k\le\sigma_j<t} \bar{I}_j(\bar{X}_k,\bar{y}_k)\Bigg) +
\end{aligned}$$

$$+h\left(\int\limits_{t_k}^{t}\bar F(s,\bar X_k,\bar y_k)ds+\sum_{t_k\le\sigma_j<t}\bar I_j(\bar X_k,\bar y_k),\int\limits_{t_k}^{t}\bar F(s,\bar X(s),\bar y(s))ds+\sum_{t_k\le\sigma_j<t}\bar I_j(\bar X(\sigma_j),\bar y(\sigma_j))\right)\Bigg]\le$$

$$\le\varepsilon\sum_{p=0}^{k-1}\Bigg[\int\limits_{t_p}^{t_{p+1}}h\left(F(s,\bar X(s),\bar y(s)),F(s,\bar X_p,\bar y_p)\right)ds+\sum_{t_p\le\tau_i<t_{p+1}}h\left(I_i(\bar X(\tau_i),\bar y(\tau_i)),I_i(\bar X_p,\bar y_p)\right)+$$

$$+h\left(\int\limits_{t_p}^{t_{p+1}}F(s,\bar X_p,\bar y_p)ds+\sum_{t_p\le\tau_i<t_{p+1}}I_i(\bar X_p,\bar y_p),\int\limits_{t_p}^{t_{p+1}}\bar F(s,\bar X_p,\bar y_p)ds+\sum_{t_p\le\sigma_j<t_{p+1}}\bar I_j(\bar X_p,\bar y_p)\right)+$$

$$+\int\limits_{t_p}^{t_{p+1}}h\left(\bar F(s,\bar X_p,\bar y_p),\bar F(s,\bar X(s),\bar y(s))ds\right)ds+\sum_{t_p\le\sigma_j<t_{p+1}}h\left(\bar I_j(\bar X(\sigma_j),\bar y(\sigma_j)),\bar I_j(\bar X_p,\bar y_p)\right)\Bigg]+$$

$$+\varepsilon\Bigg[\int\limits_{t_k}^{t}h\left(F(s,\bar X(s),\bar y(s)),F(s,\bar X_k,\bar y_k)\right)ds+\sum_{t_k\le\tau_i<t}h\left(I_i(\bar X(\tau_i),\bar y(\tau_i)),I_i(\bar X_k,\bar y_k)\right)+$$

$$+h\left(\int\limits_{t_k}^{t}F(s,\bar X_k,\bar y_k)ds+\sum_{t_k\le\tau_i<t}I_i(\bar X_k,\bar y_k),\int\limits_{t_k}^{t}\bar F(s,\bar X_k,\bar y_k)ds+\sum_{t_k\le\sigma_j<t}\bar I_j(\bar X_k,\bar y_k)\right)+$$

$$+\int\limits_{t_k}^{t}h\left(\bar F(s,\bar X_k,\bar y_k),\bar F(\bar X(s),\bar y(s))\right)ds+\sum_{t_k\le\sigma_j<t}h\left(\bar I_j(\bar X(\sigma_j),\bar y(\sigma_j)),\bar I_j(\bar X_k,\bar y_k)\right)\Bigg]\le$$

$$\le\varepsilon\sum_{p=0}^{k}\Bigg[\int\limits_{t_p}^{t_{p+1}}h\left(F(s,\bar X(s),\bar y(s)),F(s,\bar X_p,\bar y_p)\right)ds+\sum_{t_p\le\tau_i<t_{p+1}}h\left(I_i(\bar X(\tau_i),\bar y(\tau_i)),I_i(\bar X_p,\bar y_p)\right)+$$

$$+\int\limits_{t_p}^{t_{p+1}}h\left(\bar F(s,\bar X_p,\bar y_p),\bar F(s,\bar X(s),\bar y(s))\right)ds+\sum_{t_p\le\sigma_j<t_{p+1}}h\left(\bar I_j(\bar X(\sigma_j),\bar y(\sigma_j)),\bar I_j(\bar X_p,\bar y_p)\right)\Bigg]+$$

$$+\varepsilon\sum_{p=0}^{k-1}h\left(\int\limits_{t_p}^{t_{p+1}}F(s,\bar X_p,\bar y_p)ds+\sum_{t_p\le\tau_i<t_{p+1}}I_i(\bar X_p,\bar y_p),\int\limits_{t_p}^{t_{p+1}}\bar F(s,\bar X_p,\bar y_p)ds+\sum_{t_p\le\sigma_j<t_{p+1}}\bar I_j(\bar X_p,\bar y_p)\right)+$$

$$+\varepsilon h\left(\int\limits_{t_k}^{t}F(s,\bar X_k,\bar y_k)ds+\sum_{t_k\le\tau_i<t}I_i(\bar X_k,\bar y_k),\int\limits_{t_k}^{t}\bar F(s,\bar X_k,\bar y_k)ds+\sum_{t_k\le\sigma_j<t}\bar I_j(\bar X_k,\bar y_k)\right).$$

Similarly

$$\varepsilon\Bigg\|\int\limits_{0}^{t}g(s,\bar X(s),\bar y(s))ds+\sum_{0\le\tau_i<t}J_i(\bar X(\tau_i),\bar y(\tau_i))-\int\limits_{0}^{t}\bar g(s,\bar X(s),\bar y(s))ds-$$

$$-\sum_{0\le\sigma_j<t}\bar J_j(\bar X(\sigma_j),\bar y(\sigma_j))\Bigg\|\le\varepsilon\sum_{p=0}^{k}\Bigg[\int\limits_{t_p}^{t_{p+1}}\left\|g(s,\bar X(s),\bar y(s))-g(s,\bar X_p,\bar y_p)\right\|ds+$$

$$+\sum_{t_p\le\tau_i<t_{p+1}}\left\|J_i(\bar X(\tau_i),\bar y(\tau_i))-J_i(\bar X_p,\bar y_p)\right\|+\int\limits_{t_p}^{t_{p+1}}\left\|\bar g(s,\bar X_p,\bar y_p)-\bar g(s,\bar X(s),\bar y(s))\right\|ds+$$

$$+\sum_{t_p\le\sigma_j<t_{p+1}}\left\|\bar J_j(\bar X(\sigma_j),\bar y(\sigma_j))-\bar J_j(\bar X_p,\bar y_p)\right\|\Bigg]+$$

$$+\varepsilon\sum_{p=0}^{k-1}\Bigg\|\int\limits_{t_p}^{t_{p+1}}[g(s,\bar X_p,\bar y_p)-\bar g(s,\bar X_p,\bar y_p)]ds+\sum_{t_p\le\tau_i<t_{p+1}}J_i(\bar X_p,\bar y_p)-\sum_{t_p\le\sigma_j<t_{p+1}}\bar J_j(\bar X_p,\bar y_p)\Bigg\|+$$

$$+\varepsilon\Bigg\|\int\limits_{t_k}^{t}[g(s,\bar X_k,\bar y_k)-\bar g(s,\bar X_k,\bar y_k)]ds+\sum_{t_k\le\tau_i<t}J_i(\bar X_k,\bar y_k)-\sum_{t_k\le\sigma_j<t}\bar J_j(\bar X_k,\bar y_k)\Bigg\|.$$

Notice that

$$h\left(\bar{X}(s),\bar{X}_p\right)=h\left(\bar{X}(s),\bar{X}(t_p)\right)\le\varepsilon\int\limits_{t_p}^{s}h\left(\bar{F}(\bar{X}(v),\bar{y}(v)),\{0\}\right)dv+$$
$$+\sum_{t_p\le\sigma_j<s}h(\bar{I}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j)),\{0\})\le\varepsilon M(d+1)(s-t_p),$$
$$\|\bar{y}(s)-\bar{y}_p\|=\|\bar{y}(s)-\bar{y}(t_p)\|\le$$
$$\le\varepsilon\int\limits_{t_p}^{s}\left\|\bar{g}(\bar{X}(v),\bar{y}(v))\right\|dv+\sum_{t_p\le\sigma_j<s}||\bar{J}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j))||\le\varepsilon M(d+1)(s-t_p).$$

Then

$$\varepsilon\sum_{p=0}^{k}\int\limits_{t_p}^{t_{p+1}}h\left(F(s,\bar{X}(s),\bar{y}(s)),F(s,\bar{X}_p,\bar{y}_p)\right)ds\le$$
$$\le\varepsilon\sum_{p=0}^{k}\int\limits_{t_p}^{t_{p+1}}\lambda\left[h\left(\bar{X}(s),\bar{X}_p\right)+\|\bar{y}(s)-\bar{y}_p\|\right]ds\le$$
$$\le\varepsilon\lambda\cdot2\varepsilon M(d+1)\sum_{p=0}^{k}\int\limits_{t_p}^{t_{p+1}}(s-t_p)ds=2\varepsilon^2\lambda M(d+1)\sum_{p=0}^{k}\tfrac{(t_{p+1}-t_p)^2}{2}=$$
$$=\varepsilon^2\lambda M(d+1)\cdot(k+1)\cdot\left(\tfrac{L}{\varepsilon m}\right)^2\le\tfrac{\lambda ML^2(d+1)}{m},$$
$$\varepsilon\sum_{p=0}^{k}\sum_{t_p\le\tau_i<t_{p+1}}h\left(I_i(\bar{X}(\tau_i),\bar{y}(\tau_i)),I_i(\bar{X}_p,\bar{y}_p)\right)\le$$
$$\le\varepsilon\sum_{p=0}^{k}\sum_{t_p\le\tau_i<t_{p+1}}\lambda\left[h\left(\bar{X}(\tau_i),\bar{X}_p\right)+\|\bar{y}(\tau_i)-\bar{y}_p\|\right]\le$$

$$\le\varepsilon\sum_{p=0}^{k}\sum_{t_p\le\tau_i<t_{p+1}}2\lambda\varepsilon M(d+1)(\tau_i-t_p)=2\varepsilon^2\lambda M(d+1)\sum_{p=0}^{k}\sum_{t_p\le\tau_i<t_{p+1}}(\tau_i-t_p)\le$$
$$\le2\varepsilon^2\lambda M(d+1)\cdot\tfrac{L}{\varepsilon m}\sum_{p=0}^{k}\sum_{t_p\le\tau_i<t_{p+1}}1\le\tfrac{2\varepsilon\lambda ML(d+1)}{m}\cdot d\cdot\tfrac{L}{\varepsilon m}\cdot(k+1)\le\tfrac{2\lambda ML^2d(d+1)}{m},$$
$$\varepsilon\sum_{p=0}^{k}\int\limits_{t_p}^{t_{p+1}}h\left(\bar{F}(\bar{X}_p,\bar{y}_p),\bar{F}(\bar{X}(s),\bar{y}(s))\right)ds\le$$
$$\le\varepsilon\sum_{p=0}^{k}\int\limits_{t_p}^{t_{p+1}}\lambda\left[h\left(\bar{X}_p,\bar{X}(s)\right)+\|\bar{y}_p-\bar{y}(s)\|\right]ds\le\tfrac{\lambda ML^2(d+1)}{m},$$
$$\varepsilon\sum_{p=0}^{k}\sum_{t_p\le\sigma_j<t_{p+1}}h\left(\bar{I}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j)),\bar{I}_j(\bar{X}_p,\bar{y}_p)\right)\le\tfrac{2\lambda ML^2d(d+1)}{m}.$$

Similarly

$$\varepsilon\sum_{p=0}^{k}\int\limits_{t_p}^{t_{p+1}}\left\|g(s,\bar{X}(s),\bar{y}(s))-g(s,\bar{X}_p,\bar{y}_p)\right\|ds\le\tfrac{\lambda ML^2(d+1)}{m},$$
$$\varepsilon\sum_{p=0}^{k}\sum_{t_p\le\tau_i<t_{p+1}}\left\|J_i(\bar{X}(\tau_i),\bar{y}(\tau_i))-J_i(\bar{X}_p,\bar{y}_p)\right\|\le\tfrac{2\lambda ML^2d(d+1)}{m},$$
$$\varepsilon\sum_{p=0}^{k}\int\limits_{t_p}^{t_{p+1}}\left\|\bar{g}(\bar{X}_p,\bar{y}_p)-\bar{g}(\bar{X}(s),\bar{y}(s))\right\|ds\le\tfrac{\lambda ML^2(d+1)}{m},$$
$$\varepsilon\sum_{p=0}^{k}\sum_{t_p\le\sigma_j<t_{p+1}}\left\|\bar{J}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j))-\bar{J}_j(\bar{X}_p,\bar{y}_p)\right\|\le\tfrac{2\lambda ML^2d(d+1)}{m}.$$

Using condition 3) of the theorem there exist such monotone decreasing functions $f_1(t)$ and $f_2(t)$ that tend to zero as $t \to \infty$, that for all $(X, y) \in D_1 \times D_2$ we have:

$$h\left(\int_0^t F(s,\bar{X},\bar{y})ds + \sum_{0\le\tau_i<t} I_i(\bar{X},\bar{y}), \int_0^t \bar{F}(s,\bar{X},\bar{y})ds + \sum_{0\le\sigma_j<t} \bar{I}_j(\bar{X},\bar{y})\right) \le t \cdot f_1(t),$$
$$\left\|\int_0^t \big(g(s,\bar{X},\bar{y}) - \bar{g}(s,\bar{X},\bar{y})\big)\, ds + \sum_{0\le\tau_i<t} J_i(\bar{X},\bar{y}) - \sum_{0\le\sigma_j<t} \bar{J}_j(\bar{X},\bar{y})\right\| \le t \cdot f_2(t).$$

Then

$$\varepsilon h\left(\int_{t_p}^{t_{p+1}} F(s,\bar{X}_p,\bar{y}_p)ds + \sum_{t_p\le\tau_i<t_{p+1}} I_i(\bar{X}_p,\bar{y}_p), \int_{t_p}^{t_{p+1}} \bar{F}(s,\bar{X}_p,\bar{y}_p)ds + \sum_{t_p\le\sigma_j<t_{p+1}} \bar{I}_j(\bar{X}_p,\bar{y}_p)\right) =$$
$$= \varepsilon h\left(\int_0^{t_{p+1}} F(s,\bar{X}_p,\bar{y}_p)ds \overset{h}{-} \int_0^{t_p} F(s,\bar{X}_p,\bar{y}_p)ds + \sum_{0\le\tau_i<t_{p+1}} I_i(\bar{X}_p,\bar{y}_p) \overset{h}{-} \sum_{0\le\tau_i<t_p} I_i(\bar{X}_p,\bar{y}_p)\right.,$$
$$\left.\int_0^{t_{p+1}} \bar{F}(\bar{X}_p,\bar{y}_p)ds \overset{h}{-} \int_0^{t_p} \bar{F}(\bar{X}_p,\bar{y}_p)ds + \sum_{0\le\sigma_j<t_{p+1}} \bar{I}_j(\bar{X}_p,\bar{y}_p) \overset{h}{-} \sum_{0\le\sigma_j<t_p} \bar{I}_j(\bar{X}_p,\bar{y}_p)\right) \le$$
$$\le \varepsilon\left[h\left(\int_0^{t_{p+1}} F(s,\bar{X}_p,\bar{y}_p)ds + \sum_{0\le\tau_i<t_{p+1}} I_i(\bar{X}_p,\bar{y}_p), \int_0^{t_{p+1}} \bar{F}(s,\bar{X}_p,\bar{y}_p)ds + \sum_{0\le\sigma_j<t_{p+1}} \bar{I}_j(\bar{X}_p,\bar{y}_p)\right) + \right.$$
$$\left. + h\left(\int_0^{t_p} F(s,\bar{X}_p,\bar{y}_p)ds + \sum_{0\le\tau_i<t_p} I_i(\bar{X}_p,\bar{y}_p), \int_0^{t_p} \bar{F}(s,\bar{X}_p,\bar{y}_p)ds + \sum_{0\le\sigma_j<t_p} \bar{I}_j(\bar{X}_p,\bar{y}_p)\right)\right] \le$$
$$\le \varepsilon\left[t_{p+1} \cdot f_1(t_{p+1}) + t_p \cdot f_1(t_p)\right] \le 2 \sup_{\tau\in[0,L]} \tau f_1\left(\frac{\tau}{\varepsilon}\right) = \gamma_1(\varepsilon),$$

where $\tau = \varepsilon t$, $\lim\limits_{\varepsilon\to 0} \gamma_1(\varepsilon) = 0$. Similarly

$$\varepsilon\left\|\int_{t_p}^{t_{p+1}} [g(s,\bar{X}_p,\bar{y}_p) - \bar{g}(s,\bar{X}_p,\bar{y}_p)]ds + \sum_{t_p\le\tau_i<t_{p+1}} J_i(\bar{X}_p,\bar{y}_p) - \sum_{t_p\le\sigma_j<t_{p+1}} \bar{J}_j(\bar{X}_p,\bar{y}_p)\right\| \le \gamma_2(\varepsilon),$$

where $\lim\limits_{\varepsilon\to 0} \gamma_2(\varepsilon) = 0$;

$$h\left(\int_{t_k}^t F(s,\bar{X}_k,\bar{y}_k)ds + \sum_{t_k\le\tau_i<t} I_i(\bar{X}_k,\bar{y}_k), \int_{t_k}^t \bar{F}(s,\bar{X}_k,\bar{y}_k)ds + \sum_{t_k\le\sigma_j<t} \bar{I}_j(\bar{X}_k,\bar{y}_k)\right) \le \gamma_1(\varepsilon),$$
$$\varepsilon\left\|\int_{t_k}^t \big(g(s,\bar{X}_k,\bar{y}_k) - \bar{g}(\bar{X}_k,\bar{y}_k)\big)\, ds + \sum_{t_k\le\tau_i<t} J_i(\bar{X}_k,\bar{y}_k) - \sum_{t_k\le\sigma_j<t} \bar{J}_j(\bar{X}_k,\bar{y}_k)\right\| \le \gamma_2(\varepsilon).$$

So

$$\varepsilon h\left(\int_0^t F(s,\bar{X}(s),\bar{y}(s))ds + \sum_{0\le\tau_i<t} I_i(\bar{X}(\tau_i),\bar{y}(\tau_i)),\right.$$
$$\left.\int_0^t \bar{F}(s,\bar{X}(s),\bar{y}(s))ds + \sum_{0\le\sigma_j<t} \bar{I}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j))\right) \le$$

$$\leq \frac{\lambda ML^2(d+1)}{m} + \frac{4\lambda ML^2}{m}d(d+1) + \frac{\lambda ML^2(d+1)}{m} + (k+1)\gamma_1(\varepsilon) \leq$$
$$\leq \frac{2\lambda ML^2}{m}(2d+1)(d+1) + m\gamma_1(\varepsilon) \equiv \varphi_1(\varepsilon, m), \quad (11)$$

$$\varepsilon \left\| \int_0^t [g(s,\bar{X}(s),\bar{y}(s)) - \bar{g}(s,\bar{X}(s),\bar{y}(s))]ds + \right.$$
$$\left. + \sum_{0\leq\tau_i<t} J_i(\bar{X}(\tau_i),\bar{y}(\tau_i)) - \sum_{0\leq\sigma_j<t} \bar{J}_j(\bar{X}(\sigma_j),\bar{y}(\sigma_j)) \right\| \leq$$
$$\leq \frac{\lambda ML^2(d+1)}{m} + \frac{4\lambda ML^2}{m}d(d+1) + \frac{\lambda ML^2(d+1)}{m} + (k+1)\gamma_2(\varepsilon) \leq$$
$$\leq \frac{2\lambda ML^2}{m}(2d+1)(d+1) + m\gamma_2(\varepsilon) \equiv \varphi_2(\varepsilon, m). \quad (12)$$

If we substitute (11) in (9) and (12) in (10), we will get

$$h\left(X(t),\bar{X}(t)\right) \leq \varepsilon\lambda \int_0^t \left[h(X(s),\bar{X}(s)) + \|y(s)-\bar{y}(s)\|\right] ds +$$
$$+\varepsilon\lambda \sum_{0\leq\tau_i<t} \left[h(X(\tau_i),\bar{X}(\tau_i)) + \|y(\tau_i)-\bar{y}(\tau_i)\|\right] + \varphi_1(\varepsilon,m),$$
$$\|y(t)-\bar{y}(t)\| \leq \varepsilon\lambda \int_0^t \left[h(X(s),\bar{X}(s)) + \|y(s)-\bar{y}(s)\|\right] ds +$$
$$+\varepsilon\lambda \sum_{0\leq\tau_i<t} \left[h(X(\tau_i),\bar{X}(\tau_i)) + \|y(\tau_i)-\bar{y}(\tau_i)\|\right] + \varphi_2(\varepsilon,m).$$

Adding these two inequalities and applying the analogue of Gronwall—Bellmann lemma [14] we get

$$h\left(X(t),\bar{X}(t)\right) + \|y(t)-\bar{y}(t)\| \leq (\varphi_1(\varepsilon,m)+\varphi_2(\varepsilon,m))\,(1+\varepsilon\lambda)^{i(0,t)}\, e^{\varepsilon\lambda t} \leq$$
$$\leq (\varphi_1(\varepsilon,m)+\varphi_2(\varepsilon,m))\,(1+\varepsilon\lambda)^{d\cdot t}\, e^{\varepsilon\lambda t} \leq$$
$$\leq (\varphi_1(\varepsilon,m)+\varphi_2(\varepsilon,m))\,(1+\varepsilon\lambda)^{d\cdot\frac{L}{\varepsilon}}\, e^{\lambda L} \leq$$
$$\leq (\varphi_1(\varepsilon,m)+\varphi_2(\varepsilon,m))\, e^{(d+1)\lambda L} =$$
$$= \left(\tfrac{4\lambda ML^2}{m}(2d+1)(d+1) + m\gamma_1(\varepsilon) + m\gamma_2(\varepsilon)\right) e^{(d+1)\lambda L}.$$

Then for every summand the inequality holds:

$$h\left(X(t),\bar{X}(t)\right) \leq \left(\tfrac{4\lambda ML^2}{m}(2d+1)(d+1) + m\gamma_1(\varepsilon) + m\gamma_2(\varepsilon)\right) e^{(d+1)\lambda L},$$
$$\|y(t)-\bar{y}(t)\| \leq \left(\tfrac{4\lambda ML^2}{m}(2d+1)(d+1) + m\gamma_1(\varepsilon) + m\gamma_2(\varepsilon)\right) e^{(d+1)\lambda L}.$$

Let $\eta_1 = \min\{\rho,\eta,\xi\}$. Choose m to satisfy the inequality

$$e^{(d+1)\lambda L}\frac{\lambda ML^2}{m}(2d+1)(d+1) < \frac{\eta_1}{12}.$$

Then fix m and choose $\varepsilon_0 \in (0,\bar{\varepsilon}]$ such that for $\varepsilon \in (0,\varepsilon_0]$ the inequalities hold

$$e^{(d+1)\lambda L} m\gamma_1(\varepsilon) \leq \frac{\eta_1}{3},\, e^{(d+1)\lambda L} m\gamma_2(\varepsilon) \leq \frac{\eta_1}{3}.$$

Then $h\left(X(t),\bar{X}(t)\right) \le \eta_1$ and $\|y(t)-\bar{y}(t)\| \le \eta_1$ if the solution $(X(t),y(t))$ belongs to the domain $D_1 \times D_2$. And it follows from condition 3) of the theorem as $\eta_1 = \min\{\eta,\rho,\xi\}$.

So, we get that for any $\eta > 0$ and $L > 0$ there exists such $\varepsilon_0 \in (0,\bar{\varepsilon}]$ that for $\varepsilon \in (0,\varepsilon_0]$ and $t \in [0, L\varepsilon^{-1}]$ the following inequalities fulfill

$$h\left(X(t),\bar{X}(t)\right) \le \eta, \|y(t)-\bar{y}(t)\| \le \eta,$$

where $(X(t),y(t))$ and $(\bar{X}(t),\bar{y}(t))$ are the solutions of systems (2), (3) and (4), (5) with the initial conditions $X(0)=\bar{X}(0) \in D_1'$, $y(0)=\bar{y}(0) \in D_2'$.

The theorem is proved.

Example 1. *Consider the impulsive hybrid system*

$$X \in conv(R^2), y \in R.$$

$$\begin{cases} D_H X = \varepsilon\left[\begin{pmatrix} 1 & \sin t \\ \cos 2t & 2y^2 \end{pmatrix} X + \Pi_{2+e^{-t}+\sin(\ln(1+t)),1}\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}\right], \\ X_0 = S_{0.5}\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \\ \dot{y} = \varepsilon \dfrac{y}{|X|}\sin^2 t,\ y_0 = 0.1. \end{cases}$$

$$\Delta X|_{t=\tau_i} = \varepsilon X,\ \Delta y|_{t=\tau_i} = -\varepsilon y,\ \tau_i = 2\pi i,\ i = \overline{1,\infty}.$$

The averaged system is:

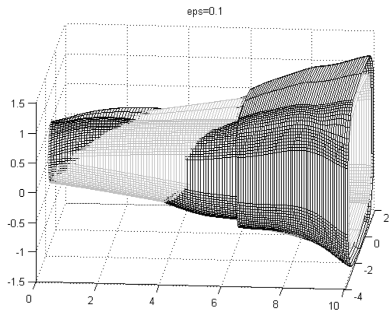
$$\begin{cases} D_H \bar{X} = \varepsilon\left[\begin{pmatrix} 1 & 0 \\ 0 & 2\bar{y}^2 \end{pmatrix} \bar{X} + \Pi_{2+\sin(\ln(1+t)),1}\begin{pmatrix} 0 \\ 0 \end{pmatrix}\right], \\ \bar{X}_0 = S_{0.5}\begin{pmatrix} -1 \\ 0 \end{pmatrix}, \\ \dot{\bar{y}} = \varepsilon \dfrac{\bar{y}}{2\left|\bar{X}\right|},\ \bar{y}_0 = 0.1. \end{cases}$$

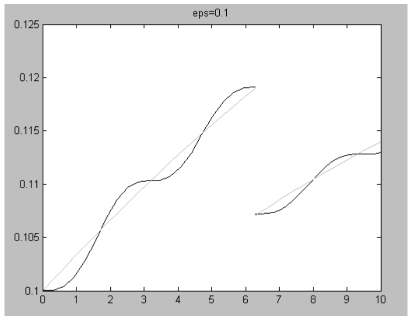
$$\Delta \bar{X}\big|_{t=\tau_j} = \varepsilon \bar{X},\ \Delta \bar{y}\big|_{t=\tau_j} = -\varepsilon \bar{y},\ \tau_j = 2\pi j,\ j = \overline{1,\infty}.$$

The graphs of the solutions of initial system and averaged system see on next page.

Conclusion. This paper contains the substantiation of the scheme of partial averaging for one class of impulsive hybrid systems where one equation is a differential equation with Hukuhara derivative and the other one is an ordinary differential equation. In case when the right-hand sides are periodic in time one can obtain a better estimate. Namely one can show that for any $L > 0$ there exist $C(L) > 0$ and $\varepsilon_0(L) > 0$ such that the conclusion of the theorem holds with $\eta = C\varepsilon$.

a) When $\varepsilon = 0.1$:

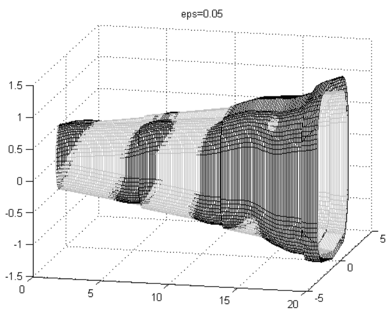


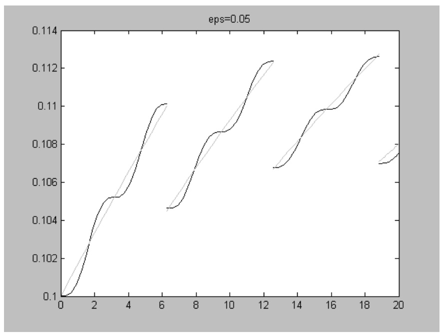


(a) (b)

Image 1. (a) The graph of the solutions of initial system $X(t)$ (black) and averaged system $\bar{X}(t)$ (gray). (b) The graph of the solutions of initial system $y(t)$ (black) and averaged system $\bar{y}(t)$ (gray).

b) When $\varepsilon = 0.05$:





(a) (b)

Image 2. (a) The graph of the solutions of initial system $X(t)$ (black) and averaged system $\bar{X}(t)$ (gray). (b) The graph of the solutions of initial system $y(t)$ (black) and averaged system $\bar{y}(t)$ (gray).

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