# MULTIDIMENSIONAL GIBBS DISTRIBUTION AND AN IDEAL GAS OF NON-RELATIVISTIC PARTICLES

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ABSTRACT. We generalize the standard methods of quantum statistical physics and thermodynamics to the multidimensional case and apply them in order to derive different thermodynamic quantities, characterizing an ideal gas of non-relativistic particles. In particular, we obtain the formula for the pressure in the internal space.

**Introduction**

Present-day observable phenomena, such as dark energy and dark matter, represent the great challenge for modern cosmology, astrophysics and theoretical physics generally. Nowadays within the scope of standard models these phenomena have no satisfactory explanation. This critical situation stimulates the search of solutions of this very complicated and overwhelmingly important problem beyond all conventional models, for example, by introducing extra spatial dimensions (ESDs). This breathtaking generalization follows directly from modern theories of unification of all known fundamental interactions (such as superstring theory, supergravity and M-theory). Indeed, these theories have the most self-consistent formulation in multidimensional space-times with ESDs [1]. Obviously, it is extremely necessary to subject these and other non-standard physical theories to a procedure of hard-edged screening concerning their compatibility with experimental data.

In the well-known Kaluza-Klein models, based on two pioneering papers [2, 3] by Theodor Kaluza and Oskar Klein respectively, all ESDs are assumed to be finite/compact and microscopic. In the recent paper [4] it was explicitly shown that Kaluza-Klein models with toroidal compactification of ESDs and a standard dust-like matter source of the gravitational field contradict experimental data of astronomical observations. In these models formulas for the classical gravitational tests of any theory of gravity (such as the perihelion shift, the deflection of light, the time delay of radar echoes [5] and PPN parameters [6, 7]) are incompatible with observations in the Solar System.

The natural topical question arises, whether Kaluza-Klein models with toroidal compactification survive, when introducing non-dust-like matter sources of the gravitational field with non-dust-like equations of state in the internal space. Such matter sources were considered in [8], where it was explicitly shown that among the exact "soliton" solutions of the vacuum Einstein equation in the 5-dimensional space-time with a single compact ESD [9-11], describing the static gravitational field of a finite spherically symmetric matter source at rest, there is only one solution, called "the black string", satisfying all observational data with the same accuracy as the Schwarzschild solution in General Relativity. This fact represents the main advantage of this solution. All ordinary non-relativistic particles must be identified exactly with the black strings. A single black string at rest is characterized by the dust-like equation of state $p_0 = 0$ in the 3-dimensional external space and the very specific, strange and even unlikely equation of state $p_1 = -\varepsilon/2$ in the 1-dimensional internal space, where $p_0$ and $p_1$ are the corresponding pressures and $\varepsilon$ is the rest energy density. Thus, the pressure $p_1$, sometimes called "tension", is negative and relativistic. Unfortunately, both these circumstances have unclear physical origin, and the corresponding burning issue remains open. This fact represents the main disadvantage of the black string.

In this work we produce consistent multidimensional generalization of standard methods of quantum mechanics, statistical physics and thermodynamics and apply it in order to derive different thermodynamic quantities, characterizing an ideal gas of black strings. Firstly, we solve exactly the 4-dimensional Schrödinger equation for the wave function of a free particle and find its energy spectrum. Secondly, we generalize the standard Gibbs distribution to the case of the multidimensional space and obtain the partition function of the considered ideal gas. Thirdly, with the help of this function and the first law of thermodynamics we arrive at the explicit expression for the pressure in the internal space and investigate its asymptotical behavior. This predictably positive and non-relativistic expression represents the usual temperature dependent contribution to the pressure. In conclusion we summarize our main results.

Let us start with the stationary 4-dimensional Schrödinger equation

$$\hat{H}_4\psi_4 = E_4\psi_4\,, \quad \hat{H}_4 = \hat{H}_3 - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial\xi^2}\,,$$

$$\hat{H}_3 = -\frac{\hbar^2}{2m}\Delta_3 = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right), \quad (1)$$

where $\hat{H}_4$ and $\hat{H}_3$ are 4- and 3-dimensional Hamilton operators respectively; $\psi_4$ is a wave function of a free non-relativistic particle (it depends on all spatial coordinates $x, y, z, \xi$, but does not depend on time $t$); the coordinate $\xi$ corresponds to the ESD and $\Delta_3$ is a 3-dimensional Laplace operator. Let us note that subscripts 4, 3 and 1 indicate everywhere that the corresponding quantity relates to the total 4-dimensional, the external 3-dimensional or the internal 1-dimensional spaces respectively. Following the variable separation method, we seek for the solution of the equation (1) in the form $\psi_4(x,y,z,\xi)=\psi_3(x,y,z)\psi_1(\xi)$ and obtain

$$\hat{H}_3\psi_3 = E_3\psi_3\,, \quad -\frac{\hbar^2}{2m}\frac{d^2\psi_1}{d\xi^2} = E_1\psi_1\,, \quad E_4 = E_3 + E_1\,, \quad (2)$$

where $E_3$ and $E_1$ represent the standard and the additional parts of the total energy $E_4$ respectively. Now our aim is to determine $E_1$. Imposing periodic boundary conditions

$$\psi_1(0)=\psi_1(a), \quad \frac{d\psi_1}{d\xi}(0)=\frac{d\psi_1}{d\xi}(a), \quad (3)$$

where $a$ is the period of the torus (the size of the ESD), one can explicitly show that

$$E_{1(n)} = \frac{2\pi^2\hbar^2}{ma^2}n^2\,, \quad n = 0,1,2,\ldots\,. \quad (4)$$

Thus, we have arrived at the additional energy spectrum, which is necessary for the subsequent determination of the corresponding partition function $Z_1$. For $n=0$ the wave function $\psi_{1(0)} = 1/\sqrt{a}$ is constant. Therefore, we can draw an important side conclusion that in the ground state ($n=0$, $E_{1(0)}=0$) the particle is uniformly smeared over the ESD. Thus, the assumption of the uniform smearing, actually made in [9-11], means that the matter source (namely, the black string) is considered in its ground state.

For $n = 1,2,3,\ldots$ the wave function $\psi_{1(n)}$ can be expressed in the form of the linear combination of two orthogonal functions

$$\psi_{1s(n)} = \sqrt{\frac{2}{a}}\sin\left(\frac{2\pi n}{a}\xi\right), \quad \psi_{1c(n)} = \sqrt{\frac{2}{a}}\cos\left(\frac{2\pi n}{a}\xi\right). \quad (5)$$

Both these functions (as well as $\psi_{1(0)}$) are real and satisfy the normalization condition $\int_0^a \psi_1^2 d\xi = 1$.

Now let us turn to the multidimensional Gibbs distribution. Proceeding from the fundamental principles of quantum statistical physics, one can show that it preserves its standard form:

$$w_\nu = \frac{1}{Z}\exp\left(-\frac{\varepsilon_\nu}{kT}\right), \quad \sum_\nu w_\nu = 1\,,$$

$$Z = \sum_\nu \exp\left(-\frac{\varepsilon_\nu}{kT}\right), \quad (6)$$

where $w_\nu$ represents the probability of finding a system, closed in the thermostat, in the $\nu$-th quantum state with the energy $\varepsilon_\nu$; $\nu$ denotes the full set of quantum numbers, unambiguously determining the considered quantum state; $k$ is the Boltzmann constant and $T$ is the temperature. Finally, $Z$ is the partition function.

Now let us consider an ideal gas of $N$ identical non-relativistic particles. Obviously, in view of (2) the partition function $Z_4$ of each of them can be expressed in the form of the product of two partition functions $Z_3$ and $Z_1$, corresponding to the external and the internal spaces respectively: $Z_4 = Z_3 Z_1$. Substituting the discrete spectrum (4) into (6), we obtain

$$\begin{aligned} Z_1 &= \sum_{n=0}^{+\infty}\exp\left(-\frac{E_{1(n)}}{kT}\right) = \sum_{n=0}^{+\infty}\exp\left(-\frac{2\pi^2\hbar^2}{ma^2kT}n^2\right) = \\ &= \sum_{n=0}^{+\infty}\exp\left(-\frac{T_c}{T}n^2\right) = \sum_{n=0}^{+\infty} q^{n^2} = \frac{1}{2}+\frac{1}{2}\theta_3(0,q) \end{aligned}, \quad (7)$$

where $\theta_3(z,q) = 1 + 2\sum_{n=1}^{+\infty} q^{n^2}\cos 2nz$ denotes the third of the theta-functions.

In (7) we have also introduced a convenient quantity $q$ and a characteristic temperature $T_c$:

$$q = \exp\left(-\frac{2\pi^2\hbar^2}{ma^2kT}\right) = \exp\left(-\frac{T_c}{T}\right), \; 0 < q < 1\,,$$

$$T_c = \frac{2\pi^2\hbar^2}{ma^2k}\,. \quad (8)$$

According to [12], the free energy $F = U - TS = -kT\ln Z$, where $U$ is the internal energy and $S$ is the entropy, preserves its standard form, while the first law of thermodynamics now reads

$$TdS = dU + p_0 a dV_3 + p_1 V_3 da\,,$$

$$dF = -SdT - p_0 a dV_3 - p_1 V_3 da\,. \quad (9)$$

It follows from (9), in particular, that

$$p_0 = -\frac{1}{a}\left(\frac{\partial F}{\partial V_3}\right)_{T,a}, \quad p_1 = -\frac{1}{V_3}\left(\frac{\partial F}{\partial a}\right)_{T,V_3},$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V_3,a}, \quad U = -T^2\left(\frac{\partial}{\partial T}\left(\frac{F}{T}\right)\right)_{V_3,a}. \quad (10)$$

For the considered ideal gas the existence of the ESD results in the additional (everywhere with respect to the standard 3-dimensional part) free energy

$$F_1 = -NkT \ln Z_1 = -NkT \ln\left[\sum_{n=0}^{+\infty} \exp\left(-\frac{T_c}{T} n^2\right)\right] = \\ = -NkT \ln\left[\frac{1}{2} + \frac{1}{2}\theta_3\left(0, \exp\left(-\frac{T_c}{T}\right)\right)\right] . \quad (11)$$

From (10) and (11) we obtain the following additional pressures:

$p_0 = 0$ ,

$$p_1 = \frac{2NkT_c}{V_3 a} \frac{\sum_{n=0}^{+\infty} n^2 q^{n^2}}{\sum_{n=0}^{+\infty} q^{n^2}} = \frac{2NkT_c}{V_3 a} \frac{\theta_3'(0,q)}{1+\theta_3(0,q)} \exp\left(-\frac{T_c}{T}\right), \quad (12)$$

where the prime denotes the derivative with respect to $q$ . It is clear that $p_1$ is positive and non-relativistic. It has the following asymptotes:

$$p_1\big|_{T<<T_c} \approx \frac{2NkT_c}{V_3 a} \exp\left(-\frac{T_c}{T}\right),$$

$$p_1\big|_{T>>T_c} \approx \frac{NkT}{V_3 a} = n_4 kT , \quad n_4 = \frac{N}{V_3 a} . \quad (13)$$

The latter asymptote is predictable, since when the temperature is high enough, we can apply the classical approach instead of the quantum one.

**Conclusion**

An ideal gas of ordinary non-relativistic particles has been described by the standard methods, generalized to the multidimensional case. In particular, the explicit expressions (11) and (12) for the additional free energy and pressures respectively have been derived. The pressure $p_1$ in the internal space is positive and temperature dependent. The relativistic, negative and temperature independent tension of each black string must be explained otherwise, for example, by the corresponding background matter perturbation.

Our results can be generalized directly to the case of the multidimensional space-time with an arbitrary number of toroidal ESDs (see our forthcoming works).

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