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ON EXISTENCE OF A SPECIAL KIND'S INTEGRAL MANIFOLD OF THE NONLINEAR DIFFERENTIAL SYSTEM WITH SLOWLY VARYING PARAMETERS

Щоголев С. А. Про існування інтегрального многовиду спеціального вигляду нелінійної диференціальної системи із повільно змінними параметрами. Для нелінійної коливної одночастотної диференціальної системи другого порядку, праві частини якої відносно кутової змінної зображувані у вигляді абсолютно та рівномірно збіжних рядів Фур'є із повільно змінними в певному сенсі коефіцієнтами, отримано умови існування інтегрального многовиду аналогічної структури.
Ключові слова: диференціальна система, многовид, повільно змінний.

Щёголев С. А. О существовании интегрального многообразия специального вида нелинейной дифференциальной системы с медленно меняющимися параметрами. Для нелинейной колебательной одночастотной дифференциальной системы второго порядка, правые части которой относительно угловой переменной представимы в виде абсолютно и равномерно сходящихся рядов Фурье с медленно меняющимися в определённом смысле коэффициентами, получены условия существования интегрального многообразия аналогичной структуры.
Ключевые слова: дифференциальная система, многообразие, медленно меняющийся.

Shchogolev S. A. On existence of a special kind's integral manifold of the nonlinear differential system with slowly varying parameters. Consider the second-order nonlinear oscillating single-frequency differential system, the right-hand parts of which with respect angular variable can be represented as an absolutely and uniformly convergent Fourier-series with slowly varying in a certain sense coefficients. Establish the conditions of existence of this system of integral manifolds of a similar structure. In this manifold system are reduced to the one differential equation with respect angular variable. Preliminary the auxiliary lemm's in which construct the transformations, which reducing researching system to the system with slowly varying, non oscillating, kind, are obtained. And coefficients of these transformations are obtained in the form of analogous Fourier-series.
Key words: differential system, manifold, slowly varying.

Introduction. One of the powerful methods of the study of nonlinear systems of differential equations is the method of integral manifolds [1,2]. Particularly important role it plays in the research of multi-frequency oscillations, in particular, in systems containing the slowly varying parameters [3]. An important object of study in the same time and are single-frequency system [4]. In this paper the problem about existence of the integral manifold, which represented by as an absolutely and uniformly convergent Fourier-series with slowly varying parameters, are researched.

Main Results.

1. Basic notation and definitions.

Let $G(\varepsilon_0) = \{t, \varepsilon : \ t \in \mathbf{R}, \ \varepsilon \in [0, \varepsilon_0], \ \varepsilon_0 \in \mathbf{R}^+\}$.

Definition 1. We say, that a function $f(t, \varepsilon)$, in general a complex-valued, belongs to the class $S_m(\varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$, if
1) $f : G(\varepsilon_0) \to \mathbf{C}$; 2) $f(t, \varepsilon) \in C^m(G(\varepsilon_0))$ with respect t;
3) $d^k f(t, \varepsilon)/dt^k = \varepsilon^k f_k^*(t, \varepsilon)$ $(0 \leq k \leq m)$,

$$\|f\|_{S_m(\varepsilon_0)} \stackrel{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |f_k^*(t, \varepsilon)| \ < +\infty.$$

Definition 2. We say, that a function $f(t, \varepsilon, \theta)$ belongs to the class $F_m^\theta(\varepsilon_0, \alpha)$ $(m \in \mathbf{N} \cup \{0\}, \ \alpha \in (0, +\infty))$ if
1) $t, \varepsilon \in G(\varepsilon_0)$, $\theta \in \mathbf{R}$; 2) $f : G(\varepsilon_0) \times \mathbf{R} \to \mathbf{R}$;

$$3) \quad f(t, \varepsilon, \theta) = \sum_{n=-\infty}^{\infty} f_n(t, \varepsilon) \exp(in\theta),$$

and:
a) $f_n(t, \varepsilon) \in S_m(\varepsilon_0)$, $f_{-n}(t, \varepsilon) \equiv \overline{f_n(t, \varepsilon)}$;
b) $\exists K \in (0, +\infty)$: $\|f_n\|_{S_m(\varepsilon_0)} \leq K \exp(-|n|\alpha)$, $n \in \mathbf{Z}$;

$$c) \quad \|f\|_{F_m^\theta(\varepsilon_0, \alpha)} \stackrel{def}{=} \sum_{n=-\infty}^{\infty} \|f_n\|_{S_m(\varepsilon_0)} < \frac{K\,(1 + e^{-\alpha})}{1 - e^{-\alpha}}.$$

So the function $f(t, \varepsilon, \theta)$ and its partial derivatives with respect t up to m-th order inclusive are analytic with respect $\theta \in \mathbf{R}$.

We state some properties of the norm $\| \cdot \|_{F_m^\theta(\varepsilon_0, \alpha)}$. Let $u, v \in F_m^\theta(\varepsilon_0, \alpha)$. Then
1) $\|ku\|_{F_m^\theta(\varepsilon_0, \alpha)} = |k| \cdot \| \ u\|_{F_m^\theta(\varepsilon_0, \alpha)}$,
2) $\|u + v\|_{F_m^\theta(\varepsilon_0, \alpha)} \leq \|u\|_{F_m^\theta(\varepsilon_0, \alpha)} + \|v\|_{F_m^\theta(\varepsilon_0, \alpha)}$,
3) $\|uv\|_{F_m^\theta(\varepsilon_0, \alpha)} \leq 2^m \|u\|_{F_m^\theta(\varepsilon_0, \alpha)} \cdot \|v\|_{F_m^\theta(\varepsilon_0, \alpha)}$.

The property 3) in [5] are proved. Functions of the class $F_m^\theta(\varepsilon_0, \alpha)$ are form a linear space, turning a complete normed space by introducing the norm $\| \cdot \|_{F_m^\theta(\varepsilon_0, \alpha)}$.

If $m_1 < m_2$, then $F_{m_2}^\theta(\varepsilon_0, \alpha) \subset F_{m_1}^\theta(\varepsilon_0, \alpha)$.

If $\varepsilon_1 < \varepsilon_2$, then $F_m^\theta(\varepsilon_2, \alpha) \subset F_m^\theta(\varepsilon_1, \alpha)$.

Definition 3. We say, that a function $f(t, \varepsilon, x)$ belongs to the class $S_m^x(\varepsilon_0, x_0, d)$, if
1) $t, \varepsilon \in G(\varepsilon_0), \ x \in \mathbf{R}$; 2) $f : G(\varepsilon_0) \times \mathbf{R} \to \mathbf{R}$;

$$3) \quad f(t, \varepsilon, x) = \sum_{l=0}^{\infty} f_l(t, \varepsilon)(x - x_0)^l,$$

and
a) $f_l : G(\varepsilon_0) \to \mathbf{R}$; b) $f_l(t, \varepsilon) \in S_m(\varepsilon_0)$;
c) the series $\sum_{l=0}^{\infty} \|f_l\|_{S_m(\varepsilon_0)}(x - x_0)^l$ is convergent if $|x - x_0| < d$.

Thus function $f(t, \varepsilon, x)$ is real, analytic with respect x, if $|x - x_0| < d$ together with its partial derivatives up to m-th order inclusive. Moreover $\forall x \in (x_0 - d, x_0 + d)$: $f(t, \varepsilon, x) \in S_m(\varepsilon_0)$.

Definition 4. We say, that a function $f(t,\varepsilon,\theta,x)$ belongs to the class $F_m^{\theta,x}(\varepsilon_0,\alpha,x_0,d)$ $(m \in \mathbf{N} \cup \{0\},\ \alpha \in (0,+\infty))$ if
1) $t,\varepsilon \in G(\varepsilon_0)$, $\theta \in \mathbf{R}$, $x \in \mathbf{R}$; 2) $f : G(\varepsilon_0) \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$;

$$3) \quad f(t,\varepsilon,\theta,x) = \sum_{n=-\infty}^{\infty} \sum_{l=0}^{\infty} f_{n,l}(t,\varepsilon) e^{in\theta}(x-x_0)^l,$$

and
a) $f_{n,l}(t,\varepsilon) \in S_m(\varepsilon_0)$, $f_{-n,l}(t,\varepsilon) \equiv \overline{f_{n,l}(t,\varepsilon)}$,
b) $\exists\ K \in (0,+\infty):\ \forall\ n \in \mathbf{Z},\ \forall\ \rho \in (0,d)$:

$$\|f_{n,l}(t,\varepsilon)\|_{S_m(\varepsilon)} \le \frac{Ke^{-|n|\alpha}}{\rho^l}.$$

Thus real function $f(t,\varepsilon,\theta,x)$ and its all partial derivatives with respect t up to m-th order inclusive are analytic with respect $\theta \in \mathbf{R}$ and x if $|x-x_0| < d$. Moreover $\forall\ x \in (x_0-d, x_0+d)$: $f(t,\varepsilon,\theta,x) \in F_m^{\theta}(\varepsilon_0,\alpha)$.

2. Statement of the Problem.

Consider the following system of differential equations:

$$\begin{aligned} \tfrac{dx}{dt} &= \mu X(t,\varepsilon,\theta,x) + \varepsilon a(t,\varepsilon,\theta,x), \\ \tfrac{d\theta}{dt} &= \omega(t,\varepsilon) + \mu\Theta(t,\varepsilon,\theta,x) + \varepsilon b(t,\varepsilon,\theta,x), \end{aligned} \tag{1}$$

where $t,\varepsilon \in G(\varepsilon_0)$, $\theta, x \in \mathbf{R}$; $X,\Theta \in F_m^{\theta,x}(\varepsilon_0,\alpha,x_0,d)$, $a,b \in F_{m-1}^{\theta,x}(\varepsilon_0,\alpha,x_0,d)$, $\omega \in S_m(\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0} \omega = \omega_0 > 0$, $\mu \in (0,\mu_0)$.

We study the question of the existence of the integral manifold $x = w(t,\varepsilon,\theta,\mu) \in$ $\in F_k^{\theta}(\varepsilon_1,\alpha_1)$ $(k < m-1,\ \varepsilon_1 < \varepsilon_0,\ \alpha_1 < \alpha)$ of the system (1).

3. Auxiliary Results.

We denote:

$$X_0(t,\varepsilon,x) = \frac{1}{2\pi}\int_0^{2\pi} X(t,\varepsilon,\theta,x)d\theta.$$

Let us assume that the following conditions.

(A). There is a real function $x_0(t,\varepsilon)$ such that
1) $X_0(t,\varepsilon,x_0(t,\varepsilon)) \equiv 0$;

$$2) \quad \inf_{G(\varepsilon_0)} \left| \frac{\partial X_0(t,\varepsilon,x_0(t,\varepsilon))}{\partial x} \right| = \gamma > 0; \tag{2}$$

3) in system (1) a function $x_0(t,\varepsilon)$ is taken as a point x_0 and is taken as d-sufficiently small positive number in the d-neighborhood of the point x_0 is no other roots of the equation $X_0(t,\varepsilon,x) = 0$, than x_0. Owing to the condition (2) the number d are exists.

(B) Parameters μ and ε are related by inequalities:

$$\mu^{r-2} \le \varepsilon^{m_1-1}, \tag{3}$$

in which $r, m_1 \in \mathbf{N}$, $r > 2m_1$, $m > 2m_1$, $m_1 \ge 1$,

$$\mu + \frac{\varepsilon}{\mu^2} < \delta, \tag{4}$$

in which $\delta \in (0, +\infty)$.

Lemma 1. *Let the condition (A). Then*
1) $\forall\ r \in \mathbf{N}\ \exists\ \mu_r \in (0, \mu_0)$ *such that* $\forall\ \mu \in (0, \mu_r)$ *exists the transformation of kind*

$$x = y + \sum_{k=1}^{r} u_k(t, \varepsilon, \varphi, y)\mu^k, \quad \theta = \varphi + \sum_{k=1}^{r} v_k(t, \varepsilon, \varphi, y)\mu^k, \tag{5}$$

where $\varphi, y \in \mathbf{R}$, $u_k, v_k \in F_m^{\varphi,y}(\varepsilon_0, \alpha, x_0, d_1)$ $(d_1 \in (0, d),\ k = \overline{1, r})$, *which reducing the system (1) to the form:*

$$\begin{gathered} \tfrac{dy}{dt} = \textstyle\sum_{k=1}^{r} Y_k(t, \varepsilon, y)\mu^k + \mu^{r+1}\widetilde{Y}_r(t, \varepsilon, \varphi, y, \mu) + \varepsilon a_r(t, \varepsilon, \varphi, y, \mu), \\ \tfrac{d\varphi}{dt} = \omega(t, \varepsilon) + \textstyle\sum_{k=1}^{r} \Phi_k(t, \varepsilon, y)\mu^k + \mu^{r+1}\widetilde{\Phi}_r(t, \varepsilon, \varphi, y, \mu) + \varepsilon b_r(t, \varepsilon, \varphi, y, \mu), \end{gathered} \tag{6}$$

where $Y_k, \Phi_k \in S_m^y(\varepsilon_0, x_0, d_1)$, $\widetilde{Y}_r, \widetilde{\Phi}_r \in F_m^{\varphi,y}(\varepsilon_0, \alpha, x_0, d_1)$, $a_r, b_r \in F_{m-1}^{\varphi,y}(\varepsilon_0, \alpha, x_0, d_1)$;
2) $\exists\ \mu_r^* \in (0, \mu_r)$ *such that* $\forall\ \mu \in (0, \mu_r^*)$ *exists inversion:*

$$y = x + p(t, \varepsilon, \theta, x, \mu), \quad \varphi = \theta + q(t, \varepsilon, \theta, x, \mu), \tag{7}$$

where $p, q \in F_m^{\theta,x}(\varepsilon, \alpha, x, d_2)$ $(d_2 \in (0, d_1))$.

Proof. The formulas determining for sufficiently small values μ the functions u_k, v_k, Y_k, Φ_k $(k = \overline{1, r})$, $\widetilde{Y}_r, \widetilde{\Phi}_r, a_r, b_r$ are obtained in [6]. From these formulas it follows that these functions belong to the specified class in the formulation of lemma. We now establish the reversibility of the transformation (5). We rewrite it in the form:

$$x = y + \mu u(t, \varepsilon, \varphi, y, \mu), \quad \theta = \varphi + \mu v(t, \varepsilon, \varphi, y, \mu), \tag{8}$$

where

$$u = \sum_{k=1}^{r} u_k(t, \varepsilon, \varphi, y)\mu^{k-1}, \quad v = \sum_{k=1}^{r} v_k(t, \varepsilon, \varphi, y)\mu^{k-1}.$$

Obviously $u, v \in F_m^{\varphi,y}(\varepsilon_0, \alpha, x_0, d_1)$.

We sustitute relations (7) in (8). Then we obtain the nonlinear system for p, q:

$$p + \mu u(t, \varepsilon, \theta + q, x + p, \mu) = 0, \quad q + \mu v(t, \varepsilon, \theta + q, x + p, \mu) = 0. \tag{9}$$

We choose some $\rho \in (0, d_1)$ and denote:

$$D_0 = \{x \in \mathbf{R};\ |x - x_0| \le \rho\},$$

$$M(\mu) = \max\left(\sup_{x \in D_0} \|u(t, \varepsilon, \theta, x, \mu)\|_{F_m^{\theta}(\varepsilon_0, \alpha)},\ \sup_{x \in D_0} \|v(t, \varepsilon, \theta, x, \mu)\|_{F_m^{\theta}(\varepsilon_0, \alpha)}\right).$$

We seek a solution of the system (9) by iterative method, identifying as an initial approximation $p_0 = q_0 = 0$, and subsequent iterations are defined by formulas:

$$p_{k+1} = -\mu u(t, \varepsilon, \theta + q_k, x + p_k, \mu), \quad q_{k+1} = -\mu v(t, \varepsilon, \theta + q_k, x + p_k, \mu). \tag{10}$$

Now we choose $d_2 \in (0, \rho)$ and denote

$$D_1 = \{x \in \mathbf{R}:\ |x - x_0| \le d_2\}.$$

We have: $p_1 = -\mu u(t, \varepsilon, \theta, x, \mu)$, $q_1 = -\mu v(t, \varepsilon, \theta, x, \mu)$.

$$\sup_{x \in D_1} \|p_1\|_{F_m^\theta(\varepsilon_0,\alpha)} = \mu \sup_{x \in D_1} \|u\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu \sup_{x \in D_0} \|u\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu),$$

$$\sup_{x \in D_1} \|q_1\|_{F_m^\theta(\varepsilon_0,\alpha)} = \mu \sup_{x \in D_1} \|v\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu \sup_{x \in D_0} \|v\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu).$$

We assume by induction that

$$\sup_{x \in D_1} \|p_k\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu), \ \sup_{x \in D_1} \|q_k\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu).$$

Then if $x \in D_1$, we have:

$$|x + p_k - x_0| \le |x - x_0| + |p_k| \le d_2 + \mu M(\mu).$$

We choose μ so small, that $\mu M(\mu) < \rho - d_2$. Then

$$\sup_{x \in D_1} \|p_{k+1}\|_{F_m^\theta(\varepsilon_0,\alpha)} = \mu \sup_{x \in D_1} \|u(t, \varepsilon, \theta + q_k, x + p_k, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le$$

$$\le \mu \sup_{x \in D_0} \|u(t, \varepsilon, \theta, x, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu),$$

$$\sup_{x \in D_1} \|q_{k+1}\|_{F_m^\theta(\varepsilon_0,\alpha)} = \mu \sup_{x \in D_1} \|v(t, \varepsilon, \theta + q_k, x + p_k, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le$$

$$\le \mu \sup_{x \in D_0} \|v(t, \varepsilon, \theta, x, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu).$$

Thus for all iterations is satisfied:

$$\sup_{x \in D_1} \|p_k(t, \varepsilon, \theta, x, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu), \ \sup_{x \in D_1} \|q_k(t, \varepsilon, \theta, x, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu).$$

Since the function $u, v \in F_m^{\varphi,y}(\varepsilon_0, \alpha, x_0, d_1)$ then $\exists\ L(\mu) \in (0, +\infty) : \forall\ \widetilde{p}, \widetilde{q}, \widetilde{\widetilde{p}}, \widetilde{\widetilde{q}} \in$ $\in F_m^{\theta,x}(\varepsilon_0, \alpha, x_0, d_2)$ such that

$$\sup_{x \in D_1} \|\widetilde{p}(t, \varepsilon, \theta, x, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu), \quad \sup_{x \in D_1} \|\widetilde{\widetilde{p}}(t, \varepsilon, \theta, x, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu),$$

$$\sup_{x \in D_1} \|\widetilde{q}(t, \varepsilon, \theta, x, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu), \quad \sup_{x \in D_1} \|\widetilde{\widetilde{q}}(t, \varepsilon, \theta, x, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le \mu M(\mu)$$

the unequalities:

$$\sup_{x \in D_1} \|u(t, \varepsilon, \theta + \widetilde{q}, x + \widetilde{p}, \mu) - u(t, \varepsilon, \theta + \widetilde{\widetilde{q}}, x + \widetilde{\widetilde{p}}, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le$$

$$\le L(\mu) \left(\sup_{x \in D_1} \|\widetilde{q} - \widetilde{\widetilde{q}}\|_{F_m^\theta(\varepsilon_0,\alpha)} + \sup_{x \in D_1} \|\widetilde{p} - \widetilde{\widetilde{p}}\|_{F_m^\theta(\varepsilon_0,\alpha)} \right),$$

$$\sup_{x \in D_1} \|v(t, \varepsilon, \theta + \widetilde{q}, x + \widetilde{p}, \mu) - v(t, \varepsilon, \theta + \widetilde{\widetilde{q}}, x + \widetilde{\widetilde{p}}, \mu)\|_{F_m^\theta(\varepsilon_0,\alpha)} \le$$

$$\le L(\mu) \left(\sup_{x \in D_1} \|\widetilde{q} - \widetilde{\widetilde{q}}\|_{F_m^\theta(\varepsilon_0,\alpha)} + \sup_{x \in D_1} \|\widetilde{p} - \widetilde{\widetilde{p}}\|_{F_m^\theta(\varepsilon_0,\alpha)} \right).$$

Hence we obtain:

$$\sup_{x\in D_1} \|p_{k+1}-p_k\|_{F^{\theta}_m(\varepsilon_0,\alpha)} =$$

$$= \mu \sup_{x\in D_1} \|u(t,\varepsilon,\theta+q_k, x+p_k,\mu 0 - u(t,\varepsilon,\theta+q_{k-1},x+p_{k-1},\mu)\|_{F^{\theta}_m(\varepsilon_0,\alpha)} \le$$

$$\le \mu L(\mu)\left(\sup_{x\in D_1}\|q_k-q_{k-1}\|_{F^{\theta}_m(\varepsilon_0,\alpha)} + \sup_{x\in D_1}\|p_k-p_{k-1}\|_{F^{\theta}_m(\varepsilon_0,\alpha)}\right);$$

$$\sup_{x\in D_1} \|q_{k+1}-q_k\|_{F^{\theta}_m(\varepsilon_0,\alpha)} =$$

$$= \mu \sup_{x\in D_1} \|v(t,\varepsilon,\theta+q_k, x+p_k,\mu 0 - v(t,\varepsilon,\theta+q_{k-1},x+p_{k-1},\mu)\|_{F^{\theta}_m(\varepsilon_0,\alpha)} \le$$

$$\le \mu L(\mu)\left(\sup_{x\in D_1}\|q_k-q_{k-1}\|_{F^{\theta}_m(\varepsilon_0,\alpha)} + \sup_{x\in D_1}\|p_k-p_{k-1}\|_{F^{\theta}_m(\varepsilon_0,\alpha)}\right).$$

Therefore, for the convergence process (10) to the solution of the system (9), which belong to the class $F^{\theta,x}_m(\varepsilon_0,\alpha,x_0,d_2)$ is sufficient condition $2\mu L(\mu) < 1$.

Lemma 1 are proved.

We consider equation:

$$Y(t,\varepsilon,y,\mu) = 0, \tag{11}$$

where $Y = \sum_{k=1}^{r} Y_k(t,\varepsilon,y)\mu^{k-1}$. In [6] shows that $Y_1(t,\varepsilon,y) = X_0(t,\varepsilon,y)$. Therefore, on the basis of the assumption (A), the equation

$$Y_1(t,\varepsilon,y) = 0 \tag{12}$$

have the root $y_0(t,\varepsilon) = x_0(t,\varepsilon) \in S_m(\varepsilon_0)$, and

$$\inf_{G(\varepsilon_0)} \left|\frac{\partial Y_1(t,\varepsilon,y_0(t,\varepsilon))}{\partial y}\right| = \gamma > 0.$$

Lemma 2. *Let suppose the assumption (A). Then* $\exists\, d_3 \in (0,+\infty)$, $\mu_{r_0} \in (0,\mu_r)$, *where* μ_r *are defined in Lemma 1, such that* $\forall\, \mu \in (0,\mu_{r_0})$ *the equation (11) have the root* $y^*(t,\varepsilon,\mu) \in S_m(\varepsilon_0)$, *such that*

$$|y^*(t,\varepsilon,\mu) - y_0(t,\varepsilon)| < \mu d_3 < d_2,$$

$$\inf_{G(\varepsilon_0)} \left|\frac{\partial Y_1(t,\varepsilon,y^*(t,\varepsilon,\mu),\mu)}{\partial y}\right| = \gamma_1(\mu) > 0.$$

Proof. We write the equation (11) in the form

$$Y_1(t,\varepsilon,y) + \mu\widetilde{Y}(t,\varepsilon,y,\mu) = 0, \tag{13}$$

where $\widetilde{Y} = \sum_{k=2}^{r} Y_k(t,\varepsilon,y)\mu^{k-2}$. Then the assertion of Lemma follows from the results [7, p. 695–702].

We denote

$$\Phi(t,\varepsilon,y,\mu) = \sum_{k=1}^{r} \Phi_k(t,\varepsilon,y)\mu^{k-1},$$

and rewrite the system (6) in form:

$$\frac{dy}{dt} = \mu Y(t,\varepsilon,y,\mu) + \mu^{r+1}\widetilde{Y}_r(t,\varepsilon,\varphi,y,\mu) + \varepsilon a_r(t,\varepsilon,\varphi,y,\mu),$$
$$\frac{d\varphi}{dt} = \omega(t,\varepsilon) + \mu\Phi(y,\varepsilon,y,\mu) + \mu^{r+1}\widetilde{\Phi}_r(t,\varepsilon,\varphi,y,\mu) + \varepsilon b_r(t,\varepsilon,\varphi,y,\mu), \tag{14}$$

where $Y,\Phi \in S_m^y(\varepsilon_0, y^*, d_2-\mu d_3)$.

Lemma 3. *Let suppose the assumption (A). Then* $\exists\, \varepsilon^* \in (0,\varepsilon_0)$, $d_4 \in (0, d_2-\mu d_3)$ *such that* $\forall\ \varepsilon \in (0,\varepsilon^*)$ *exists the chain of reversible transformations of kind:*

$$y = z_1 + \varepsilon g_1(t,\varepsilon,\psi,z_1,\mu),\ \ \varphi = \psi_1 + \varepsilon h_1(t,\varepsilon,\psi,z_1,\mu), \tag{15}$$

$$z_1 = z_2 + \varepsilon^2 g_2(t,\varepsilon,\psi_2,z_2,\mu),\ \ \psi_1 = \psi_2 + \varepsilon^2 h_2(t,\varepsilon,\psi_2,z_2,\mu), \tag{16}$$

$$\ldots$$

$$z_{m_1-2} = z_{m_1-1} + \varepsilon^{m_1-1} g_{m_1-1}(t,\varepsilon,\psi_{m_1-1},z_{m_1-1},\mu),$$
$$\psi_{m_1-2} = \psi_{m_1-1} + \varepsilon^{m_1-1} h_{m_1-1}(t,\varepsilon,\psi_{m_1-1},z_{m_1-1},\mu), \tag{17}$$

where $g_j, h_j \in F_{m-j}^{\psi_j,z_j}(\varepsilon^*,\alpha,y^*,d_4)$ $(j=\overline{1,m_1-1})$, *which reducing the system (13) to the kind:*

$$\frac{dz_{m_1-1}}{dt} = \mu Y(t,\varepsilon,z_{m_1-1},\mu) + \mu^{r+1} Z_{m_1-1}(t,\varepsilon,\psi_{m_1-1},z_{m_1-1},\mu) +$$
$$+\sum_{k=1}^{m_1-1}\varepsilon^k \alpha_k(t,\varepsilon,z_{m_1-1},\mu) + \varepsilon^{m_1}\widetilde{a}_{m_1}(t,\varepsilon,\psi_{m_1-1},z_{m_1-1},\mu),$$
$$\frac{d\psi_{m_1-1}}{dt} = \omega(t,\varepsilon) + \mu\Phi(t,\varepsilon,z_{m_1-1},\mu) + \mu^{r+1}\Phi_{m_1-1}(t,\varepsilon,\psi_{m_1-1},z_{m_1-1},\mu) +$$
$$+\sum_{k=1}^{m_1-1}\varepsilon^k \beta_k(t,\varepsilon,z_{m_1-1},\mu) + \varepsilon^{m_1}\widetilde{b}_{m_1}(t,\varepsilon,\psi_{m_1-1},z_{m_1-1},\mu), \tag{18}$$

where $Z_{m_1-1}, \Phi_{m_1-1} \in F_{m-m_1+1}^{\psi_{m_1-1},z_{m_1-1}}(\varepsilon^*,\alpha,y^*,d_4)$, $\alpha_k,\beta_k \in S_{m-k}^{z_{m_1-1}}(\varepsilon_1,y^*,d_4)$, $\widetilde{a}_{m_1},\widetilde{b}_{m_1} \in$ $\in F_{m-m_1}^{\psi_{m_1-1},z_{m_1-1}}(\varepsilon^*,\alpha,y^*,d_4)$.

Proof. We apply to the system (14) transformation (15) and require that the transformed system has the form:

$$\frac{dz_1}{dt} = \mu Y(t,\varepsilon,z_1,\mu) + \mu^{r+1} Z_1(t,\varepsilon,\psi_1,z_1,\mu) +$$
$$+\varepsilon\alpha_1(t,\varepsilon,z_1,\mu) + \varepsilon^2\widetilde{a}_2(t,\varepsilon,\psi,z_1,\mu),$$
$$\frac{d\psi_1}{dt} = \omega(t,\varepsilon) + \mu\Phi(t,\varepsilon,z_1,\mu) + \mu^{r+1}\Phi_1(t,\varepsilon,\psi_1,z_1,\mu) +$$
$$+\varepsilon\beta_1(t,\varepsilon,z_1,\mu) + \varepsilon^2\widetilde{b}_2(t,\varepsilon,\psi,z_1,\mu), \tag{19}$$

where the function $Z_1, \Phi_1, \alpha_1, \beta_1, \widetilde{a}_2, \widetilde{b}_2$ are to be determined. Then for the functions g_1, h_1 we obtain the following system of the differential equations in partial derivatives:

$$(\omega(t,\varepsilon) + \mu\Phi(t,\varepsilon,z_1,\mu))\frac{\partial g_1}{\partial\psi_1} + \mu Y(t,\varepsilon,z_1,\mu)\frac{\partial g_1}{\partial z_1} +$$
$$+\alpha_1(t,\varepsilon,z_1,\mu) = \mu\frac{\partial Y(t,\varepsilon,z_1,\mu)}{\partial z_1}\ g_1 + a_r(t,\varepsilon,\psi_1,z_1,\mu), \tag{20}$$

$$\begin{gathered}(\omega(t,\varepsilon)+\mu\Phi(t,\varepsilon,z_1,\mu))\,\tfrac{\partial h_1}{\partial\psi_1}+\mu Y(t,\varepsilon,z_1,\mu)\tfrac{\partial h_1}{\partial z_1}+\\ +\beta_1(t,\varepsilon,z_1,\mu)=\mu\tfrac{\partial\Phi(t,\varepsilon,z_1,\mu)}{\partial z_1}\;g_1+b_r(t,\varepsilon,\psi_1,z_1,\mu).\end{gathered}\tag{21}$$

The functions $Z_1,\Phi_1,\widetilde{a}_2,\widetilde{b}_2$ defined from the following systems of the linear algebraic equations:

$$\begin{gathered}\varepsilon\tfrac{\partial g_1}{\partial\psi_1}\;\Phi_1+\left(1+\varepsilon\tfrac{\partial g_1}{\partial z_1}\right)Z_1=\widetilde{Y}_r(t,\varepsilon,\psi_1+\varepsilon h_1,z_1+\varepsilon g_1,\mu),\\ \left(1+\varepsilon\tfrac{\partial h_1}{\partial\psi_1}\right)\Phi_1+\varepsilon\tfrac{\partial h_1}{\partial\psi_1}Z_1=\widetilde{\Phi}_r(t,\varepsilon,\psi_1+\varepsilon h_1,z_1+\varepsilon g_1,\mu),\end{gathered}\tag{22}$$

$$\begin{gathered}\left(1+\varepsilon\tfrac{\partial g_1}{\partial z_1}\right)\widetilde{a}_2+\varepsilon\tfrac{\partial g_1}{\partial\psi_1}\widetilde{b}_2=\\ =\tfrac{\mu}{2}\tfrac{\partial^2Y(t,\varepsilon,z_1+\nu_1\varepsilon g_1)}{\partial z_1^2}g_1^2+\tfrac{\partial a_r(t,\varepsilon,\psi+\nu_2\varepsilon h_1,z_1+\nu_2\varepsilon g_1,\mu)}{\partial\psi_1}\;h_1+\\ +\tfrac{\partial a_r(t,\varepsilon,\psi+\nu_2\varepsilon h_1,z_1+\nu_2\varepsilon g_1,\mu)}{\partial z_1}\;g_1-\tfrac{1}{\varepsilon}\tfrac{\partial g_1}{\partial t},\\ \varepsilon\tfrac{\partial h_1}{\partial z_1}\;\widetilde{a}_2+\left(1+\varepsilon\tfrac{\partial h_1}{\partial\psi_1}\right)\widetilde{b}_2=\\ =\tfrac{\mu}{2}\tfrac{\partial^2\Phi(t,\varepsilon,z_1+\nu_3\varepsilon g_1)}{\partial z_1^2}g_1^2+\tfrac{\partial b_r(t,\varepsilon,\psi+\nu_4\varepsilon h_1,z_1+\nu_4\varepsilon g_1,\mu)}{\partial\psi_1}\;h_1+\\ +\tfrac{\partial b_r(t,\varepsilon,\psi+\nu_4\varepsilon h_1,z_1+\nu_4\varepsilon g_1,\mu)}{\partial z_1}\;g_1-\tfrac{1}{\varepsilon}\tfrac{\partial h_1}{\partial t},\end{gathered}\tag{23}$$

where $\nu_1,\nu_2,\nu_3,\nu_4\in(0,1)$.

We denote $z_0=y^*(t,\varepsilon,\mu)$ and expand the functions Y,Φ in the series in z_1-z_0, which converge at $|z_1-z_0|<\rho_1$, where $\rho_1\in(0,d_2-\mu d_3)$. Due the conditions of Lemma value ρ_1 can be chosen so small that in ρ_1-neighbourhood of the point z_0 the are no, except z_0, other roots of equation $Y(t,\varepsilon,z_1,\mu)=0$.

$$\Phi(t,\varepsilon,z_1,\mu)=\sum_{l=0}^{\infty}\Phi_l^*(t,\varepsilon,\mu)(z_1-z_0)^l,\tag{24}$$

$$Y(t,\varepsilon,z_1,\mu)=\sum_{l=0}^{\infty}Y_l^*(t,\varepsilon,\mu)(z_1-z_0)^l.\tag{25}$$

In the case $\inf\limits_{G(\varepsilon_0)}|Y_1^*(t,\varepsilon,\mu)|=\gamma_1(\mu)>0$. Then

$$\frac{\partial\Phi(t,\varepsilon,z_1,\mu)}{\partial z_1}=\sum_{l=0}^{\infty}l\Phi_l^*(t,\varepsilon,\mu)(z_1-z_0)^{l-1},\tag{26}$$

$$\frac{\partial Y(t,\varepsilon,z_1,\mu)}{\partial z_1}=\sum_{l=0}^{\infty}lY_l^*(t,\varepsilon,\mu)(z_1-z_0)^{l-1}.\tag{27}$$

We expand the functions a_r, b_r in the double series, whish converge at $\psi_1 \in \mathbf{R}$ and $|z_1 - z_0| < \rho_1$:

$$a_r(t,\varepsilon,\psi_1,\mu) = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} a_{rkl}(t,\varepsilon,\mu)e^{ik\psi_1}(z_1-z_0)^l, \tag{28}$$

$$b_r(t,\varepsilon,\psi_1,\mu) = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} b_{rkl}(t,\varepsilon,\mu)e^{ik\psi_1}(z_1-z_0)^l. \tag{29}$$

We seek a solution of the system (20), (21) in the form of a double series:

$$g_1(t,\varepsilon,\psi_1,\mu) = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} g_{1kl}(t,\varepsilon,\mu)e^{ik\psi_1}(z_1-z_0)^l, \tag{30}$$

$$h_1(t,\varepsilon,\psi_1,\mu) = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} h_{1kl}(t,\varepsilon,\mu)e^{ik\psi_1}(z_1-z_0)^l. \tag{31}$$

Then

$$\frac{\partial g_1}{\partial \psi_1} = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} ikg_{1kl}e^{ik\psi_1}(z_1-z_0)^l, \tag{32}$$

$$\frac{\partial g_1}{\partial z_1} = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} lg_{1kl}e^{ik\psi_1}(z_1-z_0)^{l-1}, \tag{33}$$

$$\frac{\partial h_1}{\partial \psi_1} = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} ikh_{1kl}e^{ik\psi_1}(z_1-z_0)^l, \tag{34}$$

$$\frac{\partial h_1}{\partial z_1} = \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} lh_{1kl}e^{ik\psi_1}(z_1-z_0)^{l-1}. \tag{35}$$

We substitute expressions (30), (32), (33) in the equation (20). Using (24) we obtain:

$$\begin{gathered}\left(\omega(t,\varepsilon) + \mu\sum_{l=0}^{\infty}\Phi_l^*(t,\varepsilon,\mu)(z_1-z_0)^l\right)\sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} ikg_{1kl}e^{ik\psi_1}(z_1-z_0)^l + \\ + \left(\mu\sum_{l=0}^{\infty}Y_l^*(t,\varepsilon,\mu)(z_1-z_0)^l\right)\sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} lg_{1kl}e^{ik\psi_1}(z_1-z_0)^{l-1} + \\ +\alpha_1(t,\varepsilon,z_1,\mu) = \left(\mu\sum_{l=1}^{\infty}lY_l^*(t,\varepsilon,\mu)(z_1-z_0)^{l-1}\right)\times \\ \times\sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} g_{1kl}(t,\varepsilon,\mu)e^{ik\psi_1}(z_1-z_0)^l + \sum_{k=-\infty}^{\infty}\sum_{l=0}^{\infty} a_{rkl}(t,\varepsilon,\mu)e^{ik\psi_1}(z_1-z_0)^l.\end{gathered} \tag{36}$$

We equate in the left and right sides of the equality (36) the coefficients at $e^{ik\psi_1}$. At $k=0$ we obtain:

$$\left(\mu\sum_{l=0}^{\infty}Y_l^*(t,\varepsilon,\mu)(z_1-z_0)^l\right)\sum_{l=0}^{\infty} lg_{10l}(z_1-z_0)^{l-1} +$$

$$+\alpha_1(t,\varepsilon,z_1,\mu)=\left(\mu\sum_{l=1}^{\infty}lY_l^*(t,\varepsilon,\mu)(z_1-z_0)^{l-1}\right)\times$$

$$\times\sum_{l=0}^{\infty}g_{10l}(z_1-z_0)^l+\sum_{l=0}^{\infty}a_{r0l}(t,\varepsilon,\mu)(z_1-z_0)^l. \tag{37}$$

We denote:

$$\alpha_1(t,\varepsilon,z_1,\mu)=\sum_{l=0}^{\infty}a_{r0l}(t,\varepsilon,\mu)(z_1-z_0)^l=$$

$$=\frac{1}{2\pi}\int_0^{2\pi}a_r(t,\varepsilon,\psi_1,z_1,\mu)d\psi_1\in S_{m-1}^{z_1}(\varepsilon_0,z_0,\rho_1),$$

$$g_{10l}(t,\varepsilon,\mu)\equiv 0\ (l=0,1,2,...).$$

At $k\neq 0$ we denote:

$$g_{1k0}=-\frac{a_{rk0}}{\mu Y_1^*(t,\varepsilon,\mu)-ik(\omega(t,\varepsilon)+\mu\Phi_0^*(t,\varepsilon,\mu))}, \tag{38}$$

$$g_{1kn}=\frac{\mu\sum\limits_{j=0}^{n-1}\left[(n+1-2j)Y_{n-j+1}^*(t,\varepsilon,\mu)-ik\Phi_{n-j}^*(t,\varepsilon,\mu)\right]g_{1kj}+a_{rkn}(t,\varepsilon,\mu)}{(n-1)\mu Y_1^*(t,\varepsilon,\mu)+ik(\omega(t,\varepsilon)+\mu\Phi_0^*(t,\varepsilon,\mu))}, \tag{39}$$

$$n=1,2,\ldots;\ k\in\mathbf{Z}/\{0\}.$$

Since $\inf\limits_{G(\varepsilon_0)}\omega(t,\varepsilon)=\omega_0>0$, then

$$\inf_{G(\varepsilon_0)}|\omega(t,\varepsilon)+\mu\Phi_0^*(t,\varepsilon,\mu)|\geq\omega_0-\mu\|\Phi_0^*(t,\varepsilon,\mu)\|_{S_m(\varepsilon_0)}>\omega_1>0,$$

if $\mu\|\Phi_0^*(t,\varepsilon,\mu)\|_{S_m(\varepsilon_0)}<\omega_0-\omega_1$. From (38) follows, that $g_{1k0}(t,\varepsilon,\mu)\in S_{m-1}(\varepsilon_0)$, and

$$\|g_{1k0}(t,\varepsilon,\mu)\|_{S_{m-1}(\varepsilon_0)}\leq\frac{\|a_{rk0}(t,\varepsilon,\mu)\|_{S_{m-1}(\varepsilon_0)}}{|k|\omega_1}.$$

From (39) follows, that $\forall\ n\in\mathbf{N}$: $g_{1kn}(t,\varepsilon,\mu)\in S_{m-1}(\varepsilon_0)$.

Since series (24), (25), (28), (29) converge at $\psi_1\in\mathbf{R}$ and $|z_1-z_0|<\rho_1$, then $\exists\ \sigma\in(0,\rho_1)$, $M\in(0,+\infty)$ such that:

$$\|\Phi_l^*\|_{S_{m-1}(\varepsilon_0)}\leq\frac{M}{\sigma^l},\quad\|Y_l^*\|_{S_{m-1}(\varepsilon_0)}\leq\frac{M}{\sigma^{l-1}},$$

$$\|a_{rkl}\|_{S_{m-1}(\varepsilon_0)}\leq\frac{Me^{-|k|\alpha}}{\sigma^l},\quad\|b_{rkl}\|_{S_{m-1}(\varepsilon_0)}\leq\frac{Me^{-|k|\alpha}}{\sigma^l}.$$

Following known techniques [8], suppose by induction, that

$$\|g_{1kl}(t,\varepsilon,\mu)\|_{S_{m-1}(\varepsilon_0)}\leq\frac{P^le^{-|k|\alpha}}{\sigma^l}\quad(l=\overline{0,n-1}).$$

We show that the constant $P > 1$ can be chosen so that:

$$\|g_{1kn}(t,\varepsilon,\mu)\|_{S_{m-1}(\varepsilon_0)} \leq \frac{P^n e^{-|k|\alpha}}{\sigma^n}.$$

Let $\gamma_1(\mu)$ – the constant, which defined in Lemma 2. From (39) we obtain:

$$\|g_{1kn}\|_{S_{m-1}(\varepsilon_0)} \leq \frac{1}{(n-1)\mu\gamma_1(\mu)+|k|\omega_1}\times$$

$$\times\left[\mu 2^{m-1}\sum_{j=0}^{n-1}|n+1-2j|\cdot\|Y^*_{n-j+1}\|_{S_{m-1}(\varepsilon_0)}\cdot\|g_{1kj}\|_{S_{m-1}(\varepsilon_0)}+\right.$$

$$\left.+\mu|k|2^{m-1}\sum_{j=0}^{n-1}\|\Phi^*_{n-j}\|_{S_{m-1}(\varepsilon_0)}\cdot\|g_{1kj}\|_{S_{m-1}(\varepsilon_0)}+\|a_{rkn}\|_{S_{m-1}(\varepsilon_0)}\right]\leq$$

$$\leq\frac{2^{m-1}}{(n-1)\mu\gamma_1(\mu)+|k|\omega_1}\left[\mu\sum_{j=0}^{n-1}|n+1-2j|\frac{M}{\sigma^{n-j}}\cdot\frac{P^j e^{-|k|\alpha}}{\sigma^j}+\right.$$

$$\left.+\mu|k|\sum_{j=0}^{n-1}\frac{M}{\sigma^{n-j}}\cdot\frac{P^j e^{-|k|\alpha}}{\sigma^j}+\frac{Me^{-|k|\alpha}}{\sigma^n}\right]=$$

$$=\frac{2^{m-1}Me^{-|k|\alpha}}{((n-1)\mu\gamma_1(\mu)+|k|\omega_1)\sigma^n}\left[\mu\sum_{j=0}^{n-1}|n+1-2j|P^j+\mu|k|\sum_{j=0}^{n-1}P^j+1\right]=$$

$$=\frac{2^{m-1}Me^{-|k|\alpha}}{((n-1)\mu\gamma_1(\mu)+|k|\omega_1)\sigma^n}\times$$

$$\times\left[\mu\,\frac{(n+3)P^{n+1}-(n-1)P^n+(n+1)}{(P-1)^2}+\mu|k|\frac{P^n-1}{P-1}+1\right]\leq$$

$$\leq\frac{2^{m-1}Me^{-|k|\alpha}}{((n-1)\mu\gamma_1(\mu)+|k|\omega_1)\sigma^n}\left[\mu\frac{3(n+1)P^{n+1}}{(P-1)^2}+\mu|k|\frac{P^n}{P-1}+P^{n-1}\right]. \qquad (40)$$

Let $P \geq 1+p_0$, where $p_0 > 0$. Then from (40) we obtain:

$$\|g_{1kn}\|_{S_{m-1}(\varepsilon_0)} \leq \frac{K_0 2^{m-1}Me^{-|k|\alpha}}{((n-1)\mu\gamma_1(\mu)+|k|\omega_1)\sigma^n}\left[\mu(n+1)P^{n-1}+\right.$$

$$\left.+\mu|k|P^{n-1}+P^{n-1}\right]=\frac{K_0 2^{m-1}Me^{-|k|\alpha}}{((n-1)\mu\gamma_1(\mu)+|k|\omega_1)\sigma^n}\,(\mu(n+1)+\mu|k|+1)P^{n-1},$$

where $K_0 = 3(1+p_0)^2/p_0^2+(1+p_0)/p_0+1$.

We estimate:

$$\frac{\mu(n+1)+\mu|k|+1}{\mu\gamma_1(n-1)+\omega_1|k|}=\frac{\mu(n+1)+\mu|k|}{\mu\gamma_1(n-1)+\omega_1|k|}+\frac{1}{\mu\gamma_1(n-1)+\omega_1|k|}\leq$$

$$\leq \frac{1}{\omega_1} + \frac{\mu(n+1)+\mu|k|}{\mu\gamma_1(n-1)+\omega_1|k|} < \frac{3}{\omega_1} + \frac{\mu+\mu\tau}{\mu\gamma_1+\omega\tau},$$

where $\tau = |k|/(n-1)$.

The function $s(\tau) = (\mu+\mu\tau)/(\mu\gamma_1+\omega_1\tau)$ at $\mu\gamma_1 < \omega_1$ is monotonically decreasing, $s(0) = 1/\gamma_1$, hence $s(\tau) \leq 1/\gamma_1$. Thus for sufficiently small μ:

$$\frac{\mu(n+1)+\mu|k|+1}{\mu\gamma_1(n-1)+\omega_1|k|} \leq K_1,$$

where $K_1 = 3/\omega_1 + 1/\gamma_1$. Hence:

$$\|g_{1kn}\|_{S_{m-1}(\varepsilon_0)} \leq \frac{K_0K_1 2^{m-1} M e^{-|k|\alpha}}{\sigma^n}\ P^{n-1}.$$

We require that

$$\frac{K_0K_1 2^{m-1} M e^{-|k|\alpha}}{\sigma^n}\ P^{n-1} < \frac{e^{-|k|\alpha}}{\sigma^n}\ P^n.$$

It's enough to satisfy the inequality $P > 1 + K_0K_1 2^{m-1} M$.

Thus equation (20) have a solution $g_1(t,\varepsilon,\psi_1,z_1,\mu)$, which belong to class $F_{m-1}^{\psi_1,z_1}(\varepsilon_0,\alpha,z_0,\sigma/P)$. Since in neighbourhood $|z_1 - z_0| < \rho_1$ the are no, except z_0, other roots of equation $Y(t,\varepsilon,z_1,\mu) = 0$, then equation (18) has no singular points, except z_0, hence $g_1(t,\varepsilon,\psi_1,z_1,\mu) \in F_{m-1}^{\psi_1,z_1}(\varepsilon_0,\alpha,z_0,\rho_1)$.

Let's go to equation (21) with an already defined function $g_1(t,\varepsilon,\psi_1,z_1,\mu)$, We denote:

$$\beta_1(t,\varepsilon,z_1,\mu) = \mu\frac{\partial\Phi(t,\varepsilon,z_1,\mu)}{\partial z_1}\cdot\frac{1}{2\pi}\int_0^{2\pi} g_1(t,\varepsilon,\psi_1,z_1,\mu)d\psi_1 +$$

$$+\frac{1}{2\pi}\int_0^{2\pi} b_r(t,\varepsilon,\psi_1,z_1,\mu)d\psi_1.$$

Then $\beta_1(t,\varepsilon,z_1,\mu) \in S_{m-1}^{z_1}(\varepsilon_0,z_0,\rho_1)$.

Using arguments similar to those given for the equation (20), we see that equation (21) have a solution $h_1(t,\varepsilon,\psi_1,z_1,\mu) \in F_{m-1}^{\psi_1,z_1}(\varepsilon_0,\alpha,z_0,\rho_1)$.

Now consider the systems of the linear algebraic equations (22), (23). Obviously $\exists\ \varepsilon_1 \in (0,\varepsilon_0)$ such that $\forall\ \varepsilon \in (0,\varepsilon_1)$ the determinants of theese systems are separated from zero. Hence the system (22) have a unique solution $\Phi_1(t,\varepsilon,\psi_1,z_1,\mu)$, $Z_1(t,\varepsilon,\psi,z_1,\mu) \in F_{m-1}^{\psi_1,z_1}(\varepsilon_1,\alpha,z_0,\rho_1)$. The system (23) have a unique solution $\widetilde{a}_2(t,\varepsilon,$ $\psi_1,z_1,\mu)$, $\widetilde{b}_2(t,\varepsilon,\psi_1,z_1,\mu) \in F_{m-2}^{\psi_1,z_1}(\varepsilon_1,\alpha,z_0,\rho_1)$.

We make in the system (19) transformation (16) and require that the transformed system have the kind:

$$\begin{gathered}\frac{dz_2}{dt} = \mu Y(t,\varepsilon,z_2,\mu) + \mu^{r+1}Z_2(t,\varepsilon,\psi_2,z_2,\mu)+\\ +\varepsilon\alpha_1(t,\varepsilon,z_2,\mu) + \varepsilon^2\alpha_2(t,\varepsilon,z_2,\mu) + \varepsilon^3\widetilde{a}_3(t,\varepsilon,\psi_2,z_2,\mu),\\ \frac{d\psi_2}{dt} = \omega(t,\varepsilon) + \mu\Phi(t,\varepsilon,z_2,\mu) + \mu^{r+1}\Phi_2(t,\varepsilon,\psi_2,z_2,\mu)+\\ +\varepsilon\beta_1(t,\varepsilon,z_2,\mu) + \varepsilon^2\beta_2(t,\varepsilon,z_2,\mu) + \varepsilon^3\widetilde{b}_3(t,\varepsilon,\psi_2,z_2,\mu).\end{gathered} \tag{41}$$

Then for functions g_2, h_2 we obtain the next system of the differential equations in partial derivatives:

$$(\omega(t,\varepsilon)+\mu\Phi(t,\varepsilon,z_2,\mu))\frac{\partial g_2}{\partial\psi_2}+\mu Y(t,\varepsilon,z_2,\mu)\frac{\partial g_2}{\partial z_2}+$$

$$+\alpha_2(t,\varepsilon,z_2,\mu)=\mu\frac{\partial Y(t,\varepsilon,z_2,\mu)}{\partial z_2}\,g_2+\widetilde{a}_2(t,\varepsilon,\psi,z_2), \tag{42}$$

$$(\omega(t,\varepsilon)+\mu\Phi(t,\varepsilon,z_2,\mu))\frac{\partial h_2}{\partial\psi_2}+\mu Y(t,\varepsilon,z_2,\mu)\frac{\partial h_2}{\partial z_2}+$$

$$+\beta_2(t,\varepsilon,z_2,\mu)=\mu\frac{\partial \Phi(t,\varepsilon,z_2,\mu)}{\partial z_2}\,g_2+\widetilde{b}_2(t,\varepsilon,\psi,z_2). \tag{43}$$

The functions $Z_2, \Phi_2, \widetilde{a}_3, \widetilde{b}_3$ defined from the following systems of the linear algebraic equations:

$$\begin{gathered}\varepsilon^2\tfrac{\partial g_2}{\partial\psi_2}\,\Phi_2+\left(1+\varepsilon^2\tfrac{\partial g_2}{\partial z_2}\right)Z_2=Z_1(t,\varepsilon,\psi_2+\varepsilon^2h_2,z_2+\varepsilon^2g_2,\mu),\\ \left(1+\varepsilon^2\tfrac{\partial h_2}{\partial\psi_2}\right)\Phi_2+\varepsilon^2\tfrac{\partial h_2}{\partial\psi_2}Z_2=\Phi_1(t,\varepsilon,\psi_2+\varepsilon^2h_2,z_2+\varepsilon^2g_2,\mu),\end{gathered} \tag{44}$$

$$\left(1+\varepsilon^2\tfrac{\partial g_2}{\partial z_2}\right)\widetilde{a}_3+\varepsilon^2\tfrac{\partial g_2}{\partial\psi_2}\widetilde{b}_3=$$

$$=\tfrac{\mu\varepsilon}{2}\cdot\tfrac{\partial^2Y(t,\varepsilon,z_2+\nu_1^*\varepsilon^2g_2)}{\partial z_2^2}g_2^2+$$

$$+\tfrac{\partial\alpha_1(t,\varepsilon,z_2+\nu_2^*\varepsilon^2g_2)}{\partial z_2}\,g_2+\varepsilon\left(\tfrac{\partial\widetilde{a}_2(t,\varepsilon,\psi_2+\nu_3^*\varepsilon^2h_2,z_2+\nu_3^*\varepsilon^2h_2,\mu)}{\partial\psi_2}\,h_2+\right.$$

$$\left.+\tfrac{\partial\widetilde{a}_2(t,\varepsilon,\psi_2+\nu_3^*\varepsilon^2h_2,z_2+\nu_3^*\varepsilon^2h_2,\mu)}{\partial z_2}\,g_2\right)-\tfrac{1}{\varepsilon}\tfrac{\partial g_2}{\partial t}, \tag{45}$$

$$\varepsilon^2\tfrac{\partial h_2}{\partial z_2}\,\widetilde{a}_3+\left(1+\varepsilon^2\tfrac{\partial h_2}{\partial\psi_2}\right)\widetilde{b}_3=\tfrac{\mu\varepsilon}{2}\cdot\tfrac{\partial^2\Phi(t,\varepsilon,z_2+\nu_4^*\varepsilon^2g_2)}{\partial z_2^2}g_2^2+$$

$$+\tfrac{\partial\beta_1(t,\varepsilon,z_2+\nu_5^*\varepsilon^2g_2)}{\partial z_2}\,g_2+\varepsilon\left(\tfrac{\partial\widetilde{b}_2(t,\varepsilon,\psi_2+\nu_6^*\varepsilon^2h_2,z_2+\nu_6^*\varepsilon^2h_2,\mu)}{\partial\psi_2}\,h_2+\right.$$

$$\left.+\tfrac{\partial\widetilde{b}_2(t,\varepsilon,\psi_2+\nu_6^*\varepsilon^2h_2,z_2+\nu_6^*\varepsilon^2h_2,\mu)}{\partial z_2}\,g_2\right)-\tfrac{1}{\varepsilon}\tfrac{\partial h_2}{\partial t},$$

where $\nu_1^{;}\nu_2^*,\nu_3^*,\nu_4^*,\nu_5^*,\nu_6^*\in(0,1)$.

Exploring the systems (42), (43) and (44), (45) in the same systems (20), (21) and (22), (23) we find that $\alpha_2,\beta_2\in S_{m-2}^{z_2}(\varepsilon_1,z_0,\rho_2)$, $g_2,h_2\in F_{m-2}^{\psi_2,z_2}(\varepsilon_1,\alpha,z_0,\rho_2)$, $\Phi_2,Z_2\in F_{m-2}^{\psi_2,z_2}(\varepsilon_2,\alpha,z_0,\rho_2)$, $\widetilde{a}_3,\widetilde{b}_3\in F_{m-3}^{\psi_2,z_2}(\varepsilon_2,\alpha,z_0,\rho_2)$ $(\rho_2\in(0,\rho_1),\ \varepsilon_2\in(0,\varepsilon_1))$.

Continuing in this way, get to the transformation (17) and systems (18). We establish, that

$$\alpha_{m_1-1},\beta_{m_1-1}\in S_{m-m_1+1}^{z_{m_1-1}}(\varepsilon_{m_1-2},z_0,\rho_{m_1-1}),$$

$$g_{m_1-1},h_{m_1-1}\in F_{m-m_1+1}^{\psi_{m_1-1},z_{m_1-1}}(\varepsilon_{m_1-2},\alpha,z_0,\rho_{m_1-1}),$$

$$\Phi_{m_1-1}, Z_{m_1-1} \in F^{\psi_{m_1-1}, z_{m_1-1}}_{m-m_1+1}(\varepsilon_{m_1-1}, \alpha, z_0, \rho_{m_1-1}),$$

$$\widetilde{a}_{m_1}, \widetilde{b}_{m_1} \in F^{\psi_{m_1-1}, z_{m_1-1}}_{m-m_1}(\varepsilon_{m_1-1}, \alpha, z_0, \rho_{m_1-1}),$$

where $0 < \varepsilon_{m_1-1} < \varepsilon_{m_1-2} < \ldots < \varepsilon_2 < \varepsilon_1 < \varepsilon_0$, $0 < \rho_{m_1-1} < \rho_{m_1-2} < \ldots < \rho_2 <$ $< \rho_1 < d_1$.

Reversibility of transformations (15) – (17) for sufficiently small ε is proved similarly to Lemma 1.

Lemma 3 are proved.

4. Principal Results.

Theorem. *Suppose that the system (1) satisfies (A), (B). Then* $\exists\ \delta_0 \in (0, +\infty)$ *such, that* $\forall\ \delta \in (0, \delta_0)$ *(δ – value in condition (A)) the system (1) have the integral manifold*

$$x = w(t, \varepsilon, \theta, \mu) \in F^{\theta}_{m_1-1}(\varepsilon_1^*, \alpha^*),$$

where $\varepsilon_1^* \in (0, \varepsilon_0)$, $\alpha^* \in (0, \alpha)$, *and on this manifold the system (1) are reduces to equation:*

$$\frac{d\theta}{dt} = \omega(t, \varepsilon) + \mu\Theta(t, \varepsilon, \theta, w(t, \varepsilon, \theta, \mu)) + \varepsilon(t, \varepsilon, \theta, \mu).$$

Proof. Consider the system (18). Right-hand parts of this system are bounded at $t \in \mathbf{R}$, $\varepsilon \in (0, \varepsilon_{m_1-1})$, $\psi_{m_1-1} \in \mathbf{R}$. Therefore it is easy to see that if the $m+1 > 2m_1$, the functions $\varepsilon^{m_1-1}\widetilde{a}_{m_1}(t, \varepsilon, \psi_{m_1-1}, z_{m_1-1})$, $\varepsilon^{m_1-1}\widetilde{b}_{m_1}(t, \varepsilon, \psi_{m_1-1}, z_{m_1-1})$ belong to class $S^{z_{m_1-1}}_{m_1-1}(\varepsilon_{m_1-1}, z_0, \rho_{m_1-1})$. And thus in fact these functions are slowly varying and not oscillating, despite the dependence on ψ_{m_1-1}, thanks to the factor ε^{m_1-1}. Therefore we can to rewrite the system (18) in form:

$$\begin{aligned} \tfrac{dz}{dt} &= \mu Y(t, \varepsilon, z, \mu) + \mu^{r+1} Z(t, \varepsilon, \psi, z, \mu) + \varepsilon\widetilde{a}(t, \varepsilon, z, \mu), \\ \tfrac{d\psi}{dt} &= \omega(t, \varepsilon) + \mu\Phi(t, \varepsilon, z, \mu) + \mu^{r+1}\Psi(t, \varepsilon, \psi, z, \mu) + \varepsilon\widetilde{b}(t, \varepsilon, z, \mu), \end{aligned} \tag{46}$$

where $Y, \Phi \in S^z_{m_1-1}(\varepsilon^*, z_0, d_4)$, $Z, \Psi \in F^{\psi, z}_{m-m_1}(\varepsilon^*, \alpha_1, z_0, d_4)$, $\widetilde{a}, \widetilde{b} \in S^z_{m_1-1}(\varepsilon^*, z_0, d_4)$, $(\varepsilon^* \in (0, \varepsilon_0),\ \alpha_1 \in (0, \alpha))$.

In system (46) using the transformation:

$$z = z_0 + \mu\xi, \ \ \psi = \psi. \tag{47}$$

Since $z_0(t, \varepsilon, \mu) \in S_m(\varepsilon_0)$, then

$$\frac{dz_0}{dt} = -\varepsilon z_1(t, \varepsilon, \mu), \tag{48}$$

where $z_1 \in S_{m-1}(\varepsilon_0)$. We denote:

$$\lambda(t, \varepsilon, \mu) = \left.\frac{\partial Y(t, \varepsilon, z, \mu)}{\partial z}\right|_{z=z_0}.$$

On the basis of Lemma 2:

$$\inf_{G(\varepsilon_0)} |\lambda(t, \varepsilon, \mu)| = \gamma_1(\mu) > 0. \tag{49}$$

As a result of transformation (47) system (46) takes the form:

$$\begin{gathered}\frac{d\xi}{dt} = \mu\lambda(t,\varepsilon,\mu) + \mu^2 \frac{\partial Y(t,\varepsilon,z_0+\nu\mu\xi,\mu)}{\partial z}\ \xi^2 + \frac{\varepsilon}{\mu} z_1(t,\varepsilon,\mu) + \\ +\mu^r Z(t,\varepsilon,\psi,z_0+\mu\xi,\mu) + \tfrac{\varepsilon}{\mu}\ \widetilde{a}(t,\varepsilon,z_0+\mu\xi,\mu), \\ \frac{d\psi}{dt} = \omega(t,\varepsilon) + \mu\Phi(t,\varepsilon,z_0+\mu\xi,\mu) + \mu^{r+1}\Psi(t,\varepsilon,\psi,z_0+\mu\xi,\mu) + \\ +\varepsilon\widetilde{b}(t,\varepsilon,z_0+\mu\xi,\mu),\end{gathered} \tag{50}$$

where $0 < \nu < 1$.

Based on the assumption (B) the system (50) can be rewritten as:

$$\begin{gathered}\frac{d\xi}{dt} = \mu\lambda(t,\varepsilon,\mu)\xi + \mu^2 \frac{\partial Y(t,\varepsilon,z_0+\nu\mu\xi,\mu)}{\partial z}\ \xi^2 + \frac{\varepsilon}{\mu} z_1(t,\varepsilon,\mu) + \\ +\mu^2 \Xi_1(t,\varepsilon,\xi,\mu) + \frac{\varepsilon}{\mu}\ \widetilde{a}(t,\varepsilon,z_0+\mu\xi,\mu),\end{gathered} \tag{51}$$

$$\begin{gathered}\frac{d\psi}{dt} = \omega(t,\varepsilon) + \mu\Phi(t,\varepsilon,z_0+\mu\xi,\mu) + \mu^2 \Xi_2(t,\varepsilon,\xi,\mu) + \\ +\varepsilon\widetilde{b}(t,\varepsilon,z_0+\mu\xi,\mu),\end{gathered} \tag{52}$$

where $\Xi_1(t,\varepsilon,\xi,\mu),\ \Xi_2(t,\varepsilon,\xi,\mu) \in S^{\xi}_{m-m_1}(\varepsilon^*, z_0, \mu d_4)$.

Now we can consider equation (51) it regardless of the equation (52). Consider corresponding to equation (51), a linear non-homogeneous equation:

$$\frac{d\xi_0}{dt} = \mu\lambda(t,\varepsilon,\mu)\xi_0 + \frac{\varepsilon}{\mu}\ z_1(t,\varepsilon,mu). \tag{53}$$

Consider the next solution of this equations:

$$\xi_0(t,\varepsilon,\mu) = \frac{\varepsilon}{\mu}\ I[z_1(t,\varepsilon,\mu)], \tag{54}$$

where

$$I[z_1(t,\varepsilon,\mu)] = \int\limits_{\pm\infty}^{t} z_1(\tau,\varepsilon,\mu) \exp\left(\mu \int\limits_{\tau}^{t} \lambda(s,\varepsilon,\mu)ds\right) d\tau, \tag{55}$$

and sign at the lower limit of integration coincides with sign of $\lambda(t,\varepsilon,\mu)$. Using the unequality (49) and known estimates for integrals of the kind (55), we obtain, that $\xi_0(t,\varepsilon,\mu) \in S_{m_1-1}(\varepsilon^*)$, and $\exists\ K_2 \in (0,+\infty)$ such that

$$\|\xi_0\|_{S_{m_1-1}(\varepsilon^*)} \leq \frac{K_2\varepsilon^*}{\mu^2} \|z_1\|_{S_{m_1-1}(\varepsilon^*)}. \tag{56}$$

The solution belongs to class $S_{m_1-1}(\varepsilon^*)$ of the equation (51), we seek by iterative method, identifying as an initial approximation $\xi_0(t,\varepsilon,\mu)$, and subsequent iterations are defined by formulas:

$$\xi_{s+1}(t,\varepsilon,\mu) = I\left[\frac{\varepsilon}{\mu}\, z_1(t,\varepsilon,\mu) + \mu^2 \frac{\partial Y(t,\varepsilon,z_0+\nu\mu\xi_s,\mu)}{\partial z}\, \xi_s^2 + \right.$$

$$\left. + \mu^2 \Xi(t,\varepsilon,\xi_s,\mu) + \frac{\varepsilon}{\mu}\, \widetilde{a}(t,\varepsilon,z_0+\mu\xi_s,\mu)\right]. \tag{57}$$

We define the set:

$$\Omega = \left\{\ \xi \in S_{m_1-1}(\varepsilon^*) :\ \|\xi-\xi_0\|_{S_{m_1-1}(\varepsilon^*} \le h\ \right\}$$

and denote:

$$H_1(h,\mu) = \sup_{\xi\in\Omega} \left\| \frac{\partial Y(t,\varepsilon,z_0+\nu\mu\xi,\mu)}{\partial z}\, \xi^2 \right\|_{S_{m_1-1}(\varepsilon^*)},$$

$$H_2(h,\mu) = \sup_{\xi\in\Omega} \|\Xi(t,\varepsilon,\xi,\mu)\|_{S_{m_1-1}(\varepsilon^*)},\ H_3(h,\mu) = \sup_{\xi\in\Omega} \|\widetilde{a}(t,\varepsilon,z_0+\mu\xi,\mu)\|_{S_{m_1-1}(\varepsilon^*)},$$

$$H(h,\mu) = \max(H_1(h,\mu), H_2(h,\mu), H_3(h,\mu)).$$

Since the funcrions $Y, \Xi_1, \widetilde{a}$ are analytic with respect $\xi \in \Omega$, $\exists\ L_0(h,\mu) \in (0,+\infty)$ such that $\forall\ \xi,\eta \in \Omega$:

$$\left\| \frac{\partial Y(t,\varepsilon,z_0+\nu\mu\xi,\mu)}{\partial z}\, \xi^2 - \frac{\partial Y(t,\varepsilon,z_0+\nu\mu\eta,\mu)}{\partial z}\, \eta^2 \right\|_{S_{m_1-1}(\varepsilon^*)} \le$$

$$\le L_0(h,\mu)\|\xi-\eta\|_{S_{m_1-1}(\varepsilon^*)}.$$

Using a technique known contraction mapping principle [7], it is easy to show that if

$$K_2 H(h,\mu)\delta \le h_0 < h$$

(δ – the constant, which defined in condition (B)), all iterations (57) belongs to Ω. If

$$K_2 L_0(h,\mu)\delta < 1,$$

that process (57) is converge to solution $\xi = \xi^*(t,\varepsilon,\mu) \in S_{m_1-1}(\varepsilon_1^*)$ $(\varepsilon_1 * \in (0,+\varepsilon^*))$ of the equation (51). At the same time this solution determines the integral manifold of system (51), (52). Given Lemmas 1 – 3, this proves the theorem.

Conclusion. Thus, for the system (1) the conditions of existence of the integral manifold, which represented for sufficiently small values ε,μ by as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients, are obtained.

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