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THE STRESS STATE OF THE CONTINUOUS RECTANGULAR PLATE

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The problems on the continuous plates with the intermediate bearers are often necessary at the calculations of the building structures and constructions' elements in the mechanical engineering. As it was shown in [1, 2], the problems on the calculations of the plate shells and the folded-plate constructions are reduced to such problems also. The works of many native and foreign scientists are devoted to the calculation of the plates' stress state. The review of these works one can find in [3-5]. In the proposed article with the help of the integral transformation method [6], and the method of the three moments [7] the influence function is constructed. That allows to consider the more intricate problems on the stress state of the folded-plate constructions. The calculations of the characteristic values (the reactions and the bending moments on the bearers) are shown.

Keywords: shell, deformation, stress state, bending moment

The problem statement

The problem on the stress state of the continuous rectangular hingedly supported plate ($a < x < b, -l_1 < y < l_2$) is considered. The plate is hingedly supported along the lines $x = a_k$, $a_k = a + kl$, $l = (b-a)/n$, $k = \overline{1, n-1}$ with the help of the $n-1$ fixed bearing. The concentrated force P is applied at the point with the coordinates (ξ, n), $\xi \in (a_m, a_{m+1})$. It is necessary to estimate the bending moments' values $M_x(a_k, y)$ and the bearers' moments $N(a_k, y)$, $k = \overline{0, n}$.

Such problem is reduced to the solving of the Sophie-Germen biharmonic equation for the plates' bending

$$\Delta^2 w(x, y) = P\delta(x-\xi, y-\eta),\ a < x < b,\ x \neq a_k,\ k = \overline{1, n-1},\ |y| < \infty. \tag{1}$$

The solution of it should satisfy the conditions of the rectangular hingedly supporting

$$\begin{aligned} x = a, b &: w = 0,\ M_x = 0, \\ y = -l_1, l_2 &: w = 0,\ M_y = 0, \end{aligned} \tag{2}$$

and the conditions on the bearers

$$x = a_k,\ k = \overline{1, n-1}:\ w = 0,\ <\varphi_x> = 0,\ <M_x> = 0. \tag{3}$$

At that

$$N(a_k, y) = \langle V_x \rangle\big|_{x=a_k}\,, \tag{4}$$

where $\varphi_x(x,y)$ is the bending angle of the plate, $M_x(x,y)$, $M_y(x,y)$ are the bending moments, $V_x(x,y)$, $V_y(x,y)$ are the generalized lateral forces, which are connected with the point's by the known formulas:

$$\varphi_x = \frac{\partial w}{\partial x},$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right),$$

$$M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right), \tag{5}$$

$$V_x = -D\left(\frac{\partial^3 w}{\partial x^3} + (2-\nu)\frac{\partial^3 w}{\partial x \partial y^2}\right),$$

$$V_y = -D\left(\frac{\partial^3 w}{\partial y^3} + (2-\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\right).$$

D is the flexural rigidity of a plate $D = \dfrac{Eh^3}{12(1-\nu^2)}$, ν is the Poisson's coefficient, E is the modulus of elasticity, h is the plate's thickness.

We define as $\langle F(x,y) \rangle$, following by [1],

$$x = a : \langle F(x,y) \rangle = F(a-0, y) - F(a+0, y)$$

– is the function $F(x,y)$ jump through the line $x = a$.

The solution of the boundary problem (1)-(5) is searching with the help of the finite sin-Fourier transformation

$$w_\lambda(x) = \int_{-l_1}^{l_2} w(x,y) \sin \lambda (y + l_1) dy\,, \tag{6}$$

$$w(x,y) = \frac{1}{2\pi}\sum_{k=1}^{\infty} w_\lambda(x) \sin \lambda (y + l_1), \quad \lambda = \frac{\pi k}{2l} \tag{7}$$

As a result the problem is reduced to the one-dimensional discontinuous boundary problem [6, App. II, § 2, p.2] of the form

$$w_\lambda^{IV}(x) - 2\lambda^2 w_\lambda''(x) + \lambda^4 w_\lambda(x) = P\delta(x-\xi)\sin\lambda(\eta + l_1), \tag{8}$$

$$w_\lambda\big|_{x=a,b} = 0\,, \quad w_\lambda''\big|_{x=a,b} = 0\,, \tag{9}$$

$$w_\lambda\big|_{x=a_k}=0\,,\quad \langle w'_\lambda\rangle\big|_{x=a_k}=0\,,\quad \langle w''_\lambda\rangle\big|_{x=a_k}=0\,,\quad k=\overline{1,n-1}\,. \tag{10}$$

If the solution of this problem will be constructed, then with the help of the inverse Fourier transformation (7) one can obtain the solution of the stated problem in the form of a series.

The Green's function of the continuous problem

For the boundary problem (8, 9) solving, let's construct the Green's function $g_\lambda(x,\xi)$ of the tentative problem

$$L^2 w_\lambda = P\delta(x-\xi)\sin\lambda(\eta+l_1),\quad 0<x<l\,, \tag{11}$$

$$w_\lambda\big|_{x=0,l}=0\,,\quad Lw_\lambda\big|_{x=0,l}=0\,, \tag{12}$$

where $Lu=\partial^2 u/\partial x^2-\lambda^2 u$.

As it known [6, App. II, §1, p. 2, formula 1.17], the Green's function $g_\lambda(x,\xi)$ has the form:

$$g_\lambda(x,\xi)=\Phi_\lambda(x,\xi)-\sum_{j=0}^{3}c_i(\xi)\psi_j(x), \tag{13}$$

where $\Phi_\lambda(x,\xi)$ is the fundamental function of the equation (11), $\{\psi_j(x)\}_{j=0}^{3}$ is the fundamental basic system of the problem (11) – (12) solutions.

At that, the fundamental function of the equation (11) has the form [6, App. II, §1, p. 3, formula 1.34-35]

$$\Phi_\lambda(x,\xi)=\frac{1+\lambda|x-\xi|}{4\lambda^3}e^{-\lambda|x-\xi|}-\frac{1+\lambda(x+\xi)}{4\lambda^3}e^{-\lambda(x+\xi)}\,. \tag{14}$$

One can input the boundary functionals of the following form

$$U_0[u]=u\big|_{x=0}\,,\quad U_1[u]=Lu\big|_{x=0}\,,\quad U_2[u]=u\big|_{x=l}\,,\quad U_3[u]=Lu\big|_{x=l}\,.$$

Then, the coefficients $c_i(\xi)$, $i=\overline{0,3}$ can be estimated as

$$c_i(\xi)=U_i[\Phi_\lambda],\quad i=\overline{0,3}\,. \tag{15}$$

and the elements of the fundamental basic system of the solutions can be found as the solutions of the corresponded problems

$$\begin{cases} L^2\psi_i=0 \\ U_j[\psi_j]=\delta_{ij} \end{cases}, \tag{16}$$

$$j=\overline{0,3},\quad i=\overline{0,3}\,,$$

where δ_{ij} is the Kronecker symbol.

So, with the help of (15) and (16), one obtain

$$c_0(\xi)=0, \quad c_1(\xi)=0,$$

$$c_2(\xi)=\frac{1}{2\lambda^3}e^{-\lambda l}\left((1+\lambda l)\mathrm{sh}\lambda\xi-\lambda\xi\mathrm{ch}\lambda\xi\right), \tag{17}$$

$$c_3(\xi)=-\frac{1}{\lambda}e^{-\lambda l}\mathrm{sh}\lambda\xi .$$

$$\psi_0(x)=\frac{\mathrm{sh}\lambda(l-x)}{\mathrm{sh}\lambda l}, \quad \psi_1(x)=\frac{\mathrm{sh}\lambda x}{\mathrm{sh}\lambda l},$$

$$\psi_2(x)=-\frac{l\mathrm{ch}\lambda l}{2\lambda\mathrm{sh}^2\lambda l}sh\lambda(l-x)+\frac{(l-x)\mathrm{ch}\lambda(l-x)}{2\lambda\mathrm{sh}l}, \tag{18}$$

$$\psi_3(x)=-\frac{l\mathrm{ch}\lambda l}{2\lambda\mathrm{sh}^2\lambda l}sh\lambda x+\frac{x\mathrm{ch}\lambda x}{2\lambda\mathrm{sh}\lambda l}.$$

The Green's function of the problem (11)-(12) one can construct after the substitution of the (17) and (18) in (13).

The form of the Green's function will be

$$G_\lambda(x,\xi)=n^3 g_\lambda\left((x-a)/n,(\xi-a)/n\right).$$

The Green's function of the discontinuous solution

It is necessary to construct the Green's function of the discontinuous boundary problem (8)-(9).

In the designations of the previous part this problem will be the following

$$L^2 w_\lambda = P\delta(x-\xi)\sin\lambda(\eta+l_1), \tag{19}$$

$$w_\lambda\big|_{x=a,b}=0, \quad Lw_\lambda\big|_{x=a,b}=0, \tag{20}$$

$$w_\lambda\big|_{x=a_k}=0, \quad \langle w'_\lambda\rangle\big|_{x=a_k}=0, \quad \langle Lw_\lambda\rangle\big|_{x=a_k}=0, \quad k=\overline{1,n-1}, \tag{21}$$

The Green's function of the discontinuous problem (19)-(21) has the form [6, App. II, §2, p. 2, formula 2.19]:

$$G^*_\lambda(x,\xi)=G_\lambda(x,\xi)-\sum_{k=1}^{n-1}\mu_k G_\lambda(x,a_k), \tag{22}$$

where

$$G_\lambda(x,\xi)=\Phi_\lambda(x,\xi)-\sum_{j=0}^{3}c_j(\xi)\psi_j(x)$$

is the Green's function of the continuous problem (is constructed in the previous part), $\mu_k=N_\lambda(a_k)$.

According the three moments' theorem [7, part XVII, § 248], instead the problem (19)-(21) let's consider the $n-1$ problems of the form

$$L^2 w_\lambda=r_\lambda(x),\quad a_k<x<a_{k+1},$$

$$w_\lambda\big|_{x=a_k}=0,\quad w_\lambda\big|_{x=a_{k+1}}=0, \tag{23}$$

$$Lw_\lambda\big|_{x=a_k}=M_k,\quad Lw_\lambda\big|_{x=a_{k+1}}=M_{k+1},$$

where

$$r_\lambda(x)=\begin{cases}0, & a_k<x<a_{k+1}\\ f\delta(x-\xi), & a_m<x<a_{m+1}\end{cases}, \tag{24}$$

$$k\neq m,$$

at that $f=P\sin\lambda(\eta+l_1)$, M_k, $k=\overline{0,n}$ the transformations of the bending moments on the bearers are the unknown constants which are estimated from the second condition from (21).

$$w'_\lambda(a_k-0)=w'_\lambda(a_k+0),\quad k=\overline{1,n-1}. \tag{25}$$

The solution of the boundary problem (19)-(21), according [8], satisfies to the following correlation

$$w_\lambda(x)=\int_{a_k}^{a_{k+1}}r_\lambda(\xi)g_\lambda(x-a_k,\xi-a_k)d\xi-M_k\frac{\partial g_\lambda}{\partial\xi}(x-a_k,0)+M_{k+1}\frac{\partial g_\lambda}{\partial\xi}(x-a_k,l),$$

$$k=\overline{0,n-1}.$$

Then, with regard of the applied loading form (24), one obtain that on the each segment $a_k<x<a_{k+1}$, $k=\overline{0,n-1}$ the deflection's transformation $w_\lambda(x)$ should satisfy to ratio

$$w_\lambda(x)=\begin{cases}-M_m\dfrac{\partial g_\lambda}{\partial\xi}(x-a_m,0)+M_{m+1}\dfrac{\partial g_\lambda}{\partial\xi}(x-a_m,l)+fg_\lambda(x-a_m,\xi-a_m)\\ -M_k\dfrac{\partial g_\lambda}{\partial\xi}(x-a_k,0)+M_{k+1}\dfrac{\partial g_\lambda}{\partial\xi}(x-a_k,l)\end{cases}, \tag{26}$$

where $a_m<x<a_{m+1}$, $a_k<x<a_{k+1}$, $k=\overline{0,n-1}$, $k\neq m$.

One obtain the following equation for M_k because of the conditions (25):

$$TM_{k+1}-2BM_k+TM_{k-1}=\begin{cases}0, & 1\le k\le m-1\ \&\ m+1\le k\le n-1\\ f\dfrac{\partial}{\partial x}g_\lambda(0,\xi-a_m), & k=m\\ f\dfrac{\partial}{\partial x}g_\lambda(l,\xi-a_m), & k=m+1\end{cases}, \tag{27}$$

here $M_0=0$, $M_n=0$ in accordance with the condition (20),

$$T=-\frac{l\mathrm{ch}\lambda l}{2\mathrm{sh}^2\lambda l}+\frac{1}{2\lambda\mathrm{sh}\lambda l},$$

$$B=\frac{l\mathrm{ch}^2\lambda l}{2\mathrm{sh}^2\lambda l}-\frac{\mathrm{ch}\lambda l+\lambda l\mathrm{sh}\lambda l}{2\lambda\mathrm{sh}\lambda l}.$$

The solution of the homogeneous system (27) one can search [7] in the form

$$M_k=Cq_1^k+Dq_2^k,$$

$$q_{1,2}=B/T\pm\sqrt{(B/T)^2-1},$$

where C, D are the arbitrary constants, $q_{1,2}$ are the solutions of the characteristic equation of the homogeneous equation (27).

Then

$$\begin{aligned}M_k&=M_m\frac{q_2^k-q_1^k}{q_2^m-q_1^m}, \quad 0\le k\le m-1,\\ M_k&=M_{m+1}\frac{q_2^nq_1^k-q_1^nq_2^k}{q_1^{m+1}q_2^n-q_2^{m+1}q_1^n}, \quad m+2\le k\le n.\end{aligned} \tag{28}$$

After the substitution of the found values by $k=m$ and $k=m+1$, one obtain the system of the two linear algebraic equations relatively to M_m and M_{m+1}.

$$\begin{cases}TM_{m+1}-2BM_m+TM_m\dfrac{q_2^{m-1}-q_1^{m-1}}{q_2^m-q_1^m}=f\dfrac{\partial}{\partial x}g_\lambda(0,\xi-a_m)\\ TM_{m+1}\dfrac{q_1^{m+2}q_2^n-q_2^{m+2}q_1^n}{q_1^{m+1}q_2^n-q_2^{m+1}q_1^n}-2BM_{m+1}+TM_m=f\dfrac{\partial}{\partial x}g_\lambda(l,\xi-a_m)\end{cases}.$$

After searching of M_m and M_{m+1} as the solutions of the pointed by the formulas (28) system, one can define M_k, $k=\overline{0,n}$. One can obtain from the moments' equality condition and consider as in tin the three moments' theorem, each of the span separately:

$$(\xi-a_m)f-M_m+M_{m+1}-lV_{\lambda_m}(a_m+0)=0,$$

$$-(a_{m+1}-\xi)f-M_m+M_{m+1}-lV_{\lambda_{m+1}}(a_{m+1}-0)=0,$$

$$-M_k + M_{k+1} - lV_{\lambda_k}(a_k + 0) = 0, \quad k = \overline{0, n-1}, \quad k \neq m,$$

$$-M_k + M_{k+1} - lV_{\lambda_{k+1}}(a_{k+1} - 0) = 0, \quad k = \overline{0, n-1}, \quad k \neq m.$$

Then taking into consideration (4)

$$N_\lambda(a_k) = (-M_{k-1} + 2M_k - M_{k+1})/l, \quad k \neq m, \quad k \neq m+1;$$

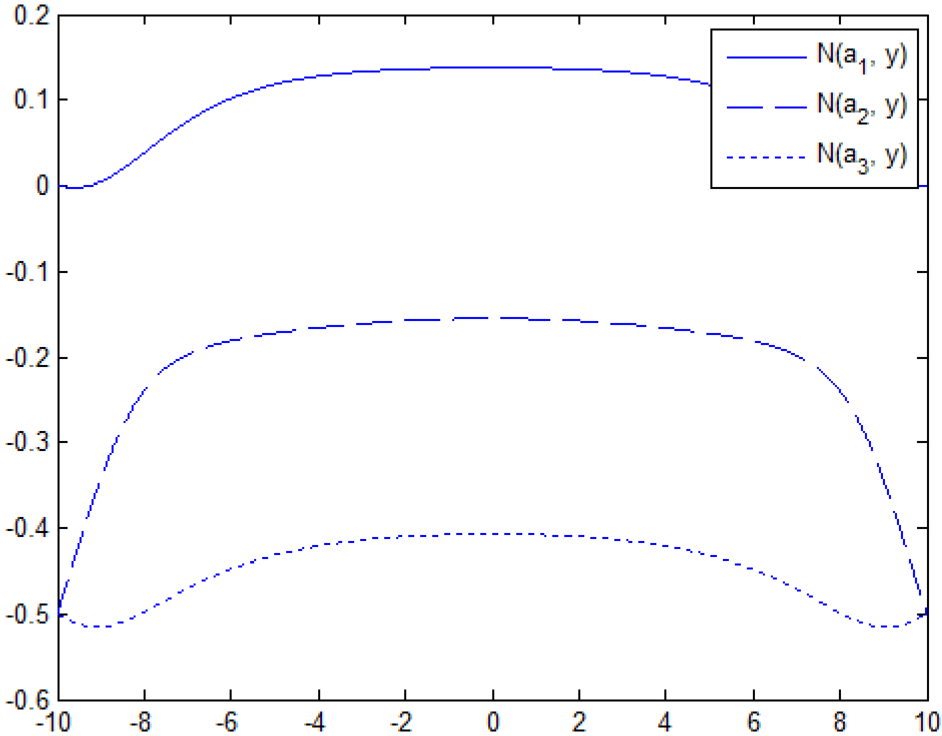
$$N_\lambda(a_m) = (-M_{m-1} + 2M_m - M_{m+1} - (a_{m+1} - \xi)f)/l;$$

$$N_\lambda(a_{m+1}) = (-M_m + 2M_{m+1} - M_{m+2} - (\xi - a_m)f)/l,$$

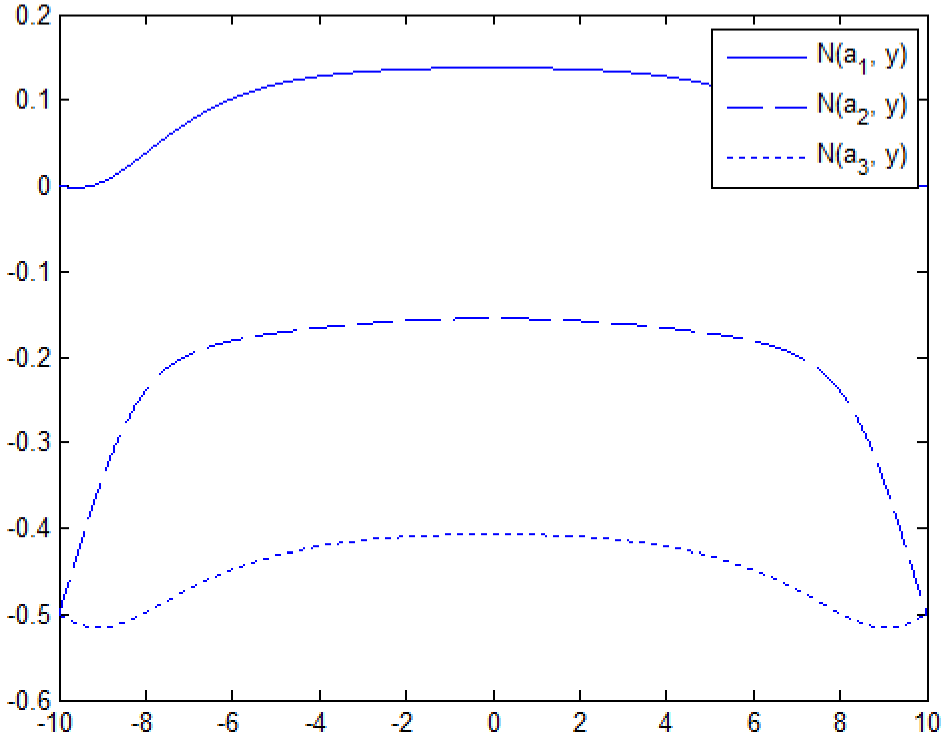
with the help of the inverse Fourier transformation (7) one can find $M_x(a_k, y)$, $N(a_k, y)$, $k = \overline{0, n}$.

The calculations

The graphics of the bending moments $M_x(a_k, y)$ and reactions of the bearings $N(a_k, y)$ are shown for the case $l_1 = 10$, $l_2 = 10$, $l = 4$, $n = 5$, $m = 3$, $\xi = 0$, $\eta = 0$ (Fig. 1). At that the solid line corresponds to the case $k = 1$, the dashdotted line to $k = 2$ and dotted line $k = 3$. The cases $k = 4,5,6$ by act of the symmetry are analogical to $k = 3,2,1$ correspondently.



a



b

Fig. 1. The graphics of the bending moments $M_x(a_k, y)$ and reactions of the bearings $N(a_k, y)$:

a – $\min\limits_{y\in(-l_1,l_2)} M_x(a_1, y) = -0.0728$, $\min\limits_{y\in(-l_1,l_2)} M_x(a_2, y) = -0.5507$, $\min\limits_{y\in(-l_1,l_2)} M_x(a_3, y) = -1.4526$;

b – $\max\limits_{y\in(-l_1,l_2)} N(a_1, y) = 0.1377$, $\max\limits_{y\in(-l_1,l_2)} N(a_2, y) = -0.1550$, $\max\limits_{y\in(-l_1,l_2)} N(a_3, y) = -0.4071$

Thus, the values of reactions and the bending moments on the bearers were obtained. The proposed approach to resolving such problems provides a solution for continuous plate with curved defects.

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НАПРУЖЕНИЙ СТАН НЕРОЗРІЗНОЇ ПРЯМОКУТНОЇ ПЛАСТИНКИ

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Задачі о нерозрізних пластинках з проміжними опорами часто виникають при розрахунках в будівельних конструкціях та елементів конструкцій в машинобудуванні. Як показано в роботах [1, 2], до таких задач зводиться і розрахунок пластинчастих оболонок і складчастих конструкцій. Розрахунку напруженого стану пластин присвячені роботи багатьох вітчизняних і зарубіжних вчених, огляд яких можна знайти в [3, 4, 5]. У даній роботі за допомогою методу інтегральних перетворень [6] і методу трьох моментів [7] побудована функція впливу, що дозволяє розглядати більш складні задачі про напружений стан пластинчастих конструкцій. Наведено розрахунки характерних величин – реакцій і величин згинальних моментів на опорах.
Ключові слова: оболонка, деформація, напружений стан, згинальний момент

НАПРЯЖЕННОЕ СОСТОЯНИЕ НЕРАЗРЕЗНОЙ ПРЯМОУГОЛЬНОЙ ПЛАСТИНКИ

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Задачи о неразрезных пластинках с промежуточными опорами часто возникают при расчетах в строительных конструкциях и элементов конструкций в машиностроении. Как показано в работах [1, 2], к таким задачам сводится и расчет пластинчатых оболочек и складчатых конструкций. Расчету напряженного состояния пластин посвящены работы многих отечественных и зарубежных ученых, обзор которых можно найти в [3, 4, 5]. В настоящей работе с помощью метода интегральных преобразований [6] и метода трёх моментов [7] построена функция влияния, что позволяет рассматривать более сложные задачи о напряжённом состоянии пластинчатых конструкций. Приведены расчёты характерных величин – реакций и величин изгибающих моментов на опорах.
Ключевые слова: оболочка, деформация, напряженное состояние, изгибающий момент