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to
FOURIER ANALYSIS

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This textbook provides a short introduction to classical Fourier Analysis. It contains basic notions and results related to trigonometric Fourier series, Fourier transforms, and orthogonal Legendre polynomials. The exposition is elementary. It is based on the standard university course of the Mathematical Analysis; in particular, the integrability of functions is treated in the sense of Riemann.

The knowledge necessary for the study of the course are given in the introductory chapter. The textbook contains also exercises to each of the main chapters.

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Preliminaries

In this section we present the background from Analysis needed for the further study of the course.

1. Even and odd functions, periodic functions

Definition 1. Let a function f be defined on a symmetric set $E \subset \mathbb{R}$, that is, such a set that for any $x \in E$ we have that $-x \in E$. This function is said to be *even* if for any $x \in E$ the equality $f(-x) = f(x)$ holds. If $f(-x) = -f(x)$ for any $x \in E$, then the function f is said to be *odd*.

For example, for $n \in \mathbb{N}$ the functions x^{2n}, $\cos nx$ are even, and the functions x^{2n-1}, $\sin nx$ are odd on $\mathbb{R}$.

The following properties are directly derived from the definitions.

(a) The graph of an even function is symmetric with respect to the y-axis, and the graph of an odd function is symmetric with respect to the origin.

(b) The product of two even functions is an even function.

(c) The product of two odd functions is an even function.

(d) The product of an even function and an odd function is an odd function.

(e) If f is an odd function, then for any $h > 0$

$$\int_{-h}^{h} f(x)\,dx = 0.$$

(f) If f is an even function, then for any $h > 0$

$$\int_{-h}^{h} f(x)\,dx = 2\int_{0}^{h} f(x)\,dx.$$

Definition 2. Let a function f be defined on $[0, b)$. Extend f to $(-b, 0)$, setting $f(x) = f(-x)$ $(x \in (-b, 0))$. Then we obtain a function f_1 which is called *the even extension* of the function f. If we set $f(x) = -f(-x)$ $(x \in (-b, 0))$, then we obtain a function f_2 which is called *odd extension* of the function f.

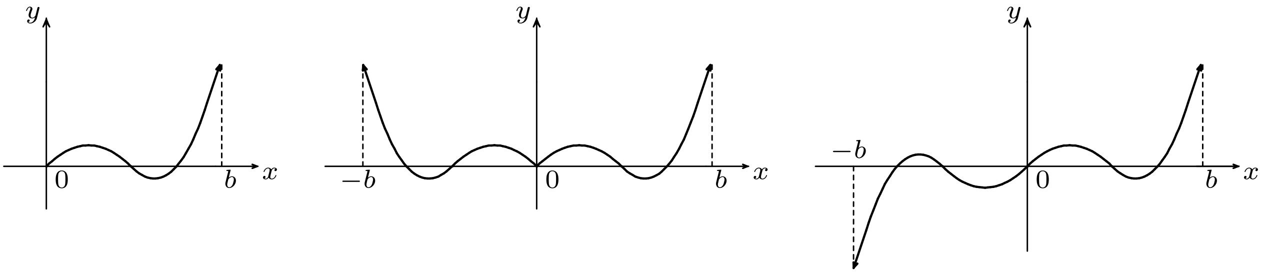

Fig. 1. The graphs of an initial function and its even and odd extensions.

Remark 1. If an odd function f is defined at the point $x = 0$, then it follows from the definition that $f(0) = 0$. Thus, an odd extension of a function f defined on $[0, b)$ is an odd function if and only if $f(0) = 0$.

Definition 3. A function f defined on $\mathbb{R}$ is said to be *periodic* with *a period* $T > 0$ if $f(x + T) = f(x)$ for all $x \in \mathbb{R}$.

The graph of a periodic function can be obtained by successive translations of its part corresponding to the interval $[0, T)$ (or any half-open interval of the length T) on nT $(n \in \mathbb{N})$ to the right and to the left. This means that the number nT also is a period. It is also clear that $f(x - nT) = f(x)$ $(n \in \mathbb{N})$.

Typical examples of 2π-periodic functions are trigonometric functions $\sin x$, $\cos x$. Another examples are 1-periodic function $\{\,x\,\}$ defined as the fractional part of x, or Dirichlet's function

$$\mathcal{D}(x) = \begin{cases} 0, & \text{if } x \text{ is irrational,} \\ 1, & \text{if } x \text{ is rational,} \end{cases}$$

for which every positive rational number is a period.

A periodic function may not be defined on the whole axis; for example, π-periodic function $\operatorname{tg} x$ isn't defined at the points $\frac{\pi}{2} + k\pi$ $(k \in \mathbb{Z})$, and π-periodic function $\operatorname{ctg} x$ isn't defined at the points $k\pi$ $(k \in \mathbb{Z})$.

Proposition 2. Assume that f is a periodic function with a period $T > 0$ and that f is integrable over $[0, T]$. Then f is integrable over any interval $I = [\alpha, \beta] \subset \mathbb{R}$. Moreover, for any $a \in \mathbb{R}$

$$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx. \tag{1}$$

PROOF. Let $k \in \mathbb{Z}$. Then

$$\int_{kT}^{(k+1)T} f(x)\,dx = \int_{kT}^{(k+1)T} f(x - kT)\,dx = \int_0^T f(u)\,du$$

(by the change of variable $u = x - kT$). This implies that f is integrable in any interval of the form $[-nT, nT]$ and therefore f is integrable in any interval $[\alpha, \beta] \subset \mathbb{R}$. Further, for any $a \in \mathbb{R}$

$$\int_a^{a+T} f(x)\,dx = \int_0^{a+T} f(x)\,dx - \int_0^a f(x)\,dx$$
$$= \int_0^T f(x)\,dx + \int_T^{a+T} f(x)\,dx - \int_0^a f(x)\,dx.$$

We also have (by the change of variable $u = x - T$) that

$$\int_T^{a+T} f(x)\,dx = \int_T^{a+T} f(x - T)\,dx = \int_0^a f(u)\,du.$$

This equality implies (1). $\square$

Definition 4. Let a function f_0 be defined on $[a, b]$ and $f_0(a) = f_0(b)$. Set $L = (b - a)/2$. *The periodic extension* of the function f_0 with the period $2L$ is the function f defined on $\mathbb{R}$ by the following equality

$$f(x + 2kL) = f_0(x) \quad (x \in [a, b],\ k \in \mathbb{Z})\,.$$

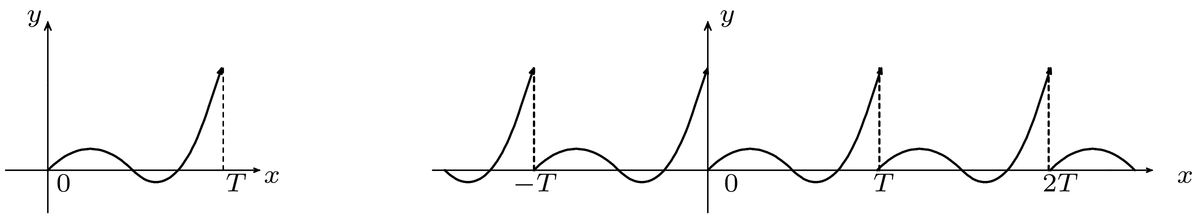

Fig. 2. The graphs of an initial function and its periodic extension.

2. Piecewise continuous and piecewise smooth functions

The class of all functions continuous on a set E is denoted by $\mathcal{C} \equiv \mathcal{C}(E)$.

Let a function f be defined on an open bounded interval (a, b).

Definition 5. The function f is said to be *piecewise continuous* on (a, b) if it has at most a finite number of points of discontinuity and, in addition, at each point of discontinuity $x_0 \in (a, b)$ the one-sided limits

$$f\,(x_0+) = \lim_{x \to x_0+} f(x), \quad f\,(x_0-) = \lim_{x \to x_0-} f(x)$$

exist and are finite.

A function f defined on $[a, b]$ is said to be piecewise continuous on $[a, b]$, if it is piecewise continuous on (a, b) and one-sided limits $f(a+)$ and $f(b-)$ exist and are finite. We shall call a function f defined on $\mathbb{R}$ piecewise continuous on $\mathbb{R}$ if it is piecewise continuous on any bounded interval. The class of all such functions we denote by $\mathcal{PC}$.

If $f\,(x_0+) \neq f\,(x_0-)$, then we say that f has *a jump* equal to $f\,(x_0+) - f\,(x_0-)$.

If a function $f \in \mathcal{PC}$, then it is integrable on any bounded interval.

Definition 6. A function f is called *piecewise smooth* on a bounded interval (a, b) if:

(i) f is piecewise continuous on (a, b);

(ii) the derivative f' exists and is continuous everywhere on (a, b), with a possible exception of a finite number of points;

(iii) f' has finite one-sided limits at every point $x \in (a, b)$.

We say that f is piecewise smooth on a closed interval $[a, b]$ if it is piecewise continuous on $[a, b]$ and there exist finite $f'(a+)$ and $f'(b-)$. We shall call f piecewise smooth on $\mathbb{R}$ if it is piecewise smooth on any bounded interval. The class of all such functions we denote by $\mathcal{PS}$.

We observe that a piecewise smooth function may have discontinuities. Further, if $f \in \mathcal{PS}$, then, giving to the derivative f' arbitrary values at the points where it does not exist, we obtain that $f' \in \mathcal{PC}$.

3. Numerical series and series of functions

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. The symbol

$$\sum_{n=1}^{\infty} a_n \equiv a_1 + a_2 + \dots + a_n + \dots \tag{2}$$

is called *series of numbers.* The numbers a_n are called *the terms*, and the numbers $S_n \equiv \sum\limits_{k=1}^{n} a_k$ $(n = 1, 2, \dots)$ are called *the partial sums* of the series (2).

If there exists $S \equiv \lim\limits_{n\to\infty} S_n$, then the series (2) is said to be *convergent*, and the number S is called *the sum* of this series. We write $S = \sum\limits_{n=1}^{\infty} a_n$. If the limit of the sequence of the partial sums of the series (2) does not exist, then this series is said to be *divergent.*

The following theorem gives *a necessary condition of convergence.*

Theorem 3. If the series (2) converges, then its terms a_n tend to zero.

We say that the series (2) is *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |a_n| \tag{3}$$

converges. If the series (2) converges, but the series (3) diverges, then we say that the series (2) converges *conditionally.*

Theorem 4. If a series absolutely converges, then it converges.

Assume that there is defined a sequence of functions $\{f_n\}_{n=1}^{\infty}$ on a set $E \subset \mathbb{R}$. We say that this sequence *converges on the set* E if for any $x \in E$ the sequence of numbers $\{f_n(x)\}_{n=1}^{\infty}$ converges. In this case the function $f(x) = \lim\limits_{n\to\infty} f_n(x)$ is called *the limit function.* We say that the sequence $\{f_n\}_{n=1}^{\infty}$ converges to the function f *uniformly* on the set E if for any positive number ε there exists a number N depending only on ε such that for any $n \geq N$ and any $x \in E$ there holds the inequality $|f_n(x) - f(x)| < \varepsilon$.

Let $\{u_n\}_{n=1}^{\infty}$ be a sequence of functions defined on a set $E \subset \mathbb{R}$. We say that the series of functions $\sum\limits_{n=1}^{\infty} u_n$ *converges on the set* E if the sequence of partial sums $S_n(x) = \sum\limits_{k=1}^{n} u_k(x)$ converges on E. In this case the function $f(x) = \lim\limits_{n\to\infty} S_n(x)$ is called *the sum* of the series $\sum\limits_{n=1}^{\infty} u_n(x)$. We write $\sum\limits_{n=1}^{\infty} u_n(x) = f(x)$. We say that the series $\sum\limits_{n=1}^{\infty} u_n$ converges *uniformly* on the set E if the sequence of its partial sums converges uniformly on E.

The basic properties of series of functions are contained in the following statements.

Theorem 5. If a series $\sum\limits_{n=1}^{\infty} u_n$ of functions u_n continuous on an interval $[a, b]$ converges uniformly on this interval to a function f, then f is continuous on $[a, b]$.

Theorem 6. If a series $\sum\limits_{n=1}^{\infty} u_n$ of functions u_n integrable on an interval $[a, b]$ converges uniformly on this interval to a function f, then f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx.$$

Theorem 7. Assume that u_n are continuously differentiable functions on an interval $[a, b]$. If the series $\sum\limits_{n=1}^{\infty} u_n$ converges at some point of $[a, b]$, and the series $\sum\limits_{n=1}^{\infty} u_n'$ converges uniformly on this interval, then the series $\sum\limits_{n=1}^{\infty} u_n$

converges uniformly on $[a, b]$ to a continuously differentiable function f and

$$f'(x) = \sum_{n=1}^{\infty} u_n'(x), \quad x \in [a, b].$$

4. Integral

The class of all Riemann integrable on an interval $[a, b]$ functions will be denoted by $\mathcal{R} \equiv \mathcal{R}[a, b]$.

Let $f \in \mathcal{R}[a, b]$. *The indefinite integral* of a function $f \in \mathcal{R}[a, b]$ is defined by the equality

$$F(x) = \int_a^x f(t)\, dt \quad (a \le x \le b).$$

It is a continuous function on $[a, b]$.

The following two theorems play the basic role in Analysis.

Theorem 8. If $f \in \mathcal{R}[a, b]$ and f is continuous at a point $x_0 \in [a, b]$, then its indefinite integral F is differentiable at the point x_0 and $F'(x_0) = f(x_0)$.

This theorem is called *the fundamental theorem of the Integral Calculus.*

Corollary 9. If $f \in \mathcal{PC}$ on $[a, b]$, then $F \in \mathcal{PS}$ and is continuous on $[a, b]$.

Theorem 10. Assume that $f \in \mathcal{PS}[a, b]$ and f is continuous on $[a, b]$. Assume that the values of f' at the points where it does not exist are defined arbitrarily. Then $f' \in \mathcal{R}$ and

$$\int_a^b f'(x)\, dx = f(b) - f(a).$$

The last equality is called *the fundamental formula of the Integral Calculus.*

We shall consider also functions defined on unbounded intervals.

Let a function f be defined on $[a, +\infty)$ and integrable on each interval

$[a, b]$ $(a < b < +\infty)$. If there exists a finite limit

$$\lim_{b \to \infty} \int_a^b f(x)\, dx,$$

then it is denoted by $\int_a^\infty f(x)\, dx$ and it is called *the improper integral* (we say also that the improper integral converges).

If the integral $\int_a^\infty |f(x)|\, dx$ converges, then we say that the integral $\int_a^\infty f(x)\, dx$ *absolutely* converges. In this case the function f is called *absolutely integrable* on $[a, \infty)$.

Theorem 11. If the integral $\int_a^\infty f(x)\, dx$ absolutely converges, then it converges.

The integral $\int_{-\infty}^b f(x)\, dx$ is defined in a similar way. If a function f is defined on $(-\infty, \infty)$, then we set

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{a} f(x)\, dx + \int_a^\infty f(x)\, dx,$$

provided that both integrals at the right-hand side converge. It is easy to see that this definition does not depend on the choice of a.

5. Approximation of integrable functions

In this section we present some statements concerning approximation of arbitrary integrable functions by continuous or step functions.

A function g defined on an interval $[a, b]$ is called *a step function* if there exists such partition $a = a_0 < a_1 < \cdots < a_m = b$ that g is constant in each interval (a_i, a_{i+1}), $i = 0, \ldots, m-1$. Obviously, any step function on $[a, b]$ is integrable in this interval.

Theorem 12. Let $f \in \mathcal{R}[a, b]$. Then for any $\varepsilon > 0$ there exists a step function g such that

$$\int\limits_a^b |f(x) - g(x)|\ dx < \varepsilon.$$

PROOF. For an arbitrary partition

$$a = x_0 < x_1 < \cdots < x_n = b,$$

denote

$$M_i = \sup_{x_i \le x \le x_{i+1}} f(x), \quad m_i = \inf_{x_i \le x \le x_{i+1}} f(x) \quad (i = 0, \ldots, n-1).$$

Let $\varepsilon > 0$. There exists a partition such that

$$\sum_{i=0}^{n-1} (M_i - m_i)\, \Delta x_i < \varepsilon.$$

Set $g(x) = m_i$ for $x \in [x_i, x_{i+1})$ $(i = 0, \ldots, n-1)$, $g(b) = m_{n-1}$. Then g is a step function and

$$\int\limits_a^b |f(x) - g(x)|\ dx = \sum_{i=0}^{n-1} \int\limits_{x_i}^{x_{i+1}} |f(x) - g(x)|\, dx$$

$$= \sum_{i=0}^{n-1} \int\limits_{x_i}^{x_{i+1}} |f(x) - m_i|\ dx \le \sum_{i=0}^{n-1} (M_i - m_i)\, \Delta x_i < \varepsilon. \quad \square$$

Theorem 13. Let $f \in \mathcal{R}[a, b]$. Then for any $\varepsilon > 0$ there exists a continuous function g such that

$$\int\limits_a^b |f(x) - g(x)|^2\ dx < \varepsilon^2.$$

PROOF. Denote $M = \sup\limits_{a \le x \le b} f(x)$, $m = \inf\limits_{a \le x \le b} f(x)$ and find a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that

$$\sum_{i=0}^{n-1} (M_i - m_i)\, \Delta x_i < \frac{\varepsilon^2}{M - m}.$$

Then

$$\sum_{i=0}^{n-1} (M_i - m_i)^2 \Delta x_i < \varepsilon^2.$$

Let $g(x) = f(x)$ for $x = x_i$ and let g be linear in each interval $[x_i, x_{i+1}]$ $(i = 0, \ldots, n-1)$. If $x \in [x_i, x_{i+1}]$, then $m_i \le f(x) \le M_i$, $m_i \le g(x) \le M_i$, and therefore $|f(x) - g(x)| \le M_i - m_i$. From here,

$$\int_a^b |f(x) - g(x)|^2 \, dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} |f(x) - g(x)|^2 \, dx$$

$$\le \sum_{i=0}^{n-1} (M_i - m_i)^2 \Delta x_i < \varepsilon^2. \quad \square$$

Remark 14. In Theorem 13 a function g can be chosen so that $g(a) = g(b) = 0$. Indeed, denote by f_1 the function which coincides with f on (a, b) and is equal to zero at the points a and b. For f_1 we construct the function g as in the proof of Theorem 13. By the definition, $g(a) = g(b) = 0$. Since the functions $f - g$ и $f_1 - g$ differ not more than in two points, then

$$\int_a^b [f(x) - g(x)]^2 \, dx = \int_a^b [f_1(x) - g(x)]^2 \, dx < \varepsilon^2.$$

A function g defined on the real line $\mathbb{R}$ is said to be *a function with a compact support* if it vanishes outside some bounded interval.

Theorem 15. Assume that a function f is defined on $\mathbb{R}$ and integrable on any interval, and f^2 is integrable on $\mathbb{R}$ in the improper sense. Then for any $\varepsilon > 0$ there exists a continuous function with a compact support g such that

$$\int_{-\infty}^{\infty} |f(x) - g(x)|^2 \, dx < \varepsilon^2. \tag{4}$$

Proof. Take $\varepsilon > 0$ and find $A > 0$ such that

$$\int_{-\infty}^{-A} f^2(x) \, dx + \int_A^{\infty} f^2(x) \, dx < \frac{\varepsilon^2}{2}.$$

Further, using Theorem 13 and Remark 14, we construct a function g_1 on $[-A, A]$, continuous and such that $g_1(-A) = g_1(A) = 0$ and

$$\int_{-A}^{A} [f(x) - g_1(x)]^2 < \frac{\varepsilon^2}{2}.$$

Define on $\mathbb{R}$ the function g equal to g_1 on $[-A, A]$ and zero outside this interval. Then g is a continuous function with a compact support on $\mathbb{R}$ satisfying inequality (4). $\square$

6. Improper integrals dependent on a parameter

Assume that a function $f(x, y)$ is defined for $x \geq a,\ y \in Y$ and for any $y \in Y$ the improper integral

$$\int_{a}^{\infty} f(x, y)\, dx \tag{5}$$

converges. This integral is called *the improper integral dependent on a parameter.* We say that the integral (5) converges *uniformly with respect to y on Y* if for any $\varepsilon > 0$ there is a number $A \geq a$ such that

$$\left| \int_{\xi}^{\infty} f(x, y)\, dx \right| < \varepsilon$$

for any $\xi \geq A$ and any $y \in Y$.

Theorem 16 (Weierstrass M-test). If there exists a non-negative on $[a, +\infty)$ function Φ such that

$$|f(x, y)| \leq |\Phi(x)| \quad (x \geq a,\ y \in Y),$$

and the integral $\int_{a}^{\infty} \Phi(x)\, dx$ converges, then the integral (5) converges uniformly with respect to y on Y.

The following theorems present properties of integrals dependent on a parameter which will be used below. We shall assume that a function f is defined on $[a, \infty) \times Y$ (Y is an interval) and a function φ is defined on $[a, \infty)$.

Theorem 17. Assume that the function f is continuous on $[a,\infty)\times Y$ and the function φ is integrable on $[a,A]$ for any $A>a$. If the integral

$$\int_a^\infty \varphi(x)f(x,y)\,dx$$

converges uniformly on Y, then it is a continuous function of the variable y on Y.

Theorem 18. Assume that f is continuous and has a continuous partial derivative $f'_y(x,y)$ on $[a,\infty)\times Y$. Let φ be integrable on any interval $[a,A]$ $(A>a)$. Assume that the integral

$$I(y)=\int_a^\infty \varphi(x)f(x,y)\,dx$$

converges for any $y\in Y$, and the integral

$$\int_a^\infty \varphi(x)f'_y(x,y)\,dx$$

converges uniformly with respect to y on Y. Then the function $I(y)$ is continuously differentiable on Y and

$$I'(y)=\int_a^\infty \varphi(x)f'_y(x,y)\,dx \quad (y\in Y).$$

Theorem 19. Let f be continuous on $[a,\infty)\times Y$, and let φ be integrable on $[a,A]$ for any $A>a$. Assume that the integral

$$\int_a^\infty \varphi(x)f(x,y)\,dx$$

converges uniformly on Y. Then:

(i) if $Y=[c,d]$, then

$$\int_c^d\left(\int_a^\infty \varphi(x)f(x,y)\,dx\right)dy=\int_a^\infty \varphi(x)\left(\int_c^d f(x,y)\,dy\right)dx;$$

(ii) if $Y = [c, \infty)$, φ is absolutely integrable on $[a, \infty)$, and the integral

$$\int_c^\infty f(x,y)\,dy$$

converges uniformly with respect to x on $[a, \infty)$, then

$$\int_c^\infty \left(\int_a^\infty \varphi(x) f(x,y)\,dx \right) dy = \int_a^\infty \varphi(x) \left(\int_c^\infty f(x,y)\,dy \right) dx$$

provided that the integral on the right-hand side converges, that is, there exists

$$\lim_{A\to\infty} \int_a^A \varphi(x) \left(\int_c^\infty f(x,y)\,dy \right) dx.$$

We observe that in standard courses of Analysis Theorems 17, 18, and 19 (i) are proved in the case $\varphi(x) \equiv 1$. The same proofs are valid in the general case, too. As for Theorem 19 (ii), it is derived from 19 (i). Generally, Theorem 19 (ii), is not true without the assumption that the function φ is absolutely integrable.

We shall need also the following theorem on interchange of the order of integrations.

Theorem 20. Let $F(x, y)$ be defined on $\mathbb{R} \times \mathbb{R}$. Assume that

(i) for any fixed value of one of the variables x, y, the function $F(x, y)$ is absolutely integrable on $\mathbb{R}$ with respect to the other variable;

(ii) the integrals

$$\int_{-\infty}^\infty F(x,y)\,dy, \qquad \int_{-\infty}^\infty |F(x,y)|\,dy,$$

(as functions of the variable x) are integrable with respect to x in each bounded interval, and integrals

$$\int_{-\infty}^\infty F(x,y)\,dx, \qquad \int_{-\infty}^\infty |F(x,y)|\,dx$$

(as functions of the variable y) are integrable with respect to y on each bounded integral.

If one of the integrals

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |F(x,y)|\, dy, \qquad \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |F(x,y)|\, dx$$

converges, then the other also converges and the equality

$$\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} F(x,y)\, dy = \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} F(x,y)\, dx$$

holds.

Usually this theorem is not included to a standard course of Mathematical Analysis, but it can be proved in the framework of this course.

7. The Cauchy – Schwarz inequality

Theorem 21 (Cauchy's inequality). Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be real numbers. Then

$$\left|\sum_{k=1}^{n} a_k b_k\right| \le \left(\sum_{k=1}^{n} a_k^2\right)^{1/2} \left(\sum_{k=1}^{n} b_k^2\right)^{1/2}. \tag{6}$$

Proof. We can assume that $a_k, b_k \ge 0$. We have

$$0 \le \sum_{k=1}^{n} (a_k - b_k)^2 = \sum_{k=1}^{n} a_k^2 - 2\sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n} b_k^2.$$

Thus,

$$2\sum_{k=1}^{n} a_k b_k \le \sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} b_k^2 \equiv A + B$$

(of course, we assume that $A, B > 0$). Replace a_k by $a_k\sqrt{\lambda}$ and b_k by $b_k/\sqrt{\lambda}$, where a number $\lambda > 0$ will be chosen later. Then the left-hand side remains the same, and we get

$$2\sum_{k=1}^{n} a_k b_k \le \lambda A + \frac{B}{\lambda}.$$

Choose λ from the equality $\lambda A = \frac{B}{\lambda}$; then $\lambda = \sqrt{\frac{B}{A}}$. We obtain

$$2\sum_{k=1}^{n} a_k b_k \le 2\sqrt{AB},$$

which is (6). $\square$

Corollary 22. If the series $\sum\limits_{n=1}^{\infty} a_n^2$ and $\sum\limits_{n=1}^{\infty} b_n^2$ converge, then the series $\sum\limits_{n=1}^{\infty} a_n b_n$ absolutely converges.

Theorem 23 (Schwarz inequality). Let $f, g \in \mathcal{R}[a,b]$. Then

$$\left|\int_a^b f(x)g(x)\,dx\right| \le \left(\int_a^b f^2(x)\,dx\right)^{1/2} \left(\int_a^b g^2(x)\,dx\right)^{1/2}.$$

For the proof, it suffices to use the same arguments as in the proof of Cauchy's inequality, replacing sums by integrals.

In the latter inequality a or b may be infinite.

8. Summability of sequences and series

Let $\{x_n\}$ be a sequence of numbers. Consider *the arithmetic means*

$$\xi_n = \frac{x_1 + \cdots + x_n}{n}.$$

We say that the sequence $\{x_n\}$ is summable to a number ξ by the method of *the arithmetic means* (or $(C,1)$-*summable*) if

$$\lim_{n\to\infty} \xi_n = \xi.$$

Theorem 24 (Cauchy). If $\{x_n\}$ converges to a number ξ, then $\{x_n\}$ is $(C,1)$-summable to ξ.

Proof. Set $\alpha_n = x_n - \xi$. Then $\alpha_n \to 0$. We have

$$\xi_n = \frac{x_1 + \cdots + x_n}{n} = \xi + \frac{\alpha_1 + \cdots + \alpha_n}{n}.$$

We shall prove that $\beta_n = (\alpha_1 + \cdots + \alpha_n)/n \to 0$. Let $\varepsilon > 0$. There exists N such that $|\alpha_k| < \varepsilon/2$ $(k \geq N)$. Let $n > N$. Then

$$|\beta_n| \leq \frac{1}{n} \sum_{k=1}^{n} |\alpha_k| < \frac{1}{n} \left(\sum_{k=1}^{N} |\alpha_k| + \frac{\varepsilon}{2}(n - N) \right) < \frac{\varepsilon}{2} + \frac{1}{n} \sum_{k=1}^{N} |\alpha_k| .$$

Now, let N' be such that

$$\frac{1}{N'} \sum_{k=1}^{N} |\alpha_k| < \frac{\varepsilon}{2}.$$

Then for all $n \geq N'$ we have $|\beta_n| < \varepsilon$. So, $\beta_n \to 0$, and hence $\xi_n \to \xi$. $\square$

Thus, convergence implies $(C, 1)$-summability to the same limit. The converse is not true.

Example 25. Let $x_n = (-1)^n$. Then $\{x_n\}$ diverges. At the same time,

$$x_1 + x_2 + \cdots + x_{2k} = 0, \quad x_1 + x_2 + \cdots + x_{2k-1} = -1 \quad (k = 1, 2, \dots).$$

Therefore $(x_1 + \cdots + x_n)/n \to 0$. $\square$

Given a series

$$\sum_{n=0}^{\infty} u_n, \tag{7}$$

we consider the sequence of its partial sums

$$S_n = \sum_{k=0}^{n} u_k \quad (n = 0, 1, \dots).$$

We say that series (7) is summable by *the method of arithmetic means* to the sum S if the sequence $\{S_n\}$ is $(C, 1)$-summable to S, that is, if

$$\sigma_n = \frac{S_0 + \cdots + S_n}{n + 1} \to S.$$

If the series (7) converges to S, then it is $(C, 1)$-summable to S. The converse is false.

Example 26. Consider the series

$$\sum_{n=0}^{\infty} (-1)^n.$$

We have $S_{2k+1} = 0$, $S_{2k} = 1$. Thus,

$$\frac{S_0 + \cdots + S_{2k+1}}{2k+2} = \frac{k+1}{2k+2} = \frac{1}{2}, \quad \frac{S_0 + \cdots + S_{2k}}{2k+1} = \frac{k+1}{2k+1} \to \frac{1}{2}.$$

Therefore, the series is $(C, 1)$-summable to $\frac{1}{2}$, while it diverges. $\square$

Remark 27. A series with positive terms converges if and only if it is $(C, 1)$-summable. This follows from the statement: *if* $x_n \to \infty$, *then*

$$\frac{x_1 + \cdots + x_n}{n} \to \infty.$$

1. Fourier series

1.1. The trigonometric system

The sequence of functions

$$1, \cos x, \sin x, \ldots, \cos nx, \sin nx, \ldots$$

is called *the trigonometric system.* These functions have period 2π.

The scalar product of functions $f, g \in \mathcal{R}$ is defined as

$$\int_{-\pi}^{\pi} f(x)g(x)\,dx.$$

Functions $f, g \in \mathcal{R}$ are said to be *orthogonal* if their scalar product is equal to zero. A system of functions is said to be *orthogonal* if for each function f of this system the integral of f^2 over $[-\pi, \pi]$ is positive, and any two different functions are orthogonal. For example, the trigonometric system is orthogonal. It is a consequence of the following equalities ($m, n \in \mathbb{N}$):

$$\int_{-\pi}^{\pi} 1\,dx = 2\pi, \quad \int_{-\pi}^{\pi} \cos nx\,dx = \int_{-\pi}^{\pi} \sin nx\,dx = \int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0,$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = \int_{-\pi}^{\pi} \sin mx \sin nx\,dx = \begin{cases} 0, & m \neq n, \\ \pi, & m = n. \end{cases}$$

A trigonometric series is called a series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right), \tag{1.1}$$

where a_n, b_n are real numbers (*coefficients*).

Proposition 1.1. Assume that the series (1.1) converges uniformly on $[-\pi, \pi]$ and let f be its sum. Then f is a continuous 2π-periodic function, and

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx \quad (m = 0, 1, \dots), \tag{1.2}$$

$$b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx \quad (m = 1, 2, \dots). \tag{1.3}$$

PROOF. The function f is continuous as the sum of a uniformly convergent series of continuous functions. Multiply the equality

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

by $\cos mx$. Since the function $\cos mx$ is bounded, on the right-hand side we obtain a uniformly convergent series. Integrating it term-by-term from $-\pi$ to π and using the orthogonality of the trigonometric system, we have for $m = 1, 2, \dots$

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} \left(\frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right) \cos mx \, dx$$

$$= \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \cos mx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx \right) = \pi a_m,$$

that is, (1.2) holds. Similarly we prove (1.2) for $m = 0$ and (1.3) for $m = 1, 2, \dots$. $\square$

Definition 1.2. Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function. Then the numbers

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad (n = 0, 1, \dots), \tag{1.4}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad (n = 1, 2, \dots) \tag{1.5}$$

are called *Fourier coefficients* of the function f. The series (1.1), where a_n and b_n are Fourier coefficients of the function f, is called *the Fourier series* of f.

We shall write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

Here the symbol $\sim$ denotes correspondence, it means only that to the function f it is assigned its Fourier series. Now Proposition 1.1 can be formulated as follows: *if a trigonometric series converges uniformly on $[-\pi, \pi]$, then it is the Fourier series of its sum.*

Example 1.3. Find the Fourier series for 2π-periodic extension of the function

$$f(x) = \begin{cases} 1, & 0 < x < \pi, \\ 0, & -\pi < x < 0, \end{cases} \qquad f(k\pi) = \frac{1}{2} \quad (k \in \mathbb{Z}).$$

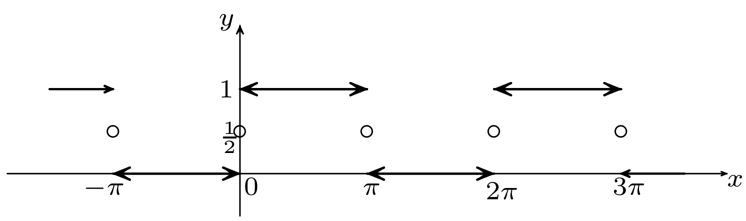

Fig. 3. The graph of 2π-periodic extension of the function $y = f(x)$.

For this, evaluate the Fourier coefficients

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi} \int_{0}^{\pi} dx = 1,$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_{0}^{\pi} \cos nx\,dx = \frac{1}{\pi n} \sin nx \Big|_0^{\pi} = 0 \quad (n = 1, 2, \dots),$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx = \frac{1}{\pi} \int_{0}^{\pi} \sin nx\,dx = \frac{1}{\pi} \cdot \frac{-\cos nx}{n} \Big|_0^{\pi}$$

$$= \frac{1}{\pi n}\left(1+(-1)^{n-1}\right) = \begin{cases} \frac{2}{\pi n}, & n = 2k-1, \\ 0, & n = 2k \end{cases} \qquad (n = 1, 2, \dots).$$

Thus, the given function has the following Fourier series

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} b_n \sin nx = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1}\sin(2k-1)x. \quad \square$$

Observe that if $f(x) = g(x)$ except a finite number of points of the interval $[-\pi, \pi]$, then the Fourier series of these functions coincide.

A function of the form

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos kx + \sin kx\right),$$

where $n \in \mathbb{N}$, a_k and b_k are real numbers and $|a_n| + |b_n| > 0$, is called *a trigonometric polynomial* of degree n. It is clear that for each such function its Fourier series is itself.

Example 1.4. Since the function $f(x) = 4 - 3\sin 2x + \cos 5x$ is a trigonometric polynomial, its Fourier series coincide with this function, that is, all the Fourier coefficients of the given function are equal to zero, except $a_0 = 8,\ b_2 = -3,\ a_5 = 1$.

Example 1.5. The function $f(x) = \sin^2 x$ can be represented as a trigonometric polynomial $f(x) = \sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$. This polynomial is also the Fourier series of the given function.

Example 1.6. Similarly, the function $f(x) = \sin^3 x$ can be represented as a trigonometric polynomial. Indeed, $\sin^3 x = \sin^2 x \sin x = \frac{1-\cos 2x}{2}\sin x = \frac{1}{2}\sin x - \frac{1}{2}\cos 2x \sin x = \frac{1}{2}\sin x - \frac{1}{4}(\sin 3x - \sin x) = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$. This polynomial is the Fourier series of the given function.

Example 1.7. We show that the function $(\cos x)^n$ is a trigonometric cosine-polynomial of the degree n, that is,

$$(\cos x)^n = \frac{a_0^{(n)}}{2} + \sum_{k=1}^{n} a_k^{(n)} \cos kx, \tag{1.6}$$

where $a_n^{(n)} \neq 0$. For this, we apply the induction. For $n = 1$ the statement is true. Assume that it is true for some n. Then, using the equality

$$\cos kx \cos x = \frac{1}{2}(\cos(k+1)x + \cos(k-1)x),$$

we easily obtain that the representation of the form (1.6) holds for $n+1$, too.

Proposition 1.8. Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function. Then:
(i) if f is even, then

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx, \tag{1.7}$$

where

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx;$$

(ii) if f is odd, then

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin nx, \tag{1.8}$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx.$$

Proof. (i) If the function f is even, then $f(x) \cos x$ also is even and therefore

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \quad (n = 0, 1, \dots).$$

Since the function $f(x) \sin x$ is odd,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = 0 \quad (n = 1, 2, \dots).$$

Similarly we get (ii). $\square$

The series (1.7) и (1.8) are called *cosine-* and *sine-series*, respectively.

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)\,.$$

Then

$$f(-x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx - b_n \sin nx)\,.$$

It follows that

$$\frac{f(x)+f(-x)}{2} \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

and

$$\frac{f(x)-f(-x)}{2} \sim \sum_{n=1}^{\infty} b_n \sin nx.$$

These functions can be called *even* and *odd parts* of the function f.

Example 1.9. The function $f(x) = |x|$ $(\pi \le x \le \pi)$ is even and therefore its Fourier series is a cosine-series.

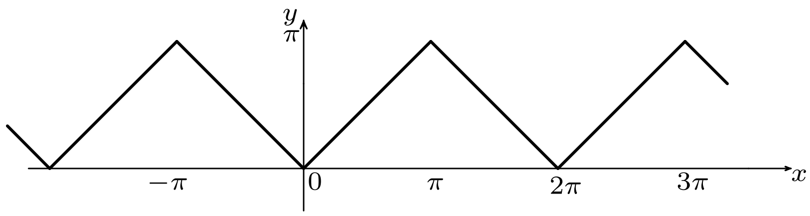

Fig. 4. The graph of the periodic extension of the function $f(x) = |x|$ $(\pi \le x \le \pi)$.

Evaluate the Fourier coefficients:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\frac{\pi^2}{2} = \pi,$$

and for $n = 1, 2, \dots,$ using the integral $\int\limits_0^{\pi} \sin nx\,dx = \begin{cases} \frac{2}{n}, & n = 2k-1, \\ 0, & n = 2k, \end{cases}$

evaluated before, we have

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\,dx = \frac{2}{\pi}\left[\begin{array}{ll} u = x & du = dx \\ dv = \cos nx & v = \frac{1}{n}\sin nx \end{array}\right]\Bigg|_0^{\pi}$$

$$= \frac{2}{\pi} \cdot \frac{-1}{n} \int_0^\pi \sin nx\, dx = \begin{cases} -\frac{4}{\pi n^2}, & n = 2k-1, \\ 0, & n = 2k. \end{cases}$$

Thus,

$$|x| \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)x \quad (|x| \le \pi). \quad \square$$

Example 1.10. The function $f(x) = \operatorname{sign} \sin x$ is odd.

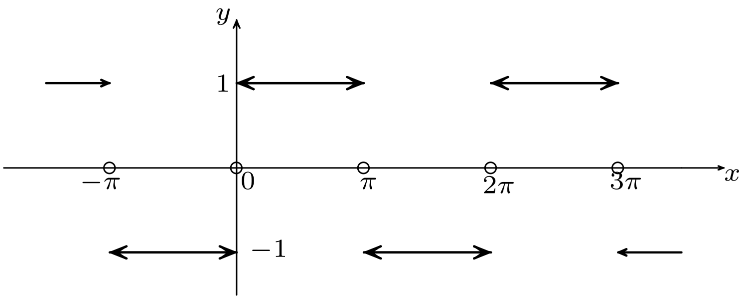

Fig. 5. The graph of the function $y = \operatorname{sign} \sin x$.

The Fourier series of f is a sine-series. Evaluate the Fourier coefficients:

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx\, dx = \frac{2}{\pi} \int_0^\pi \sin nx\, dx$$

$$= \begin{cases} \frac{4}{\pi n}, & n = 2k-1, \\ 0, & n = 2k \end{cases} \qquad (n = 1, 2, \dots).$$

Hence,

$$\operatorname{sign} \sin x \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin(2k-1)x. \quad \square$$

Example 1.11. The function $f(x) = x$ $(-\pi < x < \pi)$ is odd and therefore its Fourier series is a sine-series.

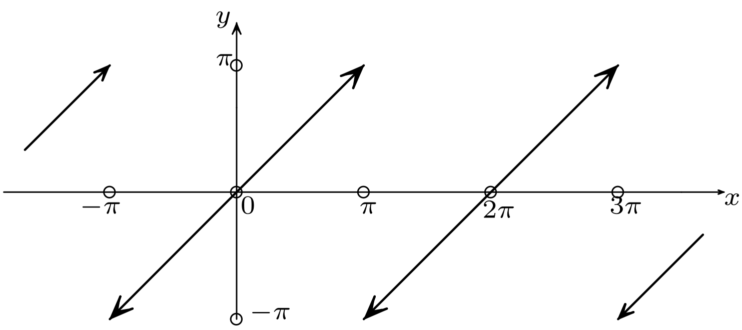

Fig. 6. The graph of the periodic extension of the function $f(x) = x$ $(-\pi < x < \pi)$.

We have:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx = \frac{2}{\pi}\left[\begin{array}{ll} u = x & du = dx \\ dv = \sin nx\,dx & v = -\frac{1}{n}\cos nx \end{array}\right]\Bigg|_0^{\pi}$$

$$= -\frac{2}{\pi n}x\cos nx\Big|_0^{\pi} = 2\frac{(-1)^{n-1}}{n}.$$

Thus,

$$x \sim 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \quad (|x| < \pi). \quad \square$$

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right).$$

Then, obviously,

$$f(-x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx - b_n\sin nx\right).$$

It is also easy to see that for $\alpha, \beta \in \mathbb{R}$

$$\alpha f(x) + \beta \sim \left(\frac{\alpha a_0}{2} + \beta\right) + \alpha\sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right).$$

Further, let $g(x) = f(\pi - x)$ and

$$g(x) \sim \frac{a_0'}{2} + \sum_{n=1}^{\infty}\left(a_n'\cos nx + b_n'\sin nx\right).$$

Then, performing the change of variable $x = \pi - t$ and using the periodicity, we get

$$a_n' = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos n(\pi - t)\,dt$$

$$= \frac{(-1)^n}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = (-1)^n a_n,$$

and, analogously, $b_n' = (-1)^{n+1}b_n$. Thus,

$$f(\pi - x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(-1)^n\left(a_n\cos nx - b_n\sin nx\right).$$

Example 1.12. We have constructed above the Fourier series

$$x \sim 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \quad (|x| < \pi).$$

If $x \in [-\pi, \pi]$, then $\pi - x \in [0, 2\pi]$, and thus we have for the 2π-periodic extension of the function $f(x) = \frac{\pi - x}{2}$ $(0 < x < 2\pi)$:

$$\frac{\pi - x}{2} \sim \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \quad (0 < x < 2\pi).$$

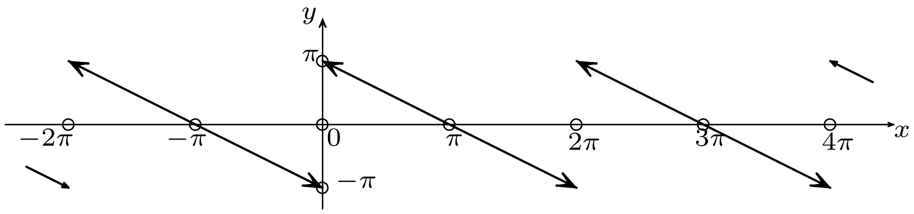

Fig. 7. The graph of the $2\pi-$periodic extension of the function $y = \frac{\pi - x}{2}$ $(0 < x < 2\pi)$.

$\square$

1.2. The complex form of the Fourier series

Recall the well-known *Euler formula*

$$\mathrm{e}^{i\varphi} = \cos\varphi + i\sin\varphi,$$

where i is the imaginary unit. It follows immediately from this formula that

$$\cos\varphi = \frac{\mathrm{e}^{i\varphi} + \mathrm{e}^{-i\varphi}}{2}, \quad \sin\varphi = \frac{\mathrm{e}^{i\varphi} - \mathrm{e}^{-i\varphi}}{2\,i}.$$

Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right). \tag{1.9}$$

Rewrite the nth partial sum of this series as

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos kx + b_k \sin kx\right)$$

$$= \frac{a_0}{2} + \frac{1}{2}\sum_{k=1}^{n}\left(a_k e^{ikx} + a_k e^{-ikx} - ib_k e^{ikx} + ib_k e^{-ikx}\right)$$
$$= \frac{a_0}{2} + \sum_{k=1}^{n}\left(\frac{a_k - ib_k}{2} e^{ikx} + \frac{a_k + ib_k}{2} e^{-ikx}\right).$$

Set

$$c_0 = \frac{a_0}{2},\; c_k = \frac{a_k - ib_k}{2}, \quad c_{-k} = \frac{a_k + ib_k}{2} \quad (k = 1, 2, \dots, n). \tag{1.10}$$

We obtain that

$$S_n(x) = c_0 + \sum_{k=1}^{n}\left(c_k e^{ikx} + c_{-k} e^{-ikx}\right) = \sum_{k=-n}^{n} c_k e^{ikx}.$$

Taking into account (1.10) and formulas for the Fourier coefficients, we have that for any integer $k \geq 0$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx - \frac{i}{2\pi}\int_{-\pi}^{\pi} f(x)\sin kx\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx.$$

Similarly, for any negative integer k

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\, dx.$$

Thus, the series (1.9), assigned to the function f, can be rewritten in the form

$$f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}, \quad \text{where} \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx}\, dx. \tag{1.11}$$

We observe that $c_{-n} = \overline{c}_n$, where the over-line means *the complex conjugation*, that is, $\overline{a + b\,i} = a - b\,i$. In particular, $\overline{e^{i\varphi}} = e^{-i\varphi}$, $\left|e^{i\varphi}\right|^2 = e^{i\varphi} \cdot \overline{e^{i\varphi}} = e^{i\varphi} \cdot e^{-i\varphi} = \cos^2\varphi + \sin^2\varphi = 1$, $e^{i\varphi} \neq 0$.

The Fourier series in the complex form can be defined for any complex-valued 2π-periodic function $f = u + iv$ integrable on $[-\pi, \pi]$, that is, such that $u, v \in \mathcal{R}[-\pi, \pi]$.

Definition 1.13. The system of functions $\left\{e^{in\varphi}\right\}$, where n runs over the set $\mathbb{Z}$ of all integers, and $\varphi \in [-\pi, \pi]$, is called *the exponential system.*

For complex-valued functions f, g defined on $[-\pi, \pi]$ *the scalar product* is given by the equality

$$\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx.$$

Proposition 1.14. The exponential system is orthogonal; more exactly,

$$\int_{-\pi}^{\pi} \mathrm{e}^{inx}\mathrm{e}^{-imx}\, dx = \begin{cases} 0, & m \neq n, \\ 2\pi, & m = n. \end{cases}$$

Proof. Let $m, n \in \mathbb{Z}$. Then, using the orthogonality of the trigonometric system, we obtain

$$\int_{-\pi}^{\pi} \mathrm{e}^{inx}\mathrm{e}^{-imx}\, dx = \int_{-\pi}^{\pi} (\cos nx + i \sin nx)(\cos mx - i \sin mx)\, dx$$

$$= \int_{-\pi}^{\pi} (\cos(n-m)x + i\sin(n-m)x)\, dx = \begin{cases} 0, & m \neq n, \\ 2\pi, & m = n. \end{cases} \qquad \square$$

Let

$$T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

be a trigonometric polynomial, and let $|a_n| + |b_n| > 0$. In the complex form

$$T_n(x) = \sum_{k=-n}^{n} c_k \mathrm{e}^{ikx}.$$

We have

$$T_n(x)\mathrm{e}^{inx} = \sum_{k=-n}^{n} c_k \mathrm{e}^{i(n+k)x} = \sum_{m=0}^{2n} c'_m \mathrm{e}^{imx}, \quad \text{where } c'_m = c_{m-n}.$$

Thus, $T_n(x) = \mathrm{e}^{-inx} P_{2n}\left(\mathrm{e}^{ix}\right)$, where

$$P_{2n}(z) = \sum_{m=0}^{2n} c'_m z^m, \quad c'_{2n} = \overline{c}_0 \neq 0,$$

is an algebraic polynomial of the degree $2n$. Since $\mathrm{e}^{-inx} \neq 0$ for all x, we have the following statement. *Every trigonometric polynomial T_n of degree n has at most $2n$ roots in $(-\pi, \pi]$.*

Example 1.15. Find the complex Fourier series of the function $f(x) = x$ $(-\pi < x < \pi)$.

Evaluate the Fourier coefficients:

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \, dx = 0;$$

if an integer $n \neq 0$, then

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\mathrm{e}^{-inx} \, dx = \frac{1}{2\pi} \left[\begin{array}{ll} u = x & du = dx \\ dv = \mathrm{e}^{-inx} \, dx & v = -\frac{1}{in}\mathrm{e}^{-inx} \end{array} \right] \Bigg|_{-\pi}^{\pi}$$

$$= \frac{-1}{2\pi in} x\mathrm{e}^{-inx} \Big|_{-\pi}^{\pi} + \frac{1}{2\pi in} \int_{-\pi}^{\pi} \mathrm{e}^{-inx} \, dx = \frac{-1}{2in} \left(\mathrm{e}^{-i\pi n} + \mathrm{e}^{i\pi n} \right) = i\frac{(-1)^n}{n}.$$

Thus,

$$x \sim i \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{(-1)^n}{n} \mathrm{e}^{inx} \quad (|x| < \pi).$$

Before we have constructed the Fourier series of this function in the real form:

$$x \sim 2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \quad (|x| < \pi). \quad \square$$

1.3. Functions of an arbitrary period

Let a function f be defined on the real line $\mathbb{R}$. Assume that f has a period $T = 2L$ $(L > 0)$. Set

$$\varphi(z) = f\left(\frac{Lz}{\pi}\right) \quad (z \in \mathbb{R}).$$

If z ranges in $[-\pi, \pi]$, then $x = \frac{Lz}{\pi}$ ranges in $[-L, L]$. The function φ has the period 2π. If $f \in \mathcal{R}[-L, L]$, then $\varphi \in \mathcal{R}[-\pi, \pi]$. Let

$$\varphi(z) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nz + b_n \sin nz\right), \tag{1.12}$$

where

$$a_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \varphi(z)\cos nz\,dz \quad (n = 0, 1, \dots),$$

$$b_n = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} \varphi(z)\sin nz\,dz \quad (n = 1, 2, \dots).$$

Now we change the variable $z = \pi x/L$. Then $\varphi(z) = f(x)$. Instead of $\cos nz$, $\sin nz$ we have functions

$$1, \cos\frac{\pi}{L}x, \sin\frac{\pi}{L}x, \dots, \cos\frac{n\pi}{L}x, \sin\frac{n\pi}{L}x, \dots \tag{1.13}$$

of the period $2L$. Functions (1.13) form the trigonometric system with the period $2L$. They are orthogonal on $[-L, L]$. Further,

$$a_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx \quad (n = 0, 1, \dots), \tag{1.14}$$

$$b_n = \frac{1}{L}\int\limits_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx \quad (n = 1, 2, \dots). \tag{1.15}$$

Numbers a_n, b_n defined by equalities (1.14) and (1.15) are called *the Fourier coefficients* of the function f. The series

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos\frac{n\pi}{L}x + b_n\sin\frac{n\pi}{L}x\Big)$$

is called *the Fourier series* of $2L$-periodic function f. It can be formally obtained from the series (1.12) by the substitution $z = \pi x/L$.

Example 1.16. The function $f(x) = |\sin x|$ is even and π-periodic, here $L = \frac{\pi}{2}$.

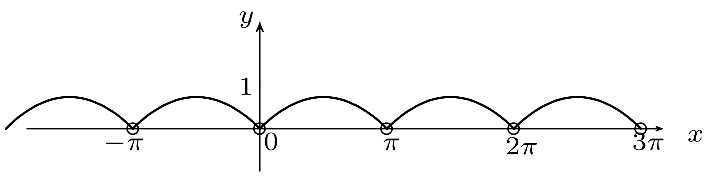

Fig. 8. The graph of the function $y = |\sin x|$.

Its sine-Fourier coefficients $b_n = 0$. We evaluate

$$a_0 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{4}{\pi}(-\cos x)\Big|_0^{\frac{\pi}{2}} = \frac{4}{\pi},$$

$$a_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \sin x\cos 2nx\,dx = -\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin(2n-1)x\,dx + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \sin(2n+1)x\,dx$$

$$= \frac{2}{\pi}\frac{1}{2n-1}\cos(2n-1)x\Big|_0^{\frac{\pi}{2}} - \frac{2}{\pi}\frac{1}{2n+1}\cos(2n+1)x\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{2}{\pi}\left(\frac{-1}{2n-1} + \frac{1}{2n+1}\right) = -\frac{4}{\pi}\frac{1}{4n^2-1} \quad (n = 1, 2, \dots).$$

Thus,

$$|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx. \quad \square$$

Example 1.17. Find the Fourier series of the 2-periodic extension of the function

$$f(x) = |x| \quad (-1 \le x \le 1).$$

Here $L = 1$. We have already constructed the Fourier series of the function $\varphi(z) = |z|$ $(-\pi \le z \le \pi)$:

$$|z| \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)z \quad (|z| \le \pi).$$

Set $z = \frac{\pi x}{L} = \pi x$ $(-1 \le x \le 1)$. We obtain

$$|x| = \frac{1}{\pi}|\pi x| = \frac{1}{\pi}\varphi(\pi x) \sim \frac{1}{\pi}\left(\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)\pi x\right).$$

Thus,

$$|x| \sim \frac{1}{2} - \frac{4}{\pi^2}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\cos(2k-1)\pi x \quad (|x| \le 1). \quad \square$$

Example 1.18. The periodic extension of the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1, \\ 1, & 1 < x < 2, \\ 3 - x, & 2 \le x < 3, \end{cases}$$

defined on the interval $[0, 3]$, is an even 3-periodic function. We find its Fourier series, taking into account that $L = \frac{3}{2}$. We have $b_n = 0$,

$$a_0 = \frac{2}{3} \int_{-\frac{3}{2}}^{\frac{3}{2}} f(x)\,dx = \frac{4}{3} \int_0^1 x\,dx + \frac{4}{3} \int_1^{\frac{3}{2}} dx = \frac{4}{3},$$

$$a_n = \frac{4}{3} \int_0^1 x \cos\frac{2\pi n x}{3}\,dx + \frac{4}{3} \int_1^{\frac{3}{2}} \cos\frac{2\pi n x}{3}\,dx = \begin{cases} 0, & n = 3k, \\ -\frac{9}{2\pi^2 n^2}, & n \ne 3k. \end{cases}$$

Thus,

$$f(x) \sim \frac{2}{3} - \frac{9}{2\pi^2} \cos\frac{2\pi x}{3}$$

$$-\frac{9}{2\pi^2} \sum_{k=1}^{\infty} \left(\frac{1}{(3k-1)^2} \cos\frac{2(3k-1)\pi x}{3} + \frac{1}{(3k+1)^2} \cos\frac{2(3k+1)\pi x}{3} \right)$$

(since the Fourier series obtained converges absolutely, we may put its terms in pairs). $\square$

Example 1.19. Let

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{\pi}{2}, \\ \pi - x, & \frac{\pi}{2} \le x \le \pi, \end{cases}$$

and let g be the even extension of f on $[-\pi, \pi]$. We construct the Fourier series of the function g on $[-\pi, \pi]$. Observe that g is π-periodic and $g(x) = \frac{1}{2} h(2x)$, where h is a 2π-periodic extension of the function $|x|$ $(-\pi \le x \le \pi)$. We have already found the Fourier series of this function,

$$h(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)x.$$

Thus,

$$g(x) = \frac{1}{2} h(2x) \sim \frac{\pi}{4} - \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos 2(2k-1)x. \quad \square$$

1.4. Orthogonal systems

Definition 1.20. A sequence $\Phi = \{\varphi_n\}_{n=0}^{\infty}$ of real-valued functions integrable on an interval $[a,b]$ is called *an orthogonal system* on $[a,b]$ if

$$\int_a^b \varphi_m(x)\varphi_n(x)\,dx = 0 \quad (m \neq n), \qquad \int_a^b \varphi_n^2(x) > 0 \quad (n = 0,1,\dots).$$

If $\int_a^b \varphi_n^2(x)\,dx = 1$ $(n = 0,1,\dots)$, then Φ is called *an orthonormal system* on $[a,b]$.

For any $f \in \mathcal{R}[a,b]$, the number

$$\|f\|_2 = \sqrt{\int_a^b f^2(x)\,dx}$$

is called *the quadratic norm* (or L^2*-norm*) of f on $[a,b]$.

The trigonometric system

$$1, \cos x, \sin x, \dots, \cos nx, \sin nx, \dots$$

is orthogonal on $[-\pi,\pi]$. The norms of these functions on $[-\pi,\pi]$ are

$$\|1\|_2 = \sqrt{2\pi},\ \|\cos nx\|_2 = \|\sin nx\|_2 = \sqrt{\pi}.$$

The corresponding orthonormal system is

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \dots. \tag{1.16}$$

There are many different orthogonal systems used in mathematics. For example, *the Rademacher system*

$$r_n(x) = \operatorname{sign}\sin 2^n \pi x \quad (n = 0,1,\dots)$$

is an orthonormal system on $[0,1]$. The systems $\{\sin nx\}_{n=1}^{\infty}$, $\{\cos nx\}_{n=0}^{\infty}$ are orthogonal systems on $[0,\pi]$.

Let $\Phi = \{\varphi_n\}_{n=0}^{\infty}$ be an orthogonal system on $[a, b]$. *The Fourier coefficients* of a function $f \in \mathcal{R}[a, b]$ are defined by the equalities

$$c_n = \frac{1}{\|\varphi_n\|_2^2} \int\limits_a^b f(x)\varphi_n(x)\,dx \quad (n = 0, 1, \dots). \tag{1.17}$$

We write

$$f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x). \tag{1.18}$$

The series at the right-hand side of (1.18) is called *the Fourier series* of f.

If $\{\varphi_n\}$ is an orthonormal system, then

$$c_n = \int\limits_a^b f(x)\varphi_n(x)\,dx \quad (n = 0, 1, \dots).$$

If $f \in \mathcal{R}[-\pi, \pi]$, then the Fourier coefficients of f with respect to the normalized trigonometric system (1.16) are

$$\widetilde{a}_0 = \frac{1}{\sqrt{2\pi}} \int\limits_{-\pi}^{\pi} f(x)\,dx,$$

$$\widetilde{a}_n = \frac{1}{\sqrt{\pi}} \int\limits_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \widetilde{b}_n = \frac{1}{\sqrt{\pi}} \int\limits_{-\pi}^{\pi} f(x)\sin nx\,dx \quad (n = 1, 2, \dots).$$

They are different from the usual trigonometric coefficients. However, the series with these coefficients in the system (1.16) is the usual Fourier series.

Let $\Phi = \{\varphi_n\}$ be an orthogonal system. We shall consider finite linear combinations (*polynomials* with respect to the system Φ):

$$P_n(x) = \sum_{k=0}^{n} \alpha_k \varphi_k(x) \quad (\alpha_k \in \mathbb{R}).$$

Lemma 1.21. We have the equality

$$\|P_n\|_2^2 = \sum_{k=0}^{n} \alpha_k^2 \|\varphi_k\|_2^2 . \tag{1.19}$$

PROOF. Indeed,

$$P_n^2(x) = \sum_{k=0}^{n} \alpha_k \varphi_k(x) \sum_{j=0}^{n} \alpha_j \varphi_j(x) = \sum_{k=0}^{n} \sum_{j=0}^{n} \alpha_k \alpha_j \varphi_k(x) \varphi_j(x).$$

Thus, by orthogonality

$$\int_a^b P_n^2(x)\, dx = \sum_{k=0}^{n} \alpha_k^2 \int_a^b \varphi_k^2(x)\, dx,$$

which is (1.19). $\square$

Remark 1.22. Equality (1.19) is a version of the Pythagorean identity

$$\|u+v\|_2^2 = \|u\|_2^2 + \|v\|_2^2 \quad (u \perp v).$$

It is easy to show that if $f \perp g$, then

$$\int_a^b [f+g]^2\, dx = \int_a^b f^2(x)\, dx + \int_a^b g^2(x)\, dx.$$

It follows immediately from Lemma 1.21

Corollary 1.23. For any trigonometric polynomial

$$T_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

we have that

$$\int_{-\pi}^{\pi} T_n^2(x)\, dx = \pi \left(\frac{a_0^2}{2} + \sum_{k=1}^{n} \left(a_k^2 + b_k^2\right) \right).$$

In the case of an orthonormal system equality (1.19) becomes

$$\int_a^b P_n^2(x)\, dx = \sum_{k=0}^{n} \alpha_k^2. \tag{1.20}$$

The quadratic (euclidean) distance between functions $f, g \in \mathcal{R}[a,b]$ is defined by

$$\|f-g\|_2 = \sqrt{\int_a^b [f(x)-g(x)]^2\, dx}$$

(it is the infinite-dimensional counterpart of the euclidean distance).

Let $\Phi = \{\varphi_n\}_{n=0}^{\infty}$ be an orthogonal system on $[a,b]$. We consider the approximation of a function f by the polynomials

$$P_n(x) = \sum_{k=1}^{n} \alpha_k \varphi_k(x)$$

in the quadratic distance.

The next theorem contains the so called *minimal property* of the partial sums of a Fourier series.

Theorem 1.24 (Gram). Let $f \in \mathcal{R}[a,b]$. Between all polynomials P_n of the degree not higher than n the best quadratic approximation $\|f - P_n\|_2$ is obtained if P_n is the nth partial sum

$$S_n(x) = \sum_{k=1}^{n} c_k \varphi_k(x)$$

of the Fourier series of f.

PROOF. We may assume that Φ is an orthonormal system. Applying (1.20), we obtain

$$\int_a^b \left[f(x) - \sum_{k=1}^{n} \alpha_k \varphi_k(x) \right]^2 dx$$

$$= \int_a^b f^2(x)\,dx - 2\sum_{k=1}^{n} \alpha_k \int_a^b f(x)\varphi_k(x)\,dx + \sum_{k=1}^{n} \alpha_k^2$$

$$= \int_a^b f^2(x)\,dx - 2\sum_{k=1}^{n} \alpha_k c_k + \sum_{k=1}^{n} \alpha_k^2$$

$$= \int_a^b f^2(x)\,dx - \sum_{k=1}^{n} c_k^2 + \sum_{k=1}^{n} (\alpha_k - c_k)^2,$$

where c_k are the Fourier coefficients. Clearly, the minimum of the right-hand side is attained for $\alpha_k = c_k$. In this case we have

$$\int_a^b [f(x) - S_n(x)]^2\,dx = \int_a^b f^2(x)\,dx - \sum_{k=1}^{n} c_k^2. \quad \square \tag{1.21}$$

Equality (1.21) is called *the Bessel identity.* Since the left-hand side of this equality is non-negative, it implies the following result.

Theorem 1.25. Given an orthogonal system $\Phi = \{\varphi_n\}$, for any function $f \in \mathcal{R}[a, b]$ it holds that

$$\sum_{n=0}^{\infty} c_n^2 \leq \int\limits_a^b f^2(x)\, dx. \tag{1.22}$$

Inequality (1.22) is called *the Bessel inequality.*

Remark 1.26. Equality (1.21) can be written as

$$\int\limits_a^b [f(x) - S_n(x)]^2\ dx = \int\limits_a^b f^2(x)\, dx - \int\limits_a^b S_n^2(x)\, dx.$$

It is *the Pythagorean identity.* It means that $f - S_n \perp S_n$.

For the trigonometric system the Bessel inequality (1.22) has the form

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} \left(a_n^2 + b_n^2\right) \leq \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f^2(x)\, dx.$$

It follows that the series at the left-hand side converges. Then, by virtue of the necessary condition of the convergence of a numerical series, its terms tend to zero. Thus, we obtain the following statement.

Lemma 1.27 (Riemann – Lebesgue). For any function $f \in \mathcal{R}[-\pi, \pi]$ its Fourier coefficients a_n, b_n tend to zero as $n \to \infty$.

1.5. Integral representation of partial sums

Let f be a function with period 2π, integrable in $[-\pi, \pi]$. We consider its Fourier series and we denote by $S_n(x)$ its partial sum

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} \left(a_k \cos kx + b_k \sin kx\right). \tag{1.23}$$

The Fourier coefficients a_k and b_k are defined by equalities

$$a_k = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t) \cos kt \, dt \quad (k = 0, 1, \dots),$$

$$b_k = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t) \sin kt \, dt \quad (k = 1, 2, \dots).$$

Putting these expressions in (1.23), we obtain

$$S_n(x) = \frac{1}{2\pi} \int\limits_{-\pi}^{\pi} f(t) \, dt$$

$$+ \sum_{k=1}^{n} \left[\frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t) \cos kt \, dt \cos kx + \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t) \sin kt \, dt \sin kx \right]$$

$$= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^{n} (\cos kt \cos kx + \sin kt \sin kx) \right] dt$$

$$= \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^{n} \cos k(t - x) \right] dt.$$

Denote

$$D_n(u) = \frac{1}{2} + \sum_{k=1}^{n} \cos ku. \tag{1.24}$$

The function $D_n(u)$ is 2π-periodic (as a sum of 2π-periodic functions).

We have

$$S_n(x) = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(t) D_n(t - x) \, dt.$$

Substituting the variable $t = x + u$ (x is fixed), we get

$$S_n(x) = \frac{1}{\pi} \int\limits_{-\pi-x}^{\pi-x} f(x + u) D_n(u) \, du.$$

Since the integrand is a function of period 2π, we have

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+u)D_n(u)\,du. \tag{1.25}$$

This is *the integral representation* of the partial sum (*Dirichlet's formula*).

The function $D_n(u)$ is called *Dirichlet's kernel.* It is an *even trigonometric polynomial* of degree n. Integrating (1.24) over $[-\pi,\pi]$, we obtain that

$$\frac{1}{\pi}\int_{-\pi}^{\pi} D_n(u)\,du = 1.$$

We will derive a concise form of Dirichlet's kernel. For this, we multiply both sides of (1.24) by $2\sin\frac{u}{2}$ and we obtain

$$2\sin\frac{u}{2}D_n(u) = \sin\frac{u}{2} + 2\sum_{k=1}^{n}\sin\frac{u}{2}\cos ku.$$

Using the equality

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta),$$

we have

$$2\sin\frac{u}{2}\cos ku = \sin\left(k+\frac{1}{2}\right)u - \sin\left(k-\frac{1}{2}\right)u.$$

This yields

$$2\sin\frac{u}{2}D_n(u) = \sin\left(n+\frac{1}{2}\right)u.$$

Thus, for $0 < |u| \le \pi$

$$D_n(u) = \frac{\sin\left(n+\frac{1}{2}\right)u}{2\sin\frac{u}{2}}. \tag{1.26}$$

We observe that the points $u = 2k\pi$, where the denominator vanishes, are removable points of discontinuity.

1.6. Pointwise convergence of Fourier series

Lemma 1.28. Let $g \in \mathcal{R}[0,\pi]$. Then

$$\lim_{n\to\infty}\int_{0}^{\pi} g(t)\sin\left(n+\frac{1}{2}\right)t\,dt = 0.$$

Proof. We define two functions

$$h_1(t) = \begin{cases} g(t)\cos\frac{t}{2}, & 0 \le t \le \pi, \\ 0, & -\pi \le t < 0 \end{cases} \quad \text{and} \quad h_2(t) = \begin{cases} g(t)\sin\frac{t}{2}, & 0 \le t \le \pi, \\ 0, & -\pi \le t < 0. \end{cases}$$

Since $h_1, h_2 \in \mathcal{R}[-\pi,\pi]$, then

$$\int_0^\pi g(t)\sin\left(n+\frac{1}{2}\right)t\,dt$$

$$= \int_0^\pi g(t)\cos\frac{t}{2}\sin nt\,dt + \int_0^\pi g(t)\sin\frac{t}{2}\cos nt\,dt$$

$$= \int_{-\pi}^\pi h_1(t)\sin nt\,dt + \int_{-\pi}^\pi h_2(t)\cos nt\,dt.$$

By the Riemann – Lebesgue Lemma (Lemma 1.27), each of the last two integrals tends to zero as $n \to \infty$. $\square$

Theorem 1.29 (Dirichlet). Let $f \in \mathcal{PS}$ be a 2π-periodic piecewise smooth function. Then for any $x \in \mathbb{R}$ the Fourier series of f converges to the value

$$S(x) = \frac{f(x-)+f(x+)}{2}.$$

Proof. We have

$$S_n(x) = \frac{1}{\pi}\int_{-\pi}^\pi f(x+t)D_n(t)\,dt$$

$$= \frac{1}{\pi}\left[\int_0^\pi f(x+t)D_n(t)\,dt + \int_{-\pi}^0 f(x+t)D_n(t)\,dt\right].$$

Changing the variable $t = -u$ in the last integral and taking into account that the Dirichlet kernel $D_n(t)$ is even, we obtain

$$S_n(x) = \frac{1}{\pi}\int_0^\pi [f(x+t)+f(x-t)]\,D_n(t)\,dt.$$

Further, since D_n is even and $\frac{1}{\pi}\int\limits_{-\pi}^{\pi} D_n(t)\,dt = 1$, we have

$$\frac{1}{\pi}\int\limits_0^{\pi} D_n(t)\,dt = \frac{1}{2}.$$

Thus,

$$\begin{aligned} S_n(x)-S(x) &= \frac{1}{\pi}\int\limits_0^{\pi}[f(x+t)-f(x+)]D_n(t)\,dt+\frac{1}{\pi}\int\limits_0^{\pi}[f(x-t)-f(x-)]D_n(t)\,dt \\ &= \frac{1}{\pi}\left(I_n^{(1)}(x)+I_n^{(2)}(x)\right). \end{aligned}$$

We will show that $I_n^{(1)}(x), I_n^{(2)}(x) \to 0$ as $n \to \infty$.

We set (x is fixed)

$$g(t) = \frac{f(x+t)-f(x+)}{2\sin\frac{t}{2}} \quad (t \in (0,\pi]).$$

The function g is piecewise continuous in $(0,\pi]$. Moreover, since the limit

$$\lambda = \lim_{t\to 0+} f'(x+t)$$

exists and is finite, applying L'Hospital's rule, we obtain that

$$\lim_{t\to 0+} g(t) = \lim_{t\to 0+}\frac{f'(x+t)}{\cos\frac{t}{2}} = \lambda.$$

Thus, g is piecewise continuous in $[0,\pi]$ and therefore g is integrable in $[0,\pi]$. By Lemma 1.28,

$$I_n^{(1)}(x) = \int\limits_0^{\pi} g(t)\sin\left(n+\frac{1}{2}\right)t\,dt \to 0 \quad (n\to\infty).$$

Similarly, $I_n^{(2)}(x) \to 0$ $(n\to\infty)$. Thus,

$$S_n(x) \to S(x) \quad (n\to\infty). \qquad \square$$

Corollary 1.30. Assume that f is a 2π-periodic piecewise continuous function. If f is differentiable at a point x_0, then the Fourier series of f converges to $f(x_0)$ at the point x_0.

Indeed, in the proof of Theorem 1.29 we used only the fact that $g(t)$ has a finite limit as $t \to 0+$ (and, similarly,

$$\frac{f(x_0 - t) - f(x_0-)}{2\sin\frac{t}{2}}$$

has a finite limit as $t \to 0+$).

Example 1.31. The function $f(x) = |\sin x|$ is piecewise smooth and continuous. Therefore it satisfies the conditions of the Dirichlet Theorem (Theorem 1.29). By this theorem, its Fourier series converges to the function at every point. We have constructed this series before (cf. Example 1.16). Thus, we have the equality

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx \quad (x \in \mathbb{R}).$$

Taking $x = 0$, we obtain

$$\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}. \quad \square$$

Example 1.32. The function $f(x) = \operatorname{sign}\sin x$ is piecewise smooth and therefore it satisfies the conditions of the Dirichlet Theorem (Theorem 1.29). It follows from this theorem that its Fourier series converges to the function at every point. This Fourier series was constructed before (Example 1.10), and thus we have the following equality

$$\operatorname{sign}\sin x = \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin(2k-1)x \quad (x \in \mathbb{R}).$$

In particular, for $x = \frac{\pi}{2}$ we have

$$\frac{\pi}{4} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2k-1}. \quad \square$$

Example 1.33. Let $f(x) = x^2$ $(x \in [-\pi, \pi])$. Then

$$x^2 \sim \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx.$$

The 2π-periodic extension of this function has finite one-sided derivatives at the point $x = \pi$. Therefore $f \in \mathcal{PS}$ and f is continuous on $\mathbb{R}$. By Dirichlet's Theorem (Theorem 1.29), the Fourier series of f converges to $f(x)$ at every point $x \in \mathbb{R}$. Thus, we obtain

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx \quad (|x| \leq \pi).$$

In particular, for $x = \pi$ we have

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos n\pi = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{1}{n^2}.$$

This implies the following useful equality

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Putting $x = 0$, we obtain one equality more

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}. \quad \square$$

Example 1.34. We have constructed above the Fourier series for the 2π-periodic extension of the function $f(x) = |x|$ $(-\pi \leq x \leq \pi)$ (Example 1.9). As in the preceding example, it is easy to see that this series converges to $f(x)$ at every point $x \in \mathbb{R}$. Thus, we have the equality

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \cos(2k-1)x \quad (|x| \leq \pi).$$

Taking $x = 0$, we obtain that

$$\frac{\pi^2}{8} = \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}. \quad \square$$

We have already observed that two 2π-periodic functions $f, g \in \mathcal{R}[-\pi, \pi]$, which differ at a finite number of points of the interval $[-\pi, \pi]$, have the same Fourier coefficients. The following theorem enable us to obtain the converse statement for piecewise smooth functions.

Theorem 1.35. Assume that all corresponding Fourier coefficients of 2π-periodic functions $f, g \in \mathcal{PS}$ coincide. Then $f(x) = g(x)$ everywhere on $[-\pi, \pi]$, with a possible exception of a finite number of points in this interval.

Proof. Indeed, on the interval $[-\pi, \pi]$ there exists at most finite number of points in which at least one of the functions f or g doesn't have a derivative. By Corollary 1.30, at every point $x_0 \in [-\pi, \pi]$, at which both the functions f and g are differentiable, their Fourier series converge to $f(x_0)$ and $g(x_0)$, correspondingly. Since the Fourier series of the functions f and g coincide, then $f(x_0) = g(x_0)$. □

1.7. Uniform convergence of Fourier series

1.7.1. Differentiation of Fourier series

In what follows we admit that a function may not be defined at a finite number of points of $[-\pi, \pi]$.

Let a 2π-periodic function $f \in \mathcal{PS}$. Then $f' \in \mathcal{PC}$ and thus $f' \in \mathcal{R}[-\pi, \pi]$. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right), \tag{1.27}$$

$$f'(x) \sim \frac{a'_0}{2} + \sum_{n=1}^{\infty} \left(a'_n \cos nx + b'_n \sin nx\right). \tag{1.28}$$

We consider the following question. *Can the series* (1.28) *be obtained by term-by-term differentiation of the series* (1.27)*?* In other words, *is it true that* $a'_0 = 0$, $a'_n = nb_n$, $b'_n = -na_n$*?*

Generally, the answer is negative. Indeed, before we have constructed the Fourier series of the function $\operatorname{sign}\sin x$:

$$\operatorname{sign}\sin x \sim \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k-1}\sin(2k-1)x.$$

Differentiating term by term, we obtain the series

$$\frac{4}{\pi}\sum_{k=1}^{\infty} \cos(2k-1)x.$$

This series is not the Fourier series of $f'(x) = 0$; actually, it is not a Fourier series of any function, since its coefficients do not tend to zero.

Such example is possible only because the function has points of discontinuity. Indeed, the following theorem holds.

Theorem 1.36. Assume that 2π-periodic function $f \in \mathcal{PS}$ is continuous on $\mathbb{R}$. If

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is the Fourier series of f, then the Fourier series of f' is given by

$$f'(x) \sim \sum_{n=1}^{\infty} (-na_n \sin nx + nb_n \cos nx).$$

PROOF. Since $f \in \mathcal{C}$, then $f(-\pi) = f(\pi)$. Therefore, by virtue of the Fundamental Theorem of Calculus (Theorem 10),

$$a_0' = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x)\, dx = \frac{1}{\pi}[f(\pi) - f(-\pi)] = 0.$$

Further, integrating by parts, we have

$$a_n' = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[f(x) \cos nx|_{-\pi}^{\pi} + n \int_{-\pi}^{\pi} f(x) \sin nx\, dx \right] = nb_n.$$

Similarly,

$$b_n' = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(x) \sin nx\, dx$$

$$= \frac{1}{\pi} \left[f(x) \sin nx|_{-\pi}^{\pi} - n \int_{-\pi}^{\pi} f(x) \cos nx\, dx \right] = -na_n. \quad \square$$

Remark 1.37. As it was shown by the example considered before Theorem 1.36, the condition of continuity in this theorem cannot be omitted.

1.7.2. Uniform convergence

Theorem 1.38. Let f be a 2π-periodic continuous and piecewise smooth function. Then the Fourier series of f converges uniformly and absolutely to the function f on the real line $\mathbb{R}$.

PROOF. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) .$$

First we shall show that

$$\sum_{n=1}^{\infty} (|a_n| + |b_n|) < \infty. \tag{1.29}$$

By the Theorem on differentiation of Fourier series (Theorem 1.36),

$$f'(x) \sim \sum_{n=1}^{\infty} (nb_n \cos nx - na_n \sin nx) .$$

Thus,

$$a_n = -\frac{1}{n} b'_n, \quad b_n = \frac{1}{n} a'_n,$$

where a'_n, b'_n are the Fourier coefficients of the derivative f'. By the Cauchy inequality (Theorem 21), this implies that for any natural N

$$\sum_{n=1}^{N} |a_n| = \sum_{n=1}^{N} \frac{|b'_n|}{n} \le \left(\sum_{n=1}^{N} \frac{1}{n^2} \right)^{1/2} \left(\sum_{n=1}^{N} (b'_n)^2 \right)^{1/2} .$$

Further, since $f' \in \mathcal{R}[-\pi, \pi]$, then, by Bessel's inequality (1.22),

$$\left(\sum_{n=1}^{N} (b'_n)^2 \right)^{1/2} \le \left(\frac{1}{\pi} \int_{-\pi}^{\pi} (f'(x))^2 \, dx \right)^{1/2} .$$

Also, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges. Denoting its sum by C_0, we obtain that for any N

$$\sum_{n=1}^{N} |a_n| \le C_1 \left(\int_{-\pi}^{\pi} (f'(x))^2 \, dx \right)^{1/2} ,$$

where $C_1 = \sqrt{C_0/\pi}$. Letting N tend to infinity, we get

$$\sum_{n=1}^{\infty} |a_n| \le C_1 \left\| f' \right\|_2 .$$

Similarly,

$$\sum_{n=1}^{\infty} |b_n| \le C_1 \left\| f' \right\|_2 .$$

Thus, we obtained (1.29). Further,

$$|a_n \cos nx + b_n \sin nx| \le |a_n| + |b_n|$$

for any $x \in \mathbb{R}$. Applying this inequality, (1.29), and the Weierstrass $M-$test of the uniform convergence, we obtain that the Fourier series of f converges uniformly and absolutely on $\mathbb{R}$. By Dirichlet's Theorem (Theorem 1.29), for any x this series converges to $f(x)$. $\square$

Remark 1.39. If f is piecewise smooth, but has jumps, then the Fourier series of f doesn't converge uniformly (otherwise f would be continuous). However, the following theorem holds.

Theorem 1.40. Let 2π-periodic function $f \in \mathcal{PS}$. Then the Fourier series of f converges to f uniformly on any closed interval which does not contain any point of discontinuity of f.

However, if f has jumps, its Fourier series cannot converge absolutely in some interval. This fact is well known in the Theory of trigonometric series.

1.7.3. Term-by-term integration of Fourier series

Remind that a piecewise continuous function may not be continuous.

Theorem 1.41. Let f be a 2π- periodic piecewise continuous function on $\mathbb{R}$ and let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

Then for any $x \in [-\pi, \pi]$

$$\int\limits_0^x f(t)\, dt = \frac{a_0}{2} x + C + \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin nx - \frac{b_n}{n} \cos nx \right),$$

where C is a constant and the series at the right hand side converges uniformly on $\mathbb{R}$.

PROOF. Set $\varphi(x) = f(x) - \frac{a_0}{2}$. Then

$$\int_{-\pi}^{\pi} \varphi(t)\,dt = 0.$$

Further, set

$$\Phi(x) = \int_0^x \varphi(t)\,dt \quad (x \in [-\pi,\pi]).$$

Then Φ is continuous and piecewise smooth on $[-\pi,\pi]$. Besides,

$$\Phi(\pi) - \Phi(-\pi) = \int_{-\pi}^{\pi} \varphi(t)\,dt = 0,$$

and therefore Φ can be extended to $\mathbb{R}$ as a 2π-periodic continuous piecewise smooth function. Let

$$\Phi(x) \sim \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(A_n \cos nx + B_n \sin nx\right).$$

By the theorem on term-by-term differentiation of Fourier series (Theorem 1.36),

$$\Phi'(x) \sim \sum_{n=1}^{\infty} \left(-nA_n \sin nx + nB_n \cos nx\right).$$

But $\Phi'(x) = \varphi(x)$ everywhere in $[-\pi,\pi]$ except a finite number of points. Moreover,

$$\varphi(x) \sim \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Thus, $A_n = -\frac{b_n}{n}$, $B_n = \frac{a_n}{n}$. We have also that $\Phi(0) = 0$ and the Fourier series of Φ converges uniformly on $\mathbb{R}$ to Φ. Thus,

$$\Phi(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} \left(\frac{a_n}{n} \sin nx - \frac{b_n}{n} \cos nx\right)$$

for any x. Taking $x = 0$, we get

$$\frac{A_0}{2} = \sum_{n=1}^{\infty} \frac{b_n}{n}. \quad \square$$

Corollary 1.42. If 2π-periodic function f is continuous on $\mathbb{R}$ and $a_0 = 0$, $a_n = b_n = 0$, then $f(x) \equiv 0$.

Indeed, $\int\limits_0^x f(t)\,dt = 0$; differentiating, we get that $f(x) = 0$ for each x.

Corollary 1.42 represents the so called *uniqueness property.* It can be also formulated as follows.

If two continuous 2π-periodic functions have the same Fourier coefficients, then these functions coincide everywhere.

1.8. Complete orthogonal systems

Let f_n be a sequence of continuous functions defined on an interval $[a, b]$. We say that this sequence *mean square converges* on $[a, b]$ to a function $f \in \mathcal{R}[a, b]$ if $\|f_n - f\|_2 \to 0$ as $n \to \infty$. We say that a series of functions $\sum\limits_{n=0}^{\infty} u_n(x)$ *mean square converges* on $[a, b]$ to a function $f \in \mathcal{R}[a, b]$ if the sequence of its partial sums $f_n(x) = \sum\limits_{k=0}^{n} u_k(x)$ mean square converges to f on $[a, b]$.

Definition 1.43. An orthogonal system $\{\varphi_n\}_{n=0}^{\infty}$ is said to be *complete* if for any continuous function f its Fourier series $\sum\limits_{n=0}^{\infty} c_n\varphi_n(x)$ mean square converges to f.

Theorem 1.44. An orthonormal system is complete if and only if for any continuous function f

$$\sum_{n=0}^{\infty} c_n^2 = \int_a^b f^2(x)\,dx. \tag{1.30}$$

Proof. Let $\{\varphi_n\}_{n=0}^{\infty}$ be an orthonormal system, let c_n be the Fourier coefficients of a continuous function f, and let S_n be the partial sums of the Fourier series of f with respect to this system. Rewrite the Bessel identity (1.21) in the following form

$$\|f - S_n\|_2^2 = \|f\|_2^2 - \sum_{k=0}^{n} c_k^2.$$

It follows that the condition $\|f - S_n\|_2 \to 0$ is equivalent to (1.30). $\square$

Equality (1.30) is called *Parseval's equality.* This is an infinite-dimensional counterpart of Pythagorean Theorem.

Theorem 1.45. Let $\Phi = \{\varphi_n\}_{n=0}^{\infty}$ be a complete orthogonal system. If a continuous function f is orthogonal to each φ_n $(n = 0, 1, \dots)$, then $f(x) = 0$ for all $x \in [a, b]$.

Proof. We may assume that $\|\varphi_n\|_2 = 1$ for all n. The Fourier coefficients of f are equal to zero. Thus, by Parseval's equality (1.30),

$$\int_a^b f^2(x)\,dx = 0.$$

Since f is continuous, this implies that $f(x) = 0$ for all $x \in [a, b]$. $\square$

Remark 1.46. The inverse statement also is true: if there is no nonzero continuous function orthogonal to all the functions φ_n, then $\{\varphi_n\}$ is complete.

The proof is out of the scope of this course.

Theorem 1.47 (uniqueness). Let $\Phi = \{\varphi_n\}$ be a complete orthogonal system. If two continuous functions have the same Fourier coefficients with respect to Φ, then these functions are identical on $[a, b]$.

Indeed, the difference of these functions is orthogonal to each φ_n, and therefore this difference is identically equal to zero.

1.9. Cesaro summation of Fourier series

Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function. Let

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1.31}$$

be its Fourier series and let $S_n(x)$ be the partial sums of the series (1.31). By Dirichlet's formula (1.25),

$$S_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) D_n(t)\,dt, \tag{1.32}$$

where
$$D_n(t) = \frac{1}{2} + \cos t + \cdots + \cos nt, \quad D_0(t) = \frac{1}{2},$$
is the Dirichlet kernel. Remind that (see (1.26))
$$D_n(t) = \frac{\sin\left(n + \frac{1}{2}\right) t}{2 \sin \frac{t}{2}}. \tag{1.33}$$

We consider the arithmetic means
$$\sigma_n(x) = \frac{S_0(x) + \cdots + S_n(x)}{n+1}$$
(they are called *Fejér means* of the order n of the function f). We shall derive an integral representation for $\sigma_n(x)$. By (1.32), we have that
$$\sigma_n(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) F_n(t)\, dt,$$
where
$$F_n(t) = \frac{D_0(t) + \cdots + D_n(t)}{n+1}.$$
The function F_n is a trigonometric polynomial of the degree n. It is called *the Fejér kernel* of the order n. We shall derive a concise formula for it. By (1.33), we have
$$F_n(t) = \frac{1}{2(n+1)\sin\frac{t}{2}} \left[\sin \frac{t}{2} + \cdots + \sin\left(n + \frac{1}{2}\right) t \right].$$
Denote
$$\Phi_n(t) = \sum_{k=0}^{n} \sin\left(k + \frac{1}{2}\right) t.$$
Multiplying by $2 \sin \frac{t}{2}$ and using the equality
$$2 \sin\left(k + \frac{1}{2}\right) t \sin \frac{t}{2} = \cos kt - \cos(k+1)t,$$
we obtain
$$2 \sin \frac{t}{2} \Phi_n(t) = \sum_{k=0}^{n} [\cos kt - \cos(k+1)t]$$
$$= 1 - \cos(n+1)t = 2 \sin^2 (n+1)\frac{t}{2}.$$

Thus,

$$F_n(t) = \frac{1}{2(n+1)} \left(\frac{\sin(n+1)\frac{t}{2}}{\sin\frac{t}{2}} \right)^2 .$$

It is easy to see that F_n has the following properties.

(1) F_n is even and non-negative.

(2) $\frac{1}{\pi} \int\limits_{-\pi}^{\pi} F_n(t)\,dt = 1.$

(3) $F_n(t) \le \frac{n+1}{2}.$

(4) $F_n(t) \le \frac{1}{2(n+1)\sin^2\frac{\delta}{2}}$ $\quad (0 < \delta \le t \le \pi)$ and thus

$$\mu_n(\delta) = \max_{\delta \le t \le \pi} F_n(t) \to 0 \quad (n \to \infty) \quad \text{for any } \delta > 0.$$

Theorem 1.48 (Fejér). Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function. Assume that at a point x the function f has one-sided limits $f(x+)$ и $f(x-)$. Then

$$\sigma_n(x) \to \frac{f(x+) + f(x-)}{2} \quad (n \to \infty).$$

Proof. Since F_n is even, we have

$$\sigma_n(x) = \frac{1}{\pi} \int\limits_{-\pi}^{\pi} f(x+t) F_n(t)\,dt = \frac{1}{\pi} \int\limits_{0}^{\pi} [f(x+t) + f(x-t)] F_n(t)\,dt.$$

We may assume that

$$f(x) = \frac{f(x+) + f(x-)}{2}.$$

Since

$$\frac{2}{\pi} \int\limits_{0}^{\pi} F_n(t)\,dt = 1,$$

we obtain that

$$\sigma_n(x) - f(x) = \frac{1}{\pi} \int\limits_{0}^{\pi} \varphi_x(t) F_n(t)\,dt,$$

where

$$\varphi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

We have

$$\varphi_x(t) \to 0 \quad (t \to 0+).$$

Let $\varepsilon > 0$. There exists a number $\delta = \delta(\varepsilon) > 0$ such that

$$|\varphi_x(t)| < \varepsilon \quad (0 \le t \le \delta).$$

Taking into account that $F_n \ge 0$, we have

$$|\sigma_n(x) - f(x)| \le \frac{1}{\pi}\left[\int_0^\delta |\varphi_x(t)|\, F_n(t)\, dt + \int_\delta^\pi |\varphi_x(t)|\, F_n(t)\, dt\right]$$

$$\le \frac{\varepsilon}{\pi}\int_0^\pi F_n(t)\, dt + \frac{\mu_n(\delta)}{\pi}\int_0^\pi |\varphi_x(t)|\, dt,$$

where $\mu_n(\delta) = \max\limits_{\delta \le t \le \pi} F_n(t)$.

We observe that

$$\frac{1}{\pi}\int_0^\pi F_n(t)\, dt = \frac{1}{2}$$

and

$$\frac{1}{\pi}\int_0^\pi |\varphi_x(t)|\, dt \le \frac{1}{\pi}\int_0^\pi (|f(x+t)| + |f(x-t)|)\, dt + 2|f(x)|$$

$$= \frac{1}{\pi}\int_{-\pi}^\pi |f(u)|\, du + 2|f(x)| = M_x.$$

Thus,

$$|\sigma_n(x) - f(x)| \le \frac{\varepsilon}{2} + M_x\mu_n(\delta).$$

Since $\mu_n(\delta) \to 0$ $(n \to \infty)$, there exists $N = N(\varepsilon, x)$ such that

$$M_x\mu_n(\delta) < \frac{\varepsilon}{2} \quad (n \ge N).$$

It follows that

$$|\sigma_n(x) - f(x)| < \varepsilon \quad (n \ge N). \quad \square$$

Corollary 1.49. Let f be a 2π-periodic piecewise continuous function. Then at every point $x \in \mathbb{R}$ the Fourier series of f is summable by the method of arithmetic means to the sum

$$\frac{f(x+) + f(x-)}{2}.$$

Corollary 1.50. Let $f \in \mathcal{R}[-\pi, \pi]$ be a 2π-periodic function. Assume that at a point x the function f has one-sides limits $f(x\pm)$. If the Fourier series of f converges at a point x, then its sum is equal to

$$\frac{f(x+) + f(x-)}{2}.$$

In particular, if the Fourier series converges at a point of continuity of f, then its sum is $f(x)$.

Theorem 1.51 (Fejér). Let f be a 2π-periodic continuous function on $\mathbb{R}$. Then the arithmetic means $\sigma_n(x)$ of the Fourier series of f converge to f uniformly on $\mathbb{R}$.

This theorem is proved by the same scheme as Theorem 1.48. We have to observe only that, by virtue of the uniform continuity of the function f, we can choose a number $\delta(\varepsilon)$ independent of x.

One of important applications of Theorem 1.51 is the classical *Weierstrass Theorem on approximation by trigonometric polynomials.*

Theorem 1.52 (Weierstrass). Let f be a 2π-periodic continuous function on $\mathbb{R}$. Then for any $\varepsilon > 0$ there exists a trigonometric polynomial T such that

$$|f(x) - T(x)| < \varepsilon \quad \text{for all} \quad x \in \mathbb{R}.$$

Indeed, by Theorem 1.51, the arithmetic means σ_n converge uniformly to the function f. Therefore for a sufficiently big n we can take σ_n as T.

Theorem 1.51 yields also the property of *completeness of trigonometric system.*

Theorem 1.53. Let f be a 2π-periodic continuous function on $\mathbb{R}$. Then the Fourier series of f mean square converges to f, that is

$$\int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 \, dx \to 0 \quad (n \to \infty).$$

Proof. Let $\varepsilon > 0$. By Theorem 1.51, there exists a number N such that

$$|f(x) - \sigma_n(x)| < \sqrt{\frac{\varepsilon}{2\pi}}$$

for all $n \geq N$ and all $x \in \mathbb{R}$. Then, by virtue of the minimal property of the partial sums S_n of a Fourier series (see Theorem 1.24),

$$\int_{-\pi}^{\pi} [f(x) - S_n(x)]^2 \, dx \leq \int_{-\pi}^{\pi} [f(x) - \sigma_n(x)]^2 \, dx < \varepsilon \quad (n \geq N). \qquad \square$$

Remark 1.54. Theorem 1.53 remains true if f is merely integrable in $[-\pi, \pi]$. For the proof it is sufficient to apply Theorem 13 on approximation of an integrable function by continuous functions.

Another formulation of Theorem 1.53 is:

The trigonometric system is complete.

Exercises to Chapter 1

1.1. Construct the graphs of the following periodic functions.

1.1.a. $$f(x) = x \quad (-1 < x < 1), \quad f(x+2) = f(x).$$

1.1.b. $$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ \cos x, & 0 < x < \pi, \end{cases} \qquad f(x+2\pi) = f(x).$$

1.1.c. $$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 2, & 0 < x < 2\pi, \end{cases} \qquad f(x+3\pi) = f(x).$$

1.2. Evaluate the Fourier coefficients of the given functions. All functions assume to be 2π-periodic. Construct the graphs of these functions.

1.2.a. $$f(x) = x \quad (-\pi < x < \pi).$$

1.2.b. $$f(x) = x^2 \quad (-\pi < x < \pi).$$

1.2.c. $$f(x) = \begin{cases} 0, & -\pi < x < 0, \\ 1, & 0 < x < \pi. \end{cases}$$

1.2.d. $$f(x) = x \quad (0 < x < 2\pi).$$

1.3. Find the Fourier series of the following functions. Sketch the graphs of their periodic extensions.

1.3.a. $$f(x) = |x| \quad (-1 < x < 1).$$

1.3.b. $$f(x) = \begin{cases} -3, & -2 < x < 0, \\ 3, & 0 < x < 2. \end{cases}$$

1.4. Construct the periodic extensions of the following functions.

1.4.a. $$f(x) = x \quad (-1 < x \leq 1).$$

1.4.b. $$f(x) = x \quad (-1 \le x < 1).$$

1.4.c. $$f(x) = x^2 \quad (-1 \le x \le 1).$$

1.4.d. $$f(x) = x^3 \quad (-2 < x \le 2).$$

1.5. Find the cosine- and sine-series for the following functions. Sketch the graphs of the odd and even extensions and their periodic extensions.

1.5.a. $$f(x) = 1 \quad \text{on} \quad (0, a).$$

1.5.b. $$f(x) = x \quad \text{on} \quad (0, a).$$

1.5.c. $$f(x) = \sin x \quad \text{on} \quad (0, 1).$$

1.5.d. $$f(x) = \sin x \quad \text{on} \quad (0, \pi).$$

1.6. Find the Fourier series of the following functions.

1.6.a. $$f(x) = x \quad \text{on} \quad (-2, 2).$$

1.6.b. $$f(x) = \begin{cases} x, & -\frac{1}{2} < x < \frac{1}{2}, \\ 1 - x, & \frac{1}{2} < x < \frac{3}{2}. \end{cases}$$

1.7. Find the Fourier series of the function $(\cos x)^n$.

1.8.a. Prove that the system of functions $\left\{ \sqrt{(1/\pi)} \sin\left(n + \frac{1}{2}\right) x \right\}_{n=0}^{\infty}$ is orthogonal on $[-\pi, \pi]$.

1.8.b. Using this result and Bessel's inequality, give an alternative proof of the equality

$$\lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \sin\left(n + \frac{1}{2}\right) x \, dx = 0 \quad (f \in \mathcal{R}[-\pi, \pi]).$$

1.9. Applying Bessel's inequality to the system $\left\{\sqrt{2/\pi}\sin nx\right\}_{n=1}^{\infty}$, show that

$$\sum_{n=1}^{\infty} B_n^2 \le \frac{2}{\pi}\int_0^{\pi} f^2(x)\,dx,$$

where

$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx.$$

1.10. Let $f(x) = \frac{\pi - x}{2}$ $(0 < x < 2\pi)$. For any $n \in \mathbb{N}$, find a trigonometric polynomial T_n of degree n, which minimizes the norm

$$\|f - T_n\|_2 = \left(\int_0^{2\pi} [f(x) - T_n(x)]^2\,dx\right)^{1/2}.$$

1.11. Let f be a continuous 2π-periodic function. For arbitrary α, β, γ, set

$$F(\alpha,\beta,\gamma) = \frac{1}{\pi}\int_{-\pi}^{\pi} [f(x) - \alpha - \beta\cos x - \gamma\cos 10x]^2\,dx.$$

Prove that F attains its minimum at one and only one point $(\alpha_0, \beta_0, \gamma_0)$, and find this point if:

1.11.a. $f(x) = \cos^2 x$;

1.11.b. $f(x) = x^2$;

1.11.c. $f(x) = \sin x$;

1.11.d. $f(x) = 1 - 2\cos x$;

1.11.e. $f(x) = |x|$;

1.11.f. $f(x) = |\sin x|$.

1.12. The same for

$$F(\alpha,\beta,\gamma) = \frac{1}{\pi}\int_{-\pi}^{\pi} [x - \alpha - \beta\cos x - \gamma\sin 2x]^2\,dx.$$

1.13. Assume that 2π-periodic continuous function f is such that for any $n \in \mathbb{N}$ there exist numbers α_n, β_n such that

$$\int_{-\pi}^{\pi} [f(x) - \alpha_n - \beta_n\sin x]^2\,dx \le \frac{1}{n}.$$

Determine f.

1.14. Assume that 2π-periodic continuous function f is such that for any $n \in \mathbb{N}$ there exist numbers $\alpha_n, \beta_n, \gamma_n$ such that

$$\int_{-\pi}^{\pi} [f(x) - \alpha_n - \beta_n \cos x - \gamma_n \sin x]^2 \, dx \le \frac{1}{n}.$$

Determine f.

1.15. Let a function f be Riemann integrable on $[a, b]$. Evaluate:

1.15.a. $\lim\limits_{n\to\infty} \int\limits_a^b f(x) \cos^2 nx \, dx$;

1.15.b. $\lim\limits_{n\to\infty} \int\limits_a^b f(x) \cos^3 nx \, dx$.

1.16. Prove that absolute values of *all* extrema (maximums and minimums) of the Dirichlet kernel D_n are not smaller than $\frac{1}{2}$. Prove that there are exactly $2n$ points of the local extrema of D_n in the half-open interval $(-\pi, \pi]$.

1.17. Let f be piecewise smooth on an interval $(0, a)$. Prove that the cosine-series and the sine-series of f at a point x_0 converge to $f(x_0)$ if x_0 is a point of continuity of f, and both converge to $\frac{1}{2}[f(x_0+) + f(x_0-)]$ if f is discontinuous at x_0.

1.18. We say that f has *the right derivative* at a point x_0 if there exists

$$\lim_{x\to x_0+} \frac{f(x) - f(x_0+)}{x - x_0}.$$

We say that f has *the left derivative* at a point x_0 if there exists

$$\lim_{x\to x_0-} \frac{f(x) - f(x_0-)}{x - x_0}.$$

Prove the following theorem.

Theorem. *Assume that 2π-periodic function f is piecewise continuous. If f has the both right and left derivatives at a point x_0, then the Fourier series of f at x_0 converges to $\frac{1}{2}[f(x_0+) + f(x_0-)]$.*

1.19. Show that the Fourier series of the function $\sqrt[3]{x}$ on the interval $(-\pi, \pi)$ converges to $\sqrt[3]{x}$ for all x in $(-\pi, \pi)$.

1.20. *Lipschitz functions.* A function f is called *a Lipschitz function* of the order $\alpha > 0$ at a point x_0 if on some open interval containing x_0 there holds the inequality

$$|f(x) - f(x_0)| \le C|x - x_0|^\alpha,$$

where C is a positive constant. [1]

1.20.a. Show that $f(x) = x^{1/3}$ is a Lipschitz function of the order $1/3$ at the point 0.

1.20.b. Let $f(x) = x\cos(1/x)$ for $x \neq 0$ and $f(0) = 0$. Show that the function f is Lipschitz of the order 1 at the point 0.

1.20.c. Show that, if f is differentiable at a point x_0, then it is a Lipschitz function of the order 1 at the point x_0.

1.20.d. Show that the functions in 1.20.a and 1.20.b are Lipschitz at every point x (the order may be different for different x).

1.21. Evaluate

1.21.a.
$$\int_{-\pi}^{\pi} D_n(t)\sin 100t\,dt;$$

1.21.b.
$$\int_{-\pi}^{\pi} D_n(t)\cos 100t\,dt.$$

1.22. Evaluate $\frac{1}{\pi}\int\limits_{-\pi}^{\pi} D_n^2(t)\,dt$ for $n = 100$.

1.23. Let $g(t) = \begin{cases} \frac{\sin(t/2)}{t}, & t \neq 0, \\ 1/2, & t = 0. \end{cases}$

Find $\lim\limits_{n\to\infty} \frac{1}{\pi}\int\limits_{-\pi}^{\pi} D_n(t)g(t)\,dt$.

1.24. Let $f(x) = 1 - x^2$ $(x \in [-\pi, \pi])$.

1.24.a. Evaluate $a_0,\ a_n,\ b_n$ $(n \in \mathbb{N})$.

1.24.b. Find the sums of the Fourier series of the function f at the points $x = 5\pi$ and $x = 6\pi$.

1.25. Find the Fourier series of 1-periodic function

$$f(x) = \begin{cases} x - [x], & x \notin \mathbb{Z}, \\ \frac{1}{2}, & x \in \mathbb{Z}. \end{cases}$$

[1] We say also that f satisfies *Hölder condition of the order* α.

Find the sums of the Fourier series at the points $x = 5$, $x = 3$, and $x = 1,5$.

1.26.a. Show that

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} = \frac{\pi - x}{2} \quad (0 < x < 2\pi).$$

1.26.b. Evaluate $\sum\limits_{n=1}^{\infty} \sin nx/n$ for $-2\pi < x < 0$.

1.26.c. Prove that

$$\frac{\pi}{4} - \frac{x}{2} = \sum_{k=1}^{\infty} \frac{\sin 2kx}{2k} \quad (0 < x < \pi).$$

1.26.d. Show that exercises 1.26.a and 1.26.c yield the equality

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1} \quad (0 < x < \pi).$$

1.26.e. Find the Fourier series of the function $\frac{\pi}{4} \operatorname{sign} x$ $(|x| < \pi)$. Prove the equality from exercise 1.26.d, using this series and Dirichlet's theorem 1.29.

1.27. Let $f_1(x) = x$ $(x \in (-\pi, \pi])$. Find the Fourier series of f_1.

1.28. Using exercise 1.27 and integration, find the Fourier series of the following functions (formulate the theorem on integration of Fourier series and verify its conditions)

1.28.a.
$$f_2(x) = x^2 \quad (x \in (-\pi, \pi]);$$

1.28.b.
$$f_3(x) = x^3 \quad (x \in (-\pi, \pi]);$$

1.28.c.
$$f_4(x) = x^4, \quad (x \in (-\pi, \pi]).$$

1.29. Using the Fourier series of the function f_2 from the exercise 1.28.a, evaluate

1.29.a.
$$\sum_{n=1}^{\infty} \frac{1}{n^2};$$

1.29.b.
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

1.30. Let f be a 2π-periodic function. Assume that $f \in \mathcal{C}^r(\mathbb{R})$ $(r \geq 1)$ (that is, f has a continuous derivative $f^{(r)}$). Prove that

$$|a_n(f)| + |b_n(f)| = \frac{1}{n^r}\left(\left|a_n\left(f^{(r)}\right)\right| + \left|b_n\left(f^{(r)}\right)\right|\right)$$

for any $n \geq 1$.

1.31. Let f be a 2π-periodic continuously differentiable function. Prove that

$$\max_{x\in\mathbb{R}} |f(x) - S_n(x)| = \frac{\varepsilon_n}{\sqrt{n}} \quad (n \geq 1),$$

where $\varepsilon_n \to 0$ as $n \to \infty$ ($S_n(x)$ denotes the nth partial sum of the Fourier series of f).

1.32. Let $f \in \mathcal{C}^r(\mathbb{R})$ $(r \geq 1)$ be a 2π-periodic function. Prove that

$$\max_{x\in\mathbb{R}} |f(x) - S_n(x)| = \frac{\varepsilon_n}{n^{r-1/2}} \quad (n \geq 1),$$

where $\varepsilon \to 0$ as $n \to \infty$.

1.33.a. Prove that the series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$$

converges uniformly on $\mathbb{R}$.

1.33.b. Let $f(x)$ be the sum of this series. Is this series the Fourier series of the function f ?

1.33.c. Prove that $f \in \mathcal{C}^1(\mathbb{R})$, and the Fourier series of f' can be obtained by the term-by-term differentiation of the initial series.

1.33.d. Show that f' is piecewise smooth and continuous, and

$$f''(x) = -\sum_{n=1}^{\infty} \frac{\sin nx}{n} \quad (x \in (0, 2\pi))$$

(apply the theorem on term-by-term integration to the series on the right-hand side and exercise 1.26.a).

1.33.e. Determine f.

1.34. Prove that if f is piecewise continuous on (a, b) and $\| f \|_2 = 0$, then $f(x) = 0$, with a possible exception of a finite number of points.

1.35. Prove the following theorem.

Theorem. *Let f be a periodic function. If f has k continuous derivatives, then for some positive constant C the Fourier coefficients of the function f satisfy inequalities*

$$|A_n| \le \frac{C}{n^k}, \quad |B_n| \le \frac{C}{n^k}$$

for all n.

1.36. Let $f(x) = \begin{cases} Ax + B, & -\pi \le x < 0, \\ \cos x, & 0 \le x \le \pi. \end{cases}$ For which values of A and B the Fourier series of f converges uniformly to f on $[-\pi, \pi]$?

1.37. Let $f(x) = \pi x - x|x|$ $(x \in (-\pi, \pi])$, and f is periodically extended with the period 2π to $\mathbb{R}$. Is the Fourier series of f uniformly convergent ?

1.38. Let $g(x) = \begin{cases} \cos x, & -\pi < x < 0, \\ \sin x, & 0 \le x \le \pi. \end{cases}$

1.38.a. Find the Fourier series of g.

1.38.b. Let

$$h(x) = \int_{-\pi}^{x} g(t)\, dt + \alpha \sin \frac{x}{2} \quad (-\pi < x \le \pi),$$

$h(x + 2\pi) = h(x)$. Find values of α for which the Fourier series of h converges uniformly to h on $[-\pi, \pi]$.

1.39.a. Which of the following series are Fourier series ?

$$1)\ \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}; \quad 2)\ \sum_{n=1}^{\infty} \frac{\cos nx}{n^3}; \quad 3)\ \sum_{n=1}^{\infty} \frac{\sin nx}{n};$$

$$4)\ \sum_{n=1}^{\infty} \sin nx; \quad 5)\ \sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n}; \quad 6)\ \sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2}.$$

1.39.b. Which of these series converge uniformly on $\mathbb{R}$?

1.40. Let $\{g_n\}$ be an orthogonal system on an interval $[a, b]$. Let f be a continuous function on $[a, b]$ and let $\{c_n\}$ be its Fourier coefficients with respect to the system $\{g_n\}$.

1.40.a. Prove that the sequence of square norms

$$\left\| f - \sum_{n=0}^{N} c_n g_n \right\|_2$$

decreases as N increases.

1.40.b. Prove that *Parseval's identity*

$$\sum_{n=0}^{\infty} c_n^2 \|g_n\|_2^2 = \int_a^b f^2(x)\,dx$$

is true for any continuous function f on $[a,b]$ if and only if $\{g_n\}$ is a complete orthogonal system on $[a,b]$.

1.41. Using the completeness of the trigonometric system, that is, the equality

$$\frac{1}{2}A_0^2 + \sum_{n=1}^{\infty}\left[A_n^2 + B_n^2\right] = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,dx,$$

where A_n, B_n are the Fourier coefficients of f, prove that trigonometric system with the period $2a$ is complete on $[-a,a]$, and the systems of sines and cosines are complete orthogonal systems on $[0,a]$.

1.42. Parseval's identity can be used for evaluation of sums of some series of numbers. Applying equality from the exercise 1.41 to the function $f(x) = (\pi^2 - 3x^2)/12$, find the sum of the series $\sum\limits_{n=1}^{\infty} n^{-4}$.

1.43. Show that the orthogonal system $\{\sin nx\}_{n=2}^{\infty}$ is not complete on $[0,\pi]$.

1.44. Parseval Theorem.

1.44.a. Prove that an orthonormal system $\{g_n\}$ on $[a,b]$ is complete if and only if for any two piecewise continuous on $[a,b]$ functions f and g there holds the equality $\sum\limits_{n=1}^{\infty} c_n d_n = \int\limits_a^b f(x)g(x)\,dx$ (where c_n and d_n are the Fourier coefficients of f and g with respect to $\{g_n\}$).

1.44.b. Extend this result on the case of orthogonal systems.

1.45. Let $f_n(x) = \sum\limits_{k=1}^{n}(\cos kx - \sin kx) + 1$. Evaluate $\int\limits_{-\pi}^{\pi} f_n^2(x)\,dx$.

1.46. Let $f(x) = \mathrm{e}^{-x}$ $(-\pi < x \le \pi)$. Evaluate $\frac{a_0^2}{2} + \sum\limits_{n=1}^{\infty} a_n^2$, where a_n are the cosine-coefficients of the function f.

1.47. Let f be a piecewise smooth continuous function on $[0,\pi]$. Assume that f satisfies at least one of the conditions

$$f(0) = f(\pi) = 0 \quad \text{or} \quad \int_0^\pi f(x)\,dx = 0.$$

Prove *Steklov's inequality*

$$\int_0^\pi f^2(x)\,dx \le \int_0^\pi \left(f'\right)^2(x)\,dx.$$

In which cases it becomes the equality ?

1.48. Let f be a 2π-periodic continuous piecewise smooth function and let

$$\int_{-\pi}^{\pi} f(x)\,dx = 0.$$

Prove *Wirtinger's inequality*

$$\int_{-\pi}^{\pi} f^2(x)\,dx \le \int_{-\pi}^{\pi} \left(f'\right)^2(x)\,dx.$$

1.49. Prove that the series $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ isn't summable in the sense of mean-arithmetic means, that is, $\lim\limits_{n\to\infty} \sigma_n$ doesn't exist.

1.50. Prove that the Fejér means satisfy the following equality

$$\sigma_n(x) = \frac{1}{2}A_0 + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right)\left[A_k \cos kx + B_k \sin kx\right],$$

where A_k, B_k are the Fourier coefficients of f.

1.51. Assume that 2π-periodic function f is piecewise continuous. Prove that the condition $m \le f(x) \le M$ implies the inequality $m \le \sigma_n(x) \le M$ for all x.

1.52. Prove that for any m

$$\left|\sum_{n=1}^{m} \frac{\sin nx}{n}\right| \le \frac{1}{2}\pi + 1.$$

Hint: use the relation between the arithmetic means of the series $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n}$ and its partial sums, applying exercise 1.50.

1.53. Let $\sum\limits_{n=0}^{\infty} u_n$ be a numerical series, let $S_n = \sum\limits_{k=0}^{n} u_k$ be its partial sums, and let

$$\sigma_n = \frac{S_0 + \dots + S_n}{n+1}$$

be the arithmetic means.

1.53.a. Show that

$$\sigma_n = \sum_{k=0}^{n} \left(1 - \frac{k}{n+1}\right) u_k.$$

1.53.b. Show that $S_n = (n+1)\sigma_n - n\sigma_{n-1}$.

1.53.c. Using exercise 1.53.b, show that, if the series $\sum\limits_{n=0}^{\infty} u_n$ is summable by the method of arithmetic means, then

$$\frac{S_n}{n} \to 0 \quad \text{and} \quad \frac{u_n}{n} \to 0 \quad (n \to \infty).$$

1.54. Show that

$$F_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) \cos kt$$

(F_n is the Fejér means of the order n).

1.55.a. Show that for the arithmetic means of the series $\sum\limits_{n=1}^{\infty} \frac{\sin nx}{n}$ we have the equality

$$\sigma_n(x) = -\frac{1}{2}x + \int\limits_0^x F_n(t)\, dt.$$

1.55.b. Show that $\sigma_n(x) < \frac{\pi - x}{2}$ for all $0 < x < \pi$.

1.56. Show that the series $\frac{1}{2} + \sum\limits_{n=1}^{\infty} \cos nx$ is summable to zero by the method of arithmetic means at every point $x \in [-\pi, \pi]$ except $x = 0$. Is this series a Fourier series? Does this series converge at some point?

1.57. Show that the arithmetic means of the series $\sum\limits_{n=1}^{\infty} \sin nx$ are equal to

$$\sigma_n(x) = \frac{1}{2} \operatorname{ctg} \frac{x}{2} - \frac{\sin(n+2)x - \sin x}{4(n+1)\sin^2 \frac{x}{2}}$$

and that $\sigma_n(x) \to \frac{1}{2} \operatorname{ctg} \frac{x}{2}$ for any $x \in [-\pi, \pi]$, $x \neq 0$.

1.58. Prove that each of the systems

$$\left\{\frac{1}{2},\ \cos x,\ \cos 2x,\ \ldots,\ \cos nx,\ \ldots\right\} \quad \text{and} \quad \{\sin x,\ \sin 2x,\ \ldots,\ \sin nx,\ \ldots\}$$

is orthogonal and complete on $[0, \pi]$.

1.59. Let f be 2π-periodic and $f \in \mathcal{R}[-\pi, \pi]$. Assume that $a_n \geq 0$, $n \in \mathbb{N}$. Prove that the series $\sum\limits_{n=1}^{\infty} a_n$ converges.

Hint: use the arithmetic means $\sigma_n(x)$. We have

$$M \geq \sigma_n(0) = \sum_{k=1}^{n} \left(1 - \frac{k}{n+1}\right) a_k + \frac{a_0}{2} \geq$$

$$\geq \sum_{k=1}^{N} \left(1 - \frac{k}{n+1}\right) a_k + \frac{a_0}{2} \quad (n \geq N).$$

2. Fourier transforms

2.1. The main properties

2.1.1. Definitions and examples

We say that a complex-valued function f defined on $\mathbb{R}$ is *absolutely integrable on* $\mathbb{R}$ if f is Riemann integrable in each bounded interval $[a, b]$ and the integral

$$\int_{\mathbb{R}} |f(x)|\, dx$$

converges. This integral is called *the norm* of f *of the order* 1 and it is denoted by $\| f \|_1$. Throughout this chapter we denote by $\mathcal{A}(\mathbb{R})$ the class of all complex-valued absolutely integrable functions on $\mathbb{R}$.

Definition 2.1. For a function $f \in \mathcal{A}(\mathbb{R})$, its *Fourier transform* $\widehat{f}$ is defined by

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-i2\pi\xi x}\, dx. \tag{2.1}$$

Since $\left|\mathrm{e}^{-i2\pi\xi x}\right| = 1$, the integral (2.1) is absolutely convergent for any $\xi \in \mathbb{R}$.

For a $2\pi-$periodic function integrable on $[0, 2\pi]$ we defined its Fourier coefficients a_n, b_n by formulas (1.4) and (1.5) (and the complex Fourier coefficients c_n by (1.11)). Clearly, the Fourier transform is an analogue of Fourier coefficients which is suitable for functions defined on $\mathbb{R}$ (non-periodic). Whereas the Fourier coefficients depend on discretely varying index n, the Fourier transform is a function of a variable ξ which ranges in the whole line $\mathbb{R}$ (that is, f and $\widehat{f}$ have the same domain $\mathbb{R}$).

First we observe that the following simple properties hold.

Theorem 2.1. Let $f \in \mathcal{A}(\mathbb{R})$. Then:
(i) if the function f is even, then $\widehat{f}$ also is an even function, and

$$\widehat{f}(\xi) = 2\int_0^\infty f(x)\cos 2\pi\xi x\,dx; \tag{2.2}$$

(ii) if f is odd, then $\widehat{f}$ also is odd, and

$$\widehat{f}(\xi) = -2i\int_0^\infty f(x)\sin 2\pi\xi x\,dx. \tag{2.3}$$

Proof. We have $\mathrm{e}^{-i2\pi\xi x} = \cos 2\pi\xi x - i\sin 2\pi\xi x$. Assume that f is even. Then $f(x)\sin 2\pi\xi x$ is an odd function of x and thus

$$\int_{-\infty}^\infty f(x)\sin 2\pi\xi x\,dx = 0.$$

On the other hand, $f(x)\cos 2\pi\xi x$ is an even function of x and thus

$$\int_{-\infty}^\infty f(x)\cos 2\pi\xi x\,dx = 2\int_0^\infty f(x)\cos 2\pi\xi x\,dx.$$

These observations imply (2.2). Similarly, we obtain (2.3). □

We consider some important examples.

Example 2.2. Let

$$\Pi(x) = \begin{cases} 1, & |x| \le 1/2, \\ 0, & |x| > 1/2. \end{cases}$$

Then

$$\widehat{\Pi}(\xi) = \frac{\sin \pi\xi}{\pi\xi} \quad (\xi \ne 0), \quad \widehat{\Pi}(0) = 1. \tag{2.4}$$

Indeed, the function Π is even and thus by (2.2),

$$\widehat{\Pi}(\xi) = 2\int_0^\infty \Pi(x)\cos 2\pi\xi x\,dx = 2\int_0^{1/2}\cos 2\pi\xi x\,dx = \frac{\sin \pi\xi}{\pi\xi}$$

for any $\xi \neq 0$. For $\xi = 0$ we have

$$\widehat{\Pi}(0) = 2\int\limits_0^\infty f(x)\,dx = 1.$$

The function (2.4) is called *the Dirichlet kernel.* Since

$$\lim_{t\to 0}\frac{\sin t}{t} = 1,$$

function (2.4) is continuous on the whole real line $\mathbb{R}$. $\square$

Example 2.3. Let

$$\Lambda(x) = \begin{cases} 1 - |x|, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Then

$$\widehat{\Lambda}(\xi) = \frac{\sin^2 \pi\xi}{\pi^2\xi^2} \quad (\xi \neq 0), \quad \widehat{\Lambda}(0) = 1. \tag{2.5}$$

Indeed, taking into account that Λ is even, and applying integration by parts, we obtain

$$\widehat{\Lambda}(\xi) = 2\int\limits_0^\infty \Lambda(x)\cos 2\pi\xi x\,dx = 2\int\limits_0^1 (1-x)\cos 2\pi\xi x\,dx$$

$$= \frac{1}{\pi\xi}\int\limits_0^1 \sin 2\pi\xi x\,dx = \frac{1-\cos 2\pi\xi}{2\pi^2\xi^2} = \frac{\sin^2 \pi\xi}{\pi^2\xi^2}$$

for any $\xi \neq 0$. For $\xi = 0$ we have

$$\widehat{\Lambda}(0) = 2\int\limits_0^\infty \Lambda(x)\,dx = 1.$$

The function (2.5) is called *the Fejér kernel.* Similarly to the Dirichlet kernel, the Fejér kernel is continuous on the whole real line $\mathbb{R}$. $\square$

Example 2.4. Let $f(x) = \mathrm{e}^{-2\pi|x|}$. Then

$$\widehat{f}(\xi) = \frac{1}{\pi\,(1+\xi^2)}. \tag{2.6}$$

Indeed, the function f is even and thus

$$\widehat{f}(\xi) = 2\int_0^\infty f(x)\cos 2\pi\xi x\,dx = 2\int_0^\infty \mathrm{e}^{-2\pi x}\cos 2\pi\xi x\,dx = \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-t}\cos\xi t\,dt.$$

Two consecutive integrations by parts give (2.6). $\square$

2.1.2. Continuity and decay at infinity of Fourier transform

Fourier transforms evaluated in examples given above are continuous functions. We shall show that this is a general property of Fourier transforms.

Theorem 2.5. Let $f \in \mathcal{A}(\mathbb{R})$. Then the Fourier transform $\widehat{f}$ is a bounded continuous function on $\mathbb{R}$, and

$$\left|\widehat{f}(\xi)\right| \le \int_{-\infty}^\infty |f(x)|\,dx \quad (\xi \in \mathbb{R}). \tag{2.7}$$

Proof. We have

$$\left|f(x)\mathrm{e}^{-2\pi i\xi x}\right| = |f(x)|. \tag{2.8}$$

Taking into account that f is absolutely integrable on $\mathbb{R}$, and applying Weierstrass M-test, we obtain that both the integrals

$$\int_{-\infty}^0 f(x)\mathrm{e}^{-2\pi i\xi x}\,dx, \quad \int_0^\infty f(x)\mathrm{e}^{-2\pi i\xi x}\,dx$$

converge uniformly in $\xi \in \mathbb{R}$. Further, $\mathrm{e}^{-2\pi i\xi x}$ is a continuous function of $(x,\xi) \in \mathbb{R}^2$. Thus, by Theorem 17, the function

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-2\pi i\xi x}\,dx$$

is continuous on $\mathbb{R}$. Moreover, by (2.8),

$$\left|\widehat{f}(\xi)\right| \le \int_{-\infty}^\infty |f(x)|\,dx,$$

and for any ξ we have (2.7). $\square$

A function $f \in \mathcal{A}(\mathbb{R})$ may be unbounded on $\mathbb{R}$ and discontinuous at some points. However, as we have shown, its Fourier transform is continuous and bounded on $\mathbb{R}$. Moreover, $\widehat{f}(\xi) \to 0$ as $|\xi| \to +\infty$. It is a corollary of the following important statement which is called *the Riemann – Lebesgue Lemma.*

Theorem 2.6. Let $f \in \mathcal{A}(\mathbb{R})$. Then

$$\lim_{\xi\to\infty} \int_{-\infty}^{\infty} f(x)\cos\xi x\, dx = 0, \tag{2.9}$$

$$\lim_{\xi\to\infty} \int_{-\infty}^{\infty} f(x)\sin\xi x\, dx = 0. \tag{2.10}$$

Proof. We prove (2.9); the proof of (2.10) is similar. First we assume that f is a step function vanishing outside some interval $[a, b]$. This means that $f(x) = 0$ for all $x \notin [a, b]$, and $[a, b]$ can be subdivided by points

$$a = x_0 < x_1 < \cdots < x_n = b$$

such that $f(x) = c_j$ for $x \in (x_j, x_{j+1})$ $(j = 0, 1, \ldots, n-1)$. Then for any $\xi \neq 0$

$$\int_{-\infty}^{\infty} f(x)\cos\xi x\, dx = \sum_{j=0}^{n-1} c_j \int_{x_j}^{x_{j+1}} \cos\xi x\, dx = \frac{1}{\xi}\sum_{j=0}^{n-1} c_j \left(\sin\xi x_{j+1} - \sin\xi x_j\right).$$

Thus,

$$\left|\int_{-\infty}^{\infty} f(x)\cos\xi x\, dx\right| \le \frac{C}{|\xi|},$$

where $C = 2\sum\limits_{j=0}^{n-1} |c_j|$. This implies (2.9) in the considered case of a step function.

Let now f be an arbitrary absolutely integrable function on $\mathbb{R}$. Let $\varepsilon > 0$. There exists $A > 0$ such that

$$\int_{-\infty}^{-A} |f(x)|\, dx + \int_{A}^{\infty} |f(x)|\, dx < \varepsilon. \tag{2.11}$$

Further, by Theorem 12, there exists a step function h, equal to zero for $|x| > A$, such that

$$\int_{-A}^{A} |f(x) - h(x)|\, dx < \varepsilon. \tag{2.12}$$

Now we have

$$\left|\int_{-\infty}^{\infty} f(x)\cos\xi x\, dx\right| \le \int_{|x|\ge A} |f(x)|\, dx + \int_{-A}^{A} |f(x) - h(x)|\, dx + \left|\int_{-A}^{A} h(x)\cos\xi x\, dx\right|$$

$$< 2\varepsilon + \left|\int_{-A}^{A} h(x)\cos\xi x\, dx\right|.$$

As it was shown above, the latter integral tends to zero as $|\xi| \to \infty$. Thus, there exists a number $E > 0$ such that

$$\left|\int_{-\infty}^{\infty} f(x)\cos\xi x\, dx\right| < 3\varepsilon \quad (|\xi| \ge E).$$

This yields (2.9) in the general case. $\square$

The Riemann – Lebesgue Lemma immediately implies

Theorem 2.7. Let $f \in \mathcal{A}(\mathbb{R})$. Then

$$\widehat{f}(\xi) \to 0 \quad (|\xi| \to \infty).$$

Remark 2.8. This theorem is an analogue of the Riemann – Lebesgue Lemma for Fourier coefficients (Lemma 1.27) which was derived from Bessel's inequality (1.22).

2.1.3. The Fourier transform of the Gaussian

The function $g(x) = \mathrm{e}^{-\pi x^2}$ is called *the Gaussian.* This function plays an extremely important role in different areas of mathematics.

We shall use the following equality which is well-known from the course of Mathematical Analysis

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\pi x^2}\, dx = 1. \tag{2.13}$$

Theorem 2.9. Let $g(x) = \mathrm{e}^{-\pi x^2}$. Then $\widehat{g}(\xi) = \mathrm{e}^{-\pi\xi^2}$.

Proof. Since g is even, we have by Theorem 2.1,

$$\widehat{g}(\xi) = 2\int_0^\infty \mathrm{e}^{-\pi x^2}\cos 2\pi\xi x\,dx.$$

Set

$$f(x,\xi) = \mathrm{e}^{-\pi x^2}\cos 2\pi\xi x \quad (x \geq 0,\ \xi \in \mathbb{R}).$$

Then $f(x,\xi)$ and $f'_\xi(x,\xi) = \mathrm{e}^{-\pi x^2}(-2\pi x)\sin 2\pi\xi x$ are continuous functions. Moreover,

$$|f(x,\xi)| \leq \mathrm{e}^{-\pi x^2}, \quad \left|f'_\xi(x,\xi)\right| \leq 2\pi x\mathrm{e}^{-\pi x^2} \quad (x \geq 0,\ \xi \in \mathbb{R}).$$

Thus, by the Weierstrass M-test, both the integrals

$$\int_0^\infty \mathrm{e}^{-\pi x^2}\cos 2\pi\xi x\,dx \quad \text{and} \quad \int_0^\infty \mathrm{e}^{-\pi x^2}x\sin 2\pi\xi x\,dx$$

converge uniformly with respect to $\xi \in \mathbb{R}$. Therefore we may differentiate under the sign of the integral. We obtain

$$\left(\widehat{g}\right)'(\xi) = -4\pi\int_0^\infty \mathrm{e}^{-\pi x^2}x\sin 2\pi\xi x\,dx.$$

Now we integrate by parts in the latter integral. This gives

$$\left(\widehat{g}\right)'(\xi) = 2\mathrm{e}^{-\pi x^2}\sin 2\pi\xi x\Big|_{x=0}^{x=\infty} - 4\pi\xi\int_0^\infty \mathrm{e}^{-\pi x^2}\cos 2\pi\xi x\,dx = -2\pi\xi\widehat{g}(\xi).$$

Thus, the function $y = \widehat{g}(\xi)$ satisfies the differential equation

$$\frac{dy}{d\xi} = -2\pi\xi y.$$

The solution is

$$y(\xi) = C\mathrm{e}^{-\pi\xi^2}.$$

Setting $\xi = 0$ and taking into account (2.13), we get

$$C = y(0) = \int_{-\infty}^\infty \mathrm{e}^{-\pi x^2}\,dx = 1.$$

Thus, $\widehat{g}(\xi) = \mathrm{e}^{-\pi\xi^2}$. $\square$

2.1.4. Basic properties of Fourier transforms

In this section we summarize some basic formulas concerning Fourier transforms.

Let a function φ be defined on $\mathbb{R}$. For any $h \in \mathbb{R}$, denote $\tau_h\varphi(x) = \varphi(x-h)$. The operation τ_h is called *translation* or *shifting.* Further, for any $\lambda > 0$, set $\delta_\lambda\varphi(x) = \varphi(\lambda x)$. The operation δ_λ is called *dilation.*

Theorem 2.10. Let f, g be complex-valued functions, absolutely integrable over $\mathbb{R}$. Then the following properties hold.

(i) Linearity.

$$\widehat{(f+g)}(\xi) = \widehat{f}(\xi) + \widehat{g}(\xi)$$

and for any complex number α

$$\widehat{(\alpha f)}(\xi) = \alpha \widehat{f}(\xi).$$

(ii) Shifting. For any $h \in \mathbb{R}$

$$\widehat{\tau_h f}(\xi) = \mathrm{e}^{-2\pi i h\xi}\widehat{f}(\xi).$$

(iii) Change of scale (dilation). For any $\lambda > 0$

$$\widehat{\delta_\lambda f}(\xi) = \frac{1}{\lambda}\widehat{f}\left(\frac{\xi}{\lambda}\right).$$

(iv) Modulation (shifting in Fourier transform). Let $f_\eta(x) = f(x)\mathrm{e}^{i2\pi\eta x}$, where $\eta \in \mathbb{R}$. Then

$$\widehat{f_\eta}(\xi) = \widehat{f}(\xi - \eta) = \tau_\eta \widehat{f}(\xi).$$

Moreover, for $g_\eta(x) = f(x)\cos 2\pi\eta x$ and $h_\eta(x) = f(x)\sin 2\pi\eta x$, we have

$$\widehat{g_\eta}(\xi) = \frac{1}{2}\left[\widehat{f}(\xi-\eta) + \widehat{f}(\xi+\eta)\right], \quad \widehat{h_\eta}(\xi) = \frac{i}{2}\left[\widehat{f}(\xi+\eta) - \widehat{f}(\xi-\eta)\right].$$

Proof. (i) By the linearity of integration, we have

$$\widehat{(f+g)}(\xi) = \int\limits_{-\infty}^{\infty} [f(x) + g(x)]\mathrm{e}^{-i2\pi\xi x}\,dx$$

$$= \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-i2\pi\xi x}\, dx + \int_{-\infty}^{\infty} g(x) \mathrm{e}^{-i2\pi u\xi x}\, dx = \widehat{f}(\xi) + \widehat{g}(\xi),$$

and

$$\widehat{\alpha f}(\xi) = \int_{-\infty}^{\infty} \alpha f(x) \mathrm{e}^{-i2\pi\xi x}\, dx = \alpha \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-i2\pi\xi x}\, dx = \alpha \widehat{f}(\xi).$$

(ii) Setting $x - h = y$, we obtain

$$\widehat{\tau_h f}(\xi) = \int_{-\infty}^{\infty} f(x-h) \mathrm{e}^{-i2\pi x} dx = \int_{-\infty}^{\infty} f(y) \mathrm{e}^{-i2\pi\xi(y+h)}\, dy$$

$$= \mathrm{e}^{-i2\pi\xi h} \int_{-\infty}^{\infty} f(y) \mathrm{e}^{-i2\pi\xi y}\, dy = \mathrm{e}^{-i2\pi\xi h} \widehat{f}(\xi).$$

(iii) Similarly, substitution $\lambda x = y$ leads to the equality

$$\widehat{\delta_\lambda f}(\xi) = \int_{-\infty}^{\infty} f(\lambda x) \mathrm{e}^{-i2\pi\xi x} dx = \frac{1}{\lambda} \int_{-\infty}^{\infty} f(y) \mathrm{e}^{-i2\pi\xi y/\lambda}\, dy = \frac{1}{\lambda} \widehat{f}\left(\frac{\xi}{\lambda}\right).$$

(iv) First, we have

$$\widehat{f_\eta}(\xi) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-i2\pi(\xi-\eta)x} dx = \widehat{f}(\xi - \eta).$$

Further,

$$\cos(2\pi\eta x) = \frac{1}{2}\left(\mathrm{e}^{i2\pi\eta x} + \mathrm{e}^{-i2\pi\eta x}\right), \quad \sin(2\pi\eta x) = \frac{i}{2}\left(\mathrm{e}^{-i2\pi\eta x} - \mathrm{e}^{i2\pi\eta x}\right).$$

Thus, formulas for $\widehat{g_\eta}$ and $\widehat{h_\eta}$ follow by linearity. □

By Theorem 2.5, for any function $f \in \mathcal{A}(\mathbb{R})$ its Fourier transform $\widehat{f}$ is continuous on $\mathbb{R}$. Now we shall show that if f has a "good" rate of decay at infinity, then $\widehat{f}$ is differentiable, and $\left(\widehat{f}\right)'$ is obtained by the differentiation with respect to the parameter ξ under the sign of the integral in (2.1).

Theorem 2.11. Assume that the functions f and $xf(x)$ are absolutely integrable over $\mathbb{R}$. Then the Fourier transform $\widehat{f}$ is continuously differentiable in $\mathbb{R}$. Moreover,

$$\left(\widehat{f}\right)'(\xi) = -2\pi i \int_{-\infty}^{\infty} x f(x) \mathrm{e}^{-2\pi i x\xi}\, dx. \tag{2.14}$$

Proof. We have

$$\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-i2\pi x\xi}\, dx. \tag{2.15}$$

Let $F(x,\xi) = \mathrm{e}^{-i2\pi x\xi}$. Then the partial derivative

$$F'_\xi(x,\xi) = -2\pi i x \mathrm{e}^{-2\pi i x\xi}$$

is a continuous function of variables $x, \xi \in \mathbb{R}$. Further,

$$\left|x f(x) \mathrm{e}^{-2\pi i x\xi}\right| \le |x f(x)|.$$

This implies that the integral at the right-hand side of (2.14) converges uniformly with respect to $\xi \in \mathbb{R}$. Thus, applying Theorem 18 to the integral (2.15), we have that this integral is a continuously differentiable function of ξ, and its derivative can be obtained by the differentiation under the sign of the integral. This completes the proof. □

Applying Theorem 2.11 and induction, we obtain the following corollary.

Corollary 2.12. Let $m \in \mathbb{N}$. Assume that the functions f and $g(x) = x^m f(x)$ are absolutely integrable over $\mathbb{R}$. Then the Fourier transform $\widehat{f}$ has the continuous derivative $\left(\widehat{f}\right)^{(m)}$ on $\mathbb{R}$, and

$$\left(\widehat{f}\right)^{(m)}(\xi) = (-2\pi i)^m \widehat{g}(\xi). \tag{2.16}$$

The next theorem gives the Fourier transform of the derivative.

Theorem 2.13. Let f be a continuous, absolutely integrable, and piecewise smooth function on $\mathbb{R}$. Assume that f' also is absolutely integrable. Then

$$\widehat{(f')}(\xi) = 2\pi i \xi \widehat{f}(\xi). \tag{2.17}$$

Proof. By the Fundamental Theorem of Calculus (see Theorem 10), we have

$$f(x) - f(0) = \int_0^x f'(t)\,dt \quad (x \in \mathbb{R}).$$

Since both the integrals $\int_0^\infty f'(t)\,dt$ and $\int_{-\infty}^0 f'(t)\,dt$ converge, there exist finite limits

$$\lim_{x\to+\infty} f(x) \quad \text{and} \quad \lim_{x\to-\infty} f(x).$$

Since f is absolutely integrable, these limits are equal to zero. Integrating by parts, we get

$$\widehat{f'}(\xi) = \int_{-\infty}^{\infty} f'(x)\mathrm{e}^{-2\pi ix\xi}\,dx = \lim_{x\to+\infty} f(x)\mathrm{e}^{-2\pi ix\xi} - \lim_{x\to-\infty} f(x)\mathrm{e}^{-2\pi ix\xi}$$

$$+2\pi i\xi \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-2\pi ix\xi}\,dx = 2\pi i\xi\widehat{f}(\xi). \qquad \square$$

Using induction, we have the following corollary.

Corollary 2.14. Let $f, f', \dots, f^{(n-1)}$ be continuous and absolutely integrable over $\mathbb{R}$. Assume also that $f^{(n-1)}$ is piecewise smooth and $f^{(n)}$ is absolutely integrable over $\mathbb{R}$. Then

$$\widehat{\left(f^{(n)}\right)}(\xi) = (2\pi i)^n \xi^n \widehat{f}(\xi).$$

2.2. Fourier inversion, Gauss – Weierstrass summation

We shall consider *the Fourier inversion formula*, which gives a solution of the following problem: recover f from $\widehat{f}$. The expected formula is

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi)\mathrm{e}^{i2\pi x\xi}\,d\xi.$$

This an analogue of Fourier series. However, the integral may not exist ($\widehat{f}$ may not be absolutely integrable). To avoid this difficulty, we multiply $\widehat{f}(\xi)$

by $\mathrm{e}^{-\pi\alpha^2\xi^2}$ $(\alpha > 0)$. Since $\widehat{f}$ is bounded on $\mathbb{R}$, this product is an absolutely integrable function on $\mathbb{R}$.

Lemma 2.15. Let $f \in \mathcal{A}(\mathbb{R})$. Then for any $\alpha > 0$

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)\mathrm{e}^{-\pi\alpha^2\xi^2}\mathrm{e}^{i2\pi x\xi}\,d\xi = \int_{-\infty}^{\infty} f(x-t)W(t,\alpha)\,dt, \tag{2.18}$$

where

$$W(t,\alpha) = \frac{1}{\alpha}\mathrm{e}^{-\pi t^2/\alpha^2}.$$

PROOF. We have

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)\mathrm{e}^{-\pi\alpha^2\xi^2}\mathrm{e}^{i2\pi x\xi}\,d\xi = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y)\mathrm{e}^{-i2\pi y\xi}\,dy\right]\mathrm{e}^{-\pi\alpha^2\xi^2}\mathrm{e}^{i2\pi x\xi}\,d\xi$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(y)\mathrm{e}^{-i2\pi\xi(y-x)}\,dy\right]\mathrm{e}^{-\pi\alpha^2\xi^2}\,d\xi.$$

Observe that for each $\alpha > 0$ the function $\mathrm{e}^{-\pi\alpha^2\xi^2}$ is absolutely integrable on $\mathbb{R}$, and for each x the integral $\int_{-\infty}^{\infty} f(y)\mathrm{e}^{-i2\pi\xi(y-x)}\,dy$ converges uniformly with respect to ξ on $\mathbb{R}$. Thus, by Theorem 19 (ii), we can interchange the order of integrations at the right-hand side of the latter equality. We shall also take into account that the integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\pi\alpha^2\xi^2}\mathrm{e}^{-i2\pi\xi(y-x)}\,d\xi \tag{2.19}$$

(with respect to the variable ξ) is equal to $\widehat{g_\alpha}(y-x)$, where $g_\alpha(t) = \mathrm{e}^{-\pi\alpha^2t^2}$. We have $g_\alpha(t) = g(\alpha t)$, where g is the Gaussian. By Theorems 2.9 and 2.10 (iii), the integral (2.19) is equal to

$$\frac{1}{\alpha}\mathrm{e}^{-\pi(y-x)^2/\alpha^2} = W(y-x,\alpha).$$

Thus,

$$\int_{-\infty}^{\infty} \widehat{f}(\xi)\mathrm{e}^{-\pi\alpha^2\xi^2}\mathrm{e}^{i2\pi x\xi}\,d\xi = \int_{-\infty}^{\infty} f(y)W(y-x,\alpha)\,dy.$$

Finally, replacing t by $x-t$, we obtain (2.18). $\square$

The function $W(t,\alpha)$ is called *the Gauss – Weierstrass kernel.*

To remove the factor $e^{-\pi\alpha^2\xi^2}$ in (2.18), we make $\alpha\to 0$. We shall use the following lemma.

Lemma 2.16. The family of functions

$$W(t,\alpha)=\frac{1}{\alpha}e^{-\pi t^2/\alpha^2}\quad(\alpha>0)$$

satisfies the following conditions:

$$\text{(i)}\qquad \int_{-\infty}^{\infty} W(t,\alpha)\,dt=1\quad(\alpha>0);$$

$$\text{(ii)}\qquad W(t,\alpha)>0;$$

$$\text{(iii)}\qquad \mu_\delta(\alpha)=\sup_{|t|\ge\delta} W(t,\alpha)\to 0\quad(\alpha\to 0+)\quad\text{for any }\delta>0;$$

$$\text{(iv)}\qquad \int_{|t|\ge\delta} W(t,\alpha)\,dt\to 0\quad(\alpha\to 0+)\quad\text{for any }\delta>0.$$

Proof. Equality (i) follows immediately from (2.13), and inequality (ii) is obvious. Let $\delta>0$. First, we have

$$W(t,\alpha)=\frac{1}{\alpha}e^{-\pi t^2/\alpha^2}\le\frac{1}{\alpha}e^{-\pi\delta^2/\alpha^2}\quad(|t|\ge\delta).$$

Applying L'Hospital's rule, we easily obtain that

$$\lim_{\alpha\to 0}\frac{1}{\alpha}e^{-\pi\delta^2/\alpha^2}=0.$$

This implies (iii). Further, changing variable $t=\alpha z$, we have

$$\frac{1}{\alpha}\int_{\delta}^{\infty}e^{-\pi t^2/\alpha^2}=\int_{\delta/\alpha}^{\infty}e^{-\pi z^2}dz,\quad \frac{1}{\alpha}\int_{-\infty}^{-\delta}e^{-\pi t^2/\alpha^2}=\int_{-\infty}^{-\delta/\alpha}e^{-\pi z^2}dz.$$

By (2.13), these equalities imply (iv). $\square$

Now we will prove a theorem which enables us to reconstruct a continuous function f from its Fourier transform (*Gauss-Weierstrass summation theorem*).

Theorem 2.17. Let f be a continuous, absolutely integrable function on $\mathbb{R}$. Then

$$\lim_{\alpha\to 0+} \int_{-\infty}^{\infty} \widehat{f}(\xi) \mathrm{e}^{-\pi\alpha^2\xi^2} \mathrm{e}^{i2\pi x\xi}\, d\xi = f(x) \tag{2.20}$$

for any $x \in \mathbb{R}$.

PROOF. Fix $x \in \mathbb{R}$. By virtue of (2.18), equality (2.20) is equivalent to the equality

$$\lim_{\alpha\to 0+} \int_{-\infty}^{\infty} f(x-t) W(t,\alpha)\, dt = f(x). \tag{2.21}$$

Set

$$\gamma(\alpha, x) = \int_{-\infty}^{\infty} f(x-t) W(t,\alpha)\, dt - f(x).$$

By Lemma 2.16 (i),

$$f(x) = \int_{-\infty}^{\infty} f(x) W(t,\alpha)\, dt.$$

From here

$$\gamma(\alpha, x) = \int_{-\infty}^{\infty} [f(x-t) - f(x)] W(t,\alpha)\, dt.$$

Let $\varepsilon > 0$. By continuity of the function f, there exists a number $\delta > 0$ such that

$$|f(x-t) - f(x)| < \varepsilon \quad (|t| \le \delta).$$

Thus, taking into account statements (ii) and (i) of Lemma 2.16, we obtain

$$|\gamma(\alpha, x)| \le \int_{-\infty}^{\infty} |f(x-t) - f(x)| W(t,\alpha)\, dt$$

$$\le \varepsilon \int_{|t|\le\delta} W(t,\alpha)\, dt + \int_{|t|\ge\delta} |f(x-t) - f(x)| W(t,\alpha)\, dt$$

$$\leq \varepsilon + \int\limits_{|t|\geq\delta} |f(x-t)|W(t,\alpha)\,dt + |f(x)| \int\limits_{|t|\geq\delta} W(t,\alpha)\,dt.$$

By Lemma 2.16 (iv), there exists $\eta > 0$ (depending on ε and x) such that

$$|f(x)| \int\limits_{|t|\geq\delta} W(t,\alpha)\,dt < \varepsilon \quad (0 < \alpha < \eta).$$

Further, we have

$$\int\limits_{|t|\geq\delta} |f(x-t)|W(t,\alpha)\,dt \leq \sup_{|t|\geq\delta} W(t,\alpha) \int\limits_{-\infty}^{\infty} |f(u)|\,du. \tag{2.22}$$

By Lemma 2.16 (iii), there exists $0 < \eta' \leq \eta$ such that for all $0 < \alpha < \eta'$ the right-hand side of inequality (2.22) is smaller than ε. Thus, we have that $|\gamma(\alpha,x)| < 3\varepsilon$ for all $0 < \alpha < \eta'$. This implies (2.21). □

Remark 2.18. The statement of Theorem 2.17 is true if f is merely piecewise continuous and

$$f(x) = \frac{f(x+) + f(x-)}{2} \quad (x \in \mathbb{R}).$$

The main result in this section is the following *Fourier inversion theorem.*

Theorem 2.19. Assume that f is continuous and absolutely integrable on $\mathbb{R}$. If $\widehat{f}$ is absolutely integrable on $\mathbb{R}$, then

$$\int\limits_{-\infty}^{\infty} \widehat{f}(\xi)\mathrm{e}^{i2\pi x\xi}\,d\xi = f(x) \quad (x \in \mathbb{R}). \tag{2.23}$$

Proof. Fix $x \in \mathbb{R}$ and set

$$\Phi_x(\xi,\alpha) = \widehat{f}(\xi)\mathrm{e}^{i2\pi x\xi}\mathrm{e}^{-\pi\alpha^2\xi^2}.$$

Then $\Phi_x(\xi,\alpha)$ is a continuous function of $(\xi,\alpha) \in \mathbb{R}^2$. Furthermore,

$$|\Phi_x(\xi,\alpha)| \leq \left|\widehat{f}(\xi)\right|.$$

Since $\widehat{f}$ is absolutely integrable, the integral

$$\int_{-\infty}^{\infty} \Phi_x(\xi,\alpha)\,d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)\mathrm{e}^{i2\pi x\xi}\mathrm{e}^{-\pi\alpha^2\xi^2}\,d\xi$$

converges uniformly with respect to $\alpha \in \mathbb{R}$ and, by Theorem 17, this integral is a continuous function of the variable $\alpha \in \mathbb{R}$. The continuity at the point $\alpha = 0$ gives that

$$\int_{-\infty}^{\infty} \Phi_x(\xi,\alpha)\,d\xi \;\to\; \int_{-\infty}^{\infty} \Phi_x(\xi,0)\,d\xi = \int_{-\infty}^{\infty} \widehat{f}(\xi)\mathrm{e}^{i2\pi x\xi}\,d\xi \quad (\alpha \to 0+).$$

On the other hand, by Theorem 2.17,

$$\int_{-\infty}^{\infty} \Phi_x(\xi,\alpha)\,d\xi \to f(x) \quad (\alpha \to 0+).$$

Thus, we have (2.23). $\square$

Remark 2.20. The integral at the left-hand side of (2.23) is equal to $\widehat{\varphi}(x)$, where $\varphi(\xi) = \widehat{f}(-\xi)$. Thus, by Theorem 2.5, this integral is a continuous function of the variable x. This shows that equality (2.23) is not true without assumption that f is continuous. Moreover, it can be shown that if f has a jump discontinuity at some point, then the Fourier transform of f cannot be absolutely integrable on $\mathbb{R}$.

We have the following *uniqueness theorem for Fourier transform.*

Theorem 2.21. Let f and g be continuous, absolutely integrable functions on $\mathbb{R}$. If $\widehat{f} = \widehat{g}$, then $f = g$.

This statement follows immediately from Theorem 2.17 (or Theorem 2.19) applied to the function $f - g$.

Theorem 2.19 can be applied to evaluate some integrals.

Example 2.22. Let $f(x) = \mathrm{e}^{-2\pi|x|}$. As it was shown in Example 2.4,

$$\widehat{f}(\xi) = \frac{1}{\pi\left(1+\xi^2\right)}.$$

Functions f and $\widehat{f}$ are absolutely integrable on $\mathbb{R}$, and f is continuous on $\mathbb{R}$. Thus, the conditions of Theorem 2.19 hold. Applying this theorem, we obtain that for any $x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} \frac{e^{i2\pi x\xi}}{\pi\,(1+\xi^2)}\,d\xi = e^{-2\pi|x|},$$

or, equivalently,

$$2\int_{0}^{\infty} \frac{\cos 2\pi x\xi}{\pi\,(1+\xi^2)}\,d\xi = e^{-2\pi x}$$

for any $x \geq 0$ *(Laplace integral).* $\square$

Example 2.23. Let

$$f(x) = \begin{cases} 1-|x|/\alpha, & |x| \leq \alpha, \\ 0, & |x| > \alpha \end{cases} \qquad (\alpha > 0).$$

By Example 2.3 and Theorem 2.10 (iii),

$$\widehat{f}(\xi) = \frac{1}{\alpha}\frac{\sin^2 \alpha\pi\xi}{(\pi\xi)^2}.$$

Since the conditions of Theorem 2.19 hold, we have that

$$\frac{1}{\alpha}\int_{-\infty}^{\infty} \frac{\sin^2 \alpha\pi\xi}{(\pi\xi)^2} e^{i2\pi\xi x}\,d\xi = f(x)$$

for all $x \in \mathbb{R}$ and all $\alpha > 0$. In particular, if $x = 0$, $\alpha = 1$, then, setting $\pi\xi = t$, we obtain

$$\int_{0}^{\infty} \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}. \quad \square$$

As we have seen, the Fourier transform of the Gaussian $g(x) = e^{-\pi x^2}$ coincides with it, that is, $\widehat{g} = g$. Using Theorem 2.19, we consider the following

Example 2.24. Let $f \in \mathcal{A}(\mathbb{R})$ be a continuous function on $\mathbb{R}$ which is not identically zero and such that $\widehat{f}(\xi) = \lambda f(\xi)$ for some constant λ. Prove that λ is equal to one of the numbers $1, -1, i$ or $-i$.

Denote $f_1 = \widehat{f}$. We have $f_1 = \lambda f$. Thus, $f_1 \in \mathcal{A}(\mathbb{R})$ and therefore the Fourier transform $\widehat{f_1}$ exists. Set $f_2 = \widehat{f_1}$. Then $f_2 = \lambda^2 f$. On the other hand, by Theorem 2.19,

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(\xi) \mathrm{e}^{i2\pi x\xi}\, d\xi = \lambda \int_{-\infty}^{\infty} f(\xi) \mathrm{e}^{i2\pi x\xi}\, d\xi = \lambda \widehat{f}(-x).$$

Hence, $f_1(x) = f(-x)/\lambda$. From here,

$$f_2(\xi) = \widehat{f_1}(\xi) = \frac{1}{\lambda} \widehat{f}(-\xi) = \frac{1}{\lambda} f_1(-\xi) = \frac{1}{\lambda^2} f(\xi).$$

Taking into account that $f_2 = \lambda^2 f$, we obtain $f = \lambda^4 f$. Since $f \not\equiv 0$, this implies that $\lambda^4 = 1$. $\square$

2.3. Fourier inversion: Dirichlet's method

Theorem 2.19 shows that a continuous function $f \in \mathcal{A}(\mathbb{R})$ can be obtained by equality (2.23), provided its Fourier transform $\widehat{f}$ also is absolutely integrable over $\mathbb{R}$. However, the latter condition may not hold, and the integral at the left-hand side of (2.23) may not exist. The following *Dirichlet's Theorem* states that for functions satisfying some good smoothness conditions, this integral may converge in the sense of *principal value*, that is, as the limit

$$\text{p.v.} \int_{-\infty}^{\infty} \widehat{f}(\xi) \mathrm{e}^{i2\pi x\xi}\, d\xi = \lim_{A\to\infty} \int_{-A}^{A} \widehat{f}(\xi) \mathrm{e}^{i2\pi \xi x}\, d\xi.$$

Theorem 2.25. Let a function f be absolutely integrable on $\mathbb{R}$. If f is differentiable at a point x, then

$$\text{p.v.} \int_{-\infty}^{\infty} \widehat{f}(\xi) \mathrm{e}^{i2\pi x\xi}\, d\xi = f(x). \tag{2.24}$$

Proof. For any $A > 0$

$$I_A(x) = \int_{-A}^{A} \widehat{f}(\xi) \mathrm{e}^{i2\pi \xi x}\, d\xi = \int_{-A}^{A} \left(\int_{-\infty}^{\infty} f(y) \mathrm{e}^{-i2\pi \xi y}\, dy \right) \mathrm{e}^{i2\pi \xi x}\, d\xi$$

$$= \int_{-A}^{A} \left(\int_{-\infty}^{\infty} f(y) \mathrm{e}^{i2\pi\xi(x-y)} \, dy \right) \, d\xi.$$

Observe that the conditions of Theorem 19 (i) on the interchange of the order of integrations hold. Indeed, the interior integral converges uniformly with respect to ξ. Thus,

$$I_A(x) = \int_{-\infty}^{\infty} f(y) \left(\int_{-A}^{A} \mathrm{e}^{i2\pi\xi(x-y)} \, d\xi \right) \, dy. \tag{2.25}$$

Applying Euler's formula

$$\sin u = \frac{\mathrm{e}^{iu} - \mathrm{e}^{-iu}}{2i},$$

we have that the interior integral in (2.25) is equal to

$$\frac{1}{i2\pi(x-y)} \mathrm{e}^{i2\pi\xi(x-y)} \Bigg|_{\xi=-A}^{\xi=A} = \frac{\mathrm{e}^{i2\pi A(x-y)} - \mathrm{e}^{-i2\pi A(x-y)}}{i2\pi(x-y)} = \frac{\sin 2\pi A(x-y)}{\pi(x-y)}.$$

Thus,

$$I_A(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(y) \frac{\sin 2\pi A(x-y)}{x-y} \, dy.$$

Setting $y - x = t$, we get

$$I_A(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(x+t) \frac{\sin 2\pi At}{t} \, dt = \frac{1}{\pi} \int_{0}^{\infty} [f(x+t) + f(x-t)] \frac{\sin 2\pi At}{t} \, dt.$$

Observe that for any $a > 0$

$$\frac{2}{\pi} \int_{0}^{\infty} \frac{\sin at}{t} \, dt = 1$$

(this is Dirichlet's integral). Hence,

$$I_A(x) - f(x) = \frac{1}{\pi} \int_{0}^{\infty} \varphi_x(t) \frac{\sin 2\pi At}{t} \, dt, \tag{2.26}$$

where

$$\varphi_x(t) = f(x+t) + f(x-t) - 2f(x).$$

We show that the integral at the right hand side of (2.26) tends to 0 as $A \to +\infty$. Let $K \geq 1$. We write this integral as the sum of two integrals: over $[0, K]$ and over $[K, +\infty)$. We have

$$\left| \int_K^\infty \varphi_x(t) \frac{\sin 2\pi At}{t}\, dt \right| \leq \frac{1}{K} \int_K^\infty |f(x+t) + f(x-t)|\, dt + \left| 2f(x) \int_K^\infty \frac{\sin 2\pi At}{t}\, dt \right|$$

$$\leq \frac{1}{K} \int_{-\infty}^\infty |f(u)|\, du + 2|f(x)| \left| \int_{2\pi AK}^\infty \frac{\sin u}{u}\, du \right|.$$

We assume that $A \geq 1$. Both the terms of the right hand side tend to 0 as $K \to +\infty$. Let $\varepsilon > 0$; then there exists K such that

$$\left| \int_K^\infty \varphi_x(t) \frac{\sin 2\pi At}{t}\, dt \right| < \frac{\varepsilon}{2}. \tag{2.27}$$

Fix this K. Since f is differentiable at the point x,

$$\frac{\varphi_x(t)}{t} \to 0 \quad (t \to 0+).$$

Thus, the function $\frac{\varphi_x(t)}{t}$ is Riemann integrable on $[0, K]$. By the Riemann – Lebesgue Lemma (Theorem 2.6),

$$\int_0^K \frac{\varphi_x(t)}{t} \sin 2\pi At\, dt \to 0 \quad (A \to +\infty).$$

Therefore there exists $A_\varepsilon > 1$ such that

$$\left| \int_0^K \frac{\varphi_x(t)}{t} \sin 2\pi At\, dt \right| < \frac{\varepsilon}{2} \quad (A \geq A_\varepsilon).$$

Together with (2.27), this shows that the integral in (2.26) tends to 0 as $A \to +\infty$ and thus

$$I_A(x) \to f(x) \quad (A \to +\infty). \qquad \square$$

Similarly, we have

Theorem 2.26. Let f be an absolutely integrable and piecewise smooth function on $\mathbb{R}$. Then at every point $x \in \mathbb{R}$

$$\text{p.v.} \int_{-\infty}^{\infty} \widehat{f}(\xi) \mathrm{e}^{i2\pi\xi x}\, d\xi = \frac{f(x+) + f(x-)}{2}.$$

Example 2.27. Let $a > 0$ and

$$f(x) = \begin{cases} \mathrm{e}^{-ax}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

Then

$$\widehat{f}(\xi) = \int_{0}^{\infty} \mathrm{e}^{-ax} \mathrm{e}^{-i2\pi\xi x}\, dx = \frac{1}{a + i2\pi\xi}.$$

Using Theorem 2.26, we obtain

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i2\pi x\xi}}{a + 2\pi i\xi}\, d\xi = \begin{cases} \mathrm{e}^{-ax}, & x > 0, \\ 1/2, & x = 0, \\ 0, & x < 0. \end{cases}$$

Denote $f_-(x) = f(-x)$. Then $\widehat{f_-}(\xi) = \widehat{f}(-\xi)$.

Let $\varphi = f + f_-$, $\varphi(x) = \mathrm{e}^{-a|x|}$ (even extension of f). Then

$$\widehat{\varphi}(\xi) = \frac{2a}{a^2 + 2\pi^2\xi^2}.$$

By Theorem 2.19,

$$2a \int_{-\infty}^{\infty} \frac{\mathrm{e}^{i2\pi x\xi}}{a^2 + 4\pi^2\xi^2}\, d\xi = \mathrm{e}^{-a|x|} \quad (x \in \mathbb{R}),$$

or

$$\frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{e}^{iux}}{a^2 + u^2}\, du = \mathrm{e}^{-a|x|} \quad (x \in \mathbb{R}).$$

Equivalently,

$$\int_{0}^{\infty} \frac{\cos ux}{a^2 + u^2}\, du = \frac{\pi}{2a} \mathrm{e}^{-a|x|}$$

(*Laplace integral*). $\square$

2.4. Convolutions

In this section we consider a new operation which plays an extremely important role in mathematics, especially for integral transforms.

Let functions f and g be defined on $\mathbb{R}$. Then their *convolution* is the function $f * g$, defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy, \tag{2.28}$$

provided that the integral exists.

First, we have the following property of *symmetry*.

Proposition 2.28. Let functions f and g be defined on $\mathbb{R}$. Then $f * g = g * f$ provided one of them exists.

PROOF. Assume that the integral (2.28) exists. Setting $x - y = u$, we obtain that $y = x - u$ and

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \int_{-\infty}^{\infty} f(u)g(x-u)\,du.$$

By the definition, the latter integral is the convolution $g * f$. $\square$

Various conditions can be imposed on f and g to insure that the integral (2.28) is absolutely convergent for all $x \in \mathbb{R}$. We shall use the following proposition.

Proposition 2.29. Assume that f and g are defined on $\mathbb{R}$ and integrable in every bounded interval. Then each of the following conditions implies that the integral

$$\int_{-\infty}^{\infty} |f(x-y)g(y)|\,dy \tag{2.29}$$

converges uniformly with respect to $x \in \mathbb{R}$:

(i) functions f^2 и g^2 are integrable on $\mathbb{R}$;

(ii) one of functions f, g belongs to the class $\mathcal{A}(\mathbb{R})$, and the other is bounded on $\mathbb{R}$;

(iii) one of functions f, g vanishes outside of a bounded interval $[a, b]$.

PROOF. First we observe that for any $x \in \mathbb{R}$ and for any bounded interval $[\alpha, \beta]$ the product $f(x-y)g(y)$ is a Riemann integrable function of y in $[\alpha, \beta]$.

(i) Applying Schwartz inequality (Theorem 23), we have

$$\int_{\alpha}^{\beta} |f(x-y)g(y)|\, dy \le \left(\int_{\alpha}^{\beta} f^2(x-y)\, dy\right)^{1/2} \left(\int_{\alpha}^{\beta} g^2(y)\, dy\right)^{1/2}$$

$$\le \left(\int_{-\infty}^{\infty} f^2(x-y)\, dy\right)^{1/2} \left(\int_{\alpha}^{\infty} g^2(y)\, dy\right)^{1/2}.$$

Setting $u = x - y$ in the first integral on the right-hand side, we obtain

$$\int_{\alpha}^{\beta} |f(x-y)g(y)|\, dy \le \left(\int_{-\infty}^{\infty} f^2(u)\, du\right)^{1/2} \left(\int_{\alpha}^{\infty} g^2(y)\, dy\right)^{1/2}.$$

The integrand on the left-hand side is a non-negative function. Hence, for any $\alpha_0 \in \mathbb{R}$ and any $x \in \mathbb{R}$ the integral

$$\int_{\alpha_0}^{\infty} |f(x-y)g(y)|\, dy \tag{2.30}$$

converges. Moreover,

$$\int_{\alpha}^{\infty} |f(x-y)g(y)|\, dy \le \left(\int_{-\infty}^{\infty} f^2(u)\, du\right)^{1/2} \left(\int_{\alpha}^{\infty} g^2(y)\, dy\right)^{1/2}$$

for any α. The right-hand side doesn't depend on x, and the second integral on the right-hand side tends to zero as $\alpha \to +\infty$. This implies that the integral (2.30) converges uniformly with respect to $x \in \mathbb{R}$.

Similarly, we obtain that for any $\beta_0 \in \mathbb{R}$ the integral

$$\int_{-\infty}^{\beta_0} |f(x-y)g(y)|\, dy \tag{2.31}$$

converges uniformly with with respect to $x \in \mathbb{R}$.

(ii) Using the property of symmetry, we may assume that $g \in \mathcal{A}(\mathbb{R})$ and f is bounded,

$$|f(x)| \leq M \quad (x \in \mathbb{R}).$$

Then for any interval $[\alpha, \beta]$

$$\int_{\alpha}^{\beta} |f(x-y)g(y)|\, dy \leq M \int_{\alpha}^{\beta} |g(y)|\, dy \leq M \int_{\alpha}^{\infty} |g(y)|\, dy.$$

As above, this implies that for any $\alpha_0 \in \mathbb{R}$ the integral (2.30) converges uniformly with respect to $x \in \mathbb{R}$. Similar arguments yield that the integral (2.31) also converges uniformly with respect to $x \in \mathbb{R}$.

(iii) We assume that g vanishes outside of a bounded interval $[a, b]$. Then for all $x \in \mathbb{R}$

$$\int_{\alpha}^{\infty} |f(x-y)g(y)|\, dy = \int_{-\infty}^{\beta} |f(x-y)g(y)|\, dy = 0$$

for any $\alpha > b$ and any $\beta < a$. This implies the uniform convergence of the integral (2.29). $\square$

In the sequel we study convolutions $f * g$ under the assumption that functions f, g satisfy one of conditions (i)-(iii). Although these functions may be discontinuous at some points, their convolution is continuous.

Proposition 2.30. Let functions f and g be defined on $\mathbb{R}$ and integrable in every bounded interval. Assume that one of conditions (i)-(iii) of Proposition 2.29 holds. Then the convolution $\varphi = f * g$ is continuous on $\mathbb{R}$.

Proof. We give the proof only for the case (i) (in which we use this statement below in the proof of the Plansherel identity). Let the functions f^2 and g^2 be integrable in $\mathbb{R}$. Set $\varphi(x) = (f * g)(x)$.

First we assume that f is a continuous function which vanishes outside an interval $[a, b]$; then f is uniformly continuous on $\mathbb{R}$. Fix a point $x_0 \in \mathbb{R}$. Applying Schwartz inequality (Theorem 23), we have

$$|\varphi(x_0 + h) - \varphi(x_0)| \leq \int_{-\infty}^{\infty} |f(x_0 + h - y) - f(x_0 - y)||g(y)|\, dy$$

$$\le \left(\int_{-\infty}^{\infty} [f(x_0 + h - y) - f(x_0 - y)]^2 \, dy \right)^{1/2} \left(\int_{-\infty}^{\infty} g^2(y) \, dy \right)^{1/2}.$$

Let $\varepsilon > 0$. Since f is uniformly continuous on $\mathbb{R}$, there exists a $0 < \delta < 1$ such that if $|h| < \delta$, then

$$|f(x+h) - f(x)| < \varepsilon \quad (x \in \mathbb{R}).$$

Thus,

$$\int_{-\infty}^{\infty} [f(x_0 + h - y) - f(x_0 - y)]^2 \, dy$$

$$= \int_{a-1}^{b+1} [f(x+h) - f(x)]^2 \, dx \le \varepsilon^2 (b - a + 2)$$

if $|h| < \delta$. Setting $A = \left(\int_{-\infty}^{\infty} g^2(y) \, dy \right)^{1/2}$, we obtain

$$|\varphi(x_0 + h) - \varphi(x_0)| < \varepsilon A \sqrt{b - a + 2} \quad (|h| < \delta).$$

This implies that φ is continuous at the point x_0.

Let now f be an arbitrary function such that f^2 is integrable on $\mathbb{R}$. Let $\varepsilon > 0$. By virtue of the theorem on approximation (Theorem 15), there exists a continuous function f_ε with a compact support such that

$$\int_{-\infty}^{\infty} [f(x) - f_\varepsilon(x)]^2 \, dx < \varepsilon^2.$$

Let $\varphi_\varepsilon = f_\varepsilon * g$. As above, applying Schwartz inequality (Theorem 23), we obtain that

$$|\varphi(x) - \varphi_\varepsilon(x)| = \left| \int_{-\infty}^{\infty} g(x-y)[f(y) - f_\varepsilon(y)] \, dy \right|$$

$$\le A \left(\int_{-\infty}^{\infty} [f(y) - f_\varepsilon(y)]^2 \, dz \right)^{1/2} < A\varepsilon$$

for any $x \in \mathbb{R}$. As it was proved, φ_ε is uniformly continuous on $\mathbb{R}$. Let $x_0 \in \mathbb{R}$. Then there exists $\delta > 0$ such that if $|h| < \delta$, then

$$|\varphi_\varepsilon(x_0 + h) - \varphi_\varepsilon(x_0)| < \varepsilon.$$

Thus,

$$|\varphi(x_0 + h) - \varphi(x_0)| < (2A + 1)\varepsilon \quad (|h| < \delta).$$

This yields that φ is continuous at the point x_0. $\square$

The main result in this section is the following *Convolution Theorem.*

Theorem 2.31. Let functions f and g belong to $\mathcal{A}(\mathbb{R})$ and satisfy one of conditions (i), (ii) or (iii). Then $f * g \in \mathcal{A}(\mathbb{R})$ and

$$\widehat{f * g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi). \tag{2.32}$$

Proof. Let

$$I(x) = \int_{-\infty}^{\infty} |f(x - y)g(y)|\, dy.$$

By Proposition 2.30, the function $I = |f| * |g|$ is continuous on $\mathbb{R}$ and therefore it is integrable in each bounded interval. Further, let

$$J(y) = \int_{-\infty}^{\infty} |f(x - y)g(y)|\, dx.$$

Then

$$J(y) = |g(y)| \int_{-\infty}^{\infty} |f(x - y)|\, dx = |g(y)|\|f\|_1.$$

Thus, $J(y)$ is integrable in each bounded interval; moreover,

$$\int_{-\infty}^{\infty} J(y)\, dy = \|f\|_1 \int_{-\infty}^{\infty} |g(y)|\, dy = \|f\|_1\|g\|_1.$$

Applying Theorem 20 to the function $F(x, y) = |f(x - y)g(y)|$, we have that the integral $\int_{-\infty}^{\infty} I(x)\, dx$ also converges. Since $|f * g(x)| \le I(x)$, it follows that $f * g \in \mathcal{A}(\mathbb{R})$.

Now we consider the Fourier transform of the convolution

$$\widehat{f*g}(\xi)=\int_{-\infty}^{\infty} f*g(x)\mathrm{e}^{-i2\pi x\xi}\,dx=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(x-y)g(y)\,dy\right)\mathrm{e}^{-i2\pi x\xi}\,dx.$$

As above, using Theorem 20, we interchange the order of integrations in the last iterated integral. Thus, we obtain

$$\widehat{f*g}(\xi)=\int_{-\infty}^{\infty} g(y)\left(\int_{-\infty}^{\infty} f(x-y)\mathrm{e}^{-i2\pi x\xi}\,dx\right)dy$$

$$=\int_{-\infty}^{\infty} g(y)\mathrm{e}^{-i2\pi y\xi}\,dy\int_{-\infty}^{\infty} f(u)\mathrm{e}^{-i2\pi u\xi}\,du=\widehat{g}(\xi)\widehat{f}(\xi).\quad\square$$

2.5. Plansherel identity

In this section we prove one of the fundamental theorems in the theory of Fourier transforms – *Plansherel's identity*. It can be interpreted as a continuous counterpart of Parseval's identity.

We shall consider complex-valued functions $f=u+iv$, where u and v are real-valued functions on $\mathbb{R}$. If $u,v\in\mathcal{A}(\mathbb{R})$, then we say that $f\in\mathcal{A}(\mathbb{R})$. In this case the Fourier transform of the function f is defined, as usual, by

$$\widehat{f}(\xi)=\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i2\pi x\xi}\,dx.$$

Theorem 2.32. Let a complex-valued function f belong to $\mathcal{A}(\mathbb{R})$. Assume that $|f|^2$ is integrable on $\mathbb{R}$. Then $\left|\widehat{f}\right|^2$ also is integrable on $\mathbb{R}$ and

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx=\int_{-\infty}^{\infty}\left|\widehat{f}(\xi)\right|^2\,d\xi. \tag{2.33}$$

Proof. Set $g(x)=\overline{f(-x)}$. Then g and $|g|^2$ are integrable on $\mathbb{R}$. Let $h=f*g$. By the Convolution Theorem (Theorem 2.31), $h\in\mathcal{A}(\mathbb{R})$ and

$\widehat{h}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi)$. But

$$\widehat{g}(\xi) = \int_{-\infty}^{\infty} \overline{f(-x)} \mathrm{e}^{-i2\pi x\xi}\, dx = \int_{-\infty}^{\infty} \overline{f(y)} \mathrm{e}^{i2\pi y\xi}\, dy = \overline{\widehat{f}(\xi)}.$$

Thus, $\widehat{h}(\xi) = \left|\widehat{f}(\xi)\right|^2$.

By Proposition 2.30, the function h is continuous on $\mathbb{R}$. Applying Theorem 2.17, we have that

$$\int_{-\infty}^{\infty} \widehat{h}(\xi) \mathrm{e}^{-\pi\alpha^2\xi^2} \mathrm{e}^{i2\pi\xi x}\, d\xi \to h(x) \quad (\alpha \to 0)$$

for any $x \in \mathbb{R}$. Taking $x = 0$, we obtain

$$\Phi(\alpha) \equiv \int_{-\infty}^{\infty} \widehat{h}(\xi) \mathrm{e}^{-\pi\alpha^2\xi^2}\, d\xi \to h(0) \quad (\alpha \to 0). \tag{2.34}$$

Observe that $\widehat{h}(\xi) = \left|\widehat{f}(\xi)\right|^2 \geq 0$. Hence, Φ is a non-negative non-increasing function on $(0, \infty)$ and, by (2.34), $\Phi(\alpha) \leq h(0)$ for $\alpha > 0$. Further, if $|\xi| \leq 1/\alpha$ $(\alpha > 0)$, then $\mathrm{e}^{-\pi\alpha^2\xi^2} \geq \mathrm{e}^{-\pi}$. Thus,

$$\int_{-1/\alpha}^{1/\alpha} \widehat{h}(\xi)\, d\xi \leq \mathrm{e}^{\pi} h(0)$$

for any $\alpha > 0$. This implies that the integral

$$\int_{-\infty}^{\infty} \widehat{h}(\xi)\, d\xi$$

converges. Hence, we can apply Theorem 2.19 (Fourier inversion). By this theorem,

$$h(x) = \int_{-\infty}^{\infty} \mathrm{e}^{i2\pi x\xi} \widehat{h}(\xi)\, d\xi$$

for any $x \in \mathbb{R}$. For $x = 0$ this gives that

$$h(0) = \int_{-\infty}^{\infty} \widehat{h}(\xi)\, d\xi.$$

Since $\widehat{h}(\xi) = \left|\widehat{f}(\xi)\right|^2$, then $\left|\widehat{f}\right|^2$ is integrable on $\mathbb{R}$ and

$$h(0) = \int_{-\infty}^{\infty} \left|\widehat{f}(\xi)\right|^2 \, d\xi.$$

On the other hand,

$$h(0) = \int_{-\infty}^{\infty} g(-t) f(t) \, dt = \int_{-\infty}^{\infty} |f(t)|^2 \, dt.$$

This implies (2.33). $\square$

Example 2.33. Let

$$f(x) = \begin{cases} 1, & |x| \le \frac{1}{2}, \\ 0, & |x| > \frac{1}{2}. \end{cases}$$

Then $\int\limits_{-\infty}^{\infty} f^2(x) \, dx = 1$. Further,

$$\widehat{f}(\xi) = \frac{\sin \pi\xi}{\pi\xi}.$$

Thus,

$$\int_{-\infty}^{\infty} \frac{\sin^2 \pi\xi}{(\pi\xi)^2} \, d\xi = \int_{-\infty}^{\infty} \left|\widehat{f}(\xi)\right|^2 \, d\xi = \int_{-\infty}^{\infty} f^2(x) \, dx = 1.$$

From here,

$$\int_{-\infty}^{\infty} \frac{\sin^2 z}{z^2} \, dz = \pi. \quad \square$$

This equality has been obtained above (see Example 2.23).

Exercises to Chapter 2

2.1. Evaluate Fourier transforms of the following functions

2.1.a.
$$f(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| > 1; \end{cases}$$

2.1.b.
$$f(x) = \begin{cases} 1, & 0 < x < 1, \\ -1, & -1 < x < 0, \\ 0, & |x| > 1; \end{cases}$$

2.1.c.
$$f(x) = \mathrm{e}^{-|x|};$$

2.1.d.
$$f(x) = \mathrm{e}^{-|x|} \cos 2\pi x;$$

2.1.e.
$$f(x) = \mathrm{e}^{-|x|} + i\Pi(x)$$

(the definition of the function Π see in Example 2.2);

2.1.f.
$$f(x) = \Lambda(x) + i\mathrm{e}^{-3|x|} \cos 2\pi x$$

(the definition of the function Λ see in Example 2.3).

2.2. *Cosine-transform.*

2.2.a. Prove that if a function $f \in \mathcal{A}(\mathbb{R})$ is even, then

$$\widehat{f}(u) = 2 \int_0^\infty f(x) \cos 2\pi ux \, dx.$$

2.2.b. For a function $f \in \mathcal{A}(0, \infty)$ its *cosine-transform* $\widehat{f}_C$ is defined by $\widehat{f}_C(u) = 2 \int_0^\infty f(x) \cos 2\pi ux \, dx$. Prove that the cosine-transform of a function f is equal to the Fourier transform of the even extension of f.

2.3. Evaluate cosine-transforms of the following functions

2.3.a.
$$f(x) = \mathrm{e}^{-x};$$

2.3.b.
$$h(x) = \begin{cases} 1, & 0 < x < 2, \\ 0, & x > 2; \end{cases}$$

2.3.c.
$$k(x) = \begin{cases} 1 - 2x, & 0 < x < 1/2, \\ 0, & x > 1/2. \end{cases}$$

2.4. Evaluate Fourier transforms of the following functions

2.4.a.
$$f(x) = \begin{cases} x, & |x| \le a, \\ 0, & |x| > a; \end{cases}$$

2.4.b.
$$f(x) = \begin{cases} \cos x, & |x| \le \pi, \\ 0, & |x| > \pi; \end{cases}$$

2.4.c.
$$f(x) = \begin{cases} \sin x, & |x| \le \pi, \\ 0, & |x| > \pi; \end{cases}$$

2.4.d.
$$f(x) = |x|\mathrm{e}^{-|x|};$$

2.4.e.
$$f(x) = \begin{cases} \mathrm{e}^{x}, & x < 0, \\ -\mathrm{e}^{-x}, & x \ge 0. \end{cases}$$

2.5. Evaluate Fourier transforms of the following functions

2.5.a.
$$f(x) = \Pi\left(x - \frac{1}{2}\right);$$

2.5.b.
$$f(x) = \Pi\left(\frac{x - \frac{1}{2}a}{a}\right) \quad (a > 0);$$

2.5.c.
$$f(x) = \Pi(x)\operatorname{sign} x;$$

2.5.d.
$$f(x) = \mathrm{e}^{-c|x-b|} \quad (c > 0,\ b \in \mathbb{R});$$

2.5.e. $$f(x) = \mathrm{e}^{-(x-b)^2/c} \quad (c > 0,\ b \in \mathbb{R});$$

2.5.f. $$f(x) = \mathrm{e}^{-c|x|} \sin bx \quad (c > 0,\ b \in \mathbb{R}).$$

2.6. Prove that $\int\limits_0^\infty \mathrm{e}^{-ax} \sin cx\, dx = \frac{c}{a^2+c^2}$ and $\int\limits_0^\infty \mathrm{e}^{-ax} \cos cx\, dx = \frac{a}{a^2+c^2}$, where $a > 0$, $c \in \mathbb{R}$.

2.7. Evaluate Fourier transforms of the following functions

2.7.a. $$f(x) = \begin{cases} 1, & |x| < \frac{1}{2}c, \\ 0, & |x| > \frac{1}{2}c; \end{cases}$$

2.7.b. $$f(x) = \begin{cases} \cos \frac{\pi x}{c}, & |x| < \frac{1}{2}c, \\ 0, & |x| > \frac{1}{2}c; \end{cases}$$

2.7.c. $$f(x) = \begin{cases} \cos^2 \frac{\pi x}{c}, & |x| < \frac{1}{2}c, \\ 0, & |x| > \frac{1}{2}c; \end{cases}$$

2.7.d. $$f(x) = \frac{1}{c} \Lambda\left(\frac{x}{c}\right) \quad (c > 0);$$

2.7.e. $$f(x) = x^2 \mathrm{e}^{-\pi x^2};$$

2.7.f. $$f(x) = \left(4\pi x^2 - 1\right) \mathrm{e}^{-\pi x^2}.$$

2.8. Using the equality $\int\limits_0^\infty \frac{\sin t}{t}\, dt = \frac{\pi}{2}$, evaluate the Fourier transform of the function $f(x) = \frac{\sin ax}{x} \quad (a > 0)$.

2.9. Evaluate the Fourier transforms of the following functions

2.9.a. $$f(x) = \mathrm{e}^{-a|x|} \quad (a > 0);$$

2.9.b. $$f(x) = \mathrm{e}^{-4x^2-4x-1};$$

2.9.c.
$$f(x) = x\mathrm{e}^{-x^2}.$$

2.10. Let $H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$ Evaluate the Fourier transforms of the following functions

2.10.a.
$$f(x) = H(x)\mathrm{e}^{-ax} \quad (a > 0);$$

2.10.b.
$$f(x) = H(x)\mathrm{e}^{-ax} \cos bx \quad (a > 0,\ b \neq 0);$$

2.10.c.
$$f(x) = H(x)\mathrm{e}^{-ax} \sin bx \quad (a > 0,\ b \neq 0).$$

2.11. Let $f \in \mathcal{A}(\mathbb{R})$. Evaluate the Fourier transforms of the following functions

2.11.a.
$$f(-x);$$

2.11.b.
$$f\left(x - x_0\right) \quad (x_0 \in \mathbb{R});$$

2.11.c.
$$f(x)\mathrm{e}^{i\xi_0 x} \quad (\xi_0 \in \mathbb{R});$$

2.11.d.
$$f(x) \sin \xi_0 x;$$

2.11.e.
$$f(3x)\mathrm{e}^{ix};$$

2.11.f.
$$f(2x).$$

2.12. Assume that f is continuously differentiable, f' is piecewise smooth, f, f', f'' and $xf'(x)$ are absolutely integrable on $\mathbb{R}$. Apply Fourier transform to solve the equation

$$f''(x) + xf'(x) + f(x) = 0, \quad f(0) = 1,\ f'(0) = 0.$$

2.13. We shall write $f(x) \supset \widehat{f}(u)$ to denote that $\widehat{f}(u)$ is the Fourier transform of a function $f(x)$. Prove that if f and $\widehat{f}$ are absolutely integrable, and f is continuous, then $\widehat{f}(x) \supset f(-u)$. Using this result, verify that

2.13.a.
$$\frac{1}{1+x^2} \supset \pi \mathrm{e}^{-2\pi|u|};$$

2.13.b.
$$\frac{\sin^2 \pi x}{\pi^2 x^2} \supset \Lambda(u).$$

2.14. Using 2.13.b, prove the equality $\int\limits_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi.$

2.15. Prove that if the second derivative f'' belongs to $\mathcal{A}(\mathbb{R})$ and is piecewise continuous, then $\widehat{f} \in \mathcal{A}(\mathbb{R})$.
Hint: Apply Corollary 2.14.

2.16. Assume that functions $f,\ g \in \mathcal{A}(\mathbb{R})$ are piecewise continuous. Prove that if $\widehat{f} = \widehat{g}$, then $f = g$, with a possible exception of a set which is finite in each bounded interval.

2.17. Give an example of a continuous bounded function on $\mathbb{R}$ which is not absolutely integrable. Conversely, give an example of a continuous absolutely integrable function which is unbounded.

2.18. Let a piecewise continuous function $f \in \mathcal{A}(\mathbb{R})$ have points of unremovable discontinuity and is such that $0 < \int\limits_{-\infty}^{\infty} |f(x)|\, dx < \infty$. Prove that $\widehat{f}$ cannot be absolutely integrable.

2.19.a. Prove that $i\pi \mathrm{e}^{-2\pi|x|} \operatorname{sign} x \supset \frac{u}{1+u^2}$.

2.19.b. Show that $g(u) = \frac{u}{1+u^2}$ is not absolutely integrable.

2.20.a. Using Example 2.2 and basic properties of Fourier transforms, show that $\frac{i}{2}\Pi\left(x - \frac{1}{2}\right) - \frac{i}{2}\Pi\left(x + \frac{1}{2}\right) \supset \frac{\sin^2 \pi u}{\pi u}$.

2.20.b. Show that $g(u) = \frac{\sin^2 \pi u}{\pi u}$ is not absolutely integrable.

2.21. A function f is said to satisfy *two-sided Lipschitz condition at a point* x if

$$|f(x+s) - f(x+)| \le A\, s^{\alpha} \quad (\delta_1 > s > 0),$$

$$|f(x+s) - f(x-)| \le B\, s^{\beta} \quad (-\delta_2 < s < 0),$$

where A, B, α, β, δ_1, δ_2 are some positive constants. Prove the following theorem.

Theorem. *Let $f \in \mathcal{A}(\mathbb{R})$ be a piecewise continuous function. If f satisfies two-sided Lipschitz condition at a point x, then*

$$\text{p.v.} \int_{-\infty}^{\infty} \widehat{f}(u) \mathrm{e}^{2\pi i u x}\, du = \frac{1}{2}[f(x+) + f(x-)].$$

2.22. Let $f \in \mathcal{A}(\mathbb{R})$ be a continuous function. Assume that

$$\widehat{f}(\xi) = \begin{cases} 1 - \xi^2, & |\xi| \le 1, \\ 0, & |\xi| > 1. \end{cases}$$

Find f.

2.23. Let $f \in \mathcal{A}(\mathbb{R})$ be a continuous function. Let

$$\widehat{f}(\xi) = \begin{cases} 1 - |\xi|, & |\xi| \le 1, \\ 0, & |\xi| > 1. \end{cases}$$

Find f.

2.24. Using the Fourier inversion formula, prove the following equalities

2.24.a.
$$\int_0^{\infty} \frac{\cos \xi x}{a^2 + \xi^2}\, d\xi = \frac{\pi}{2a} \mathrm{e}^{-a|x|} \quad (x \in \mathbb{R},\ a > 0);$$

2.24.b.
$$\int_0^{\infty} \mathrm{e}^{-\frac{1}{2}\sigma^2\xi^2} \cos \xi x\, d\xi = \frac{1}{\sigma}\sqrt{\frac{\pi}{2}} \mathrm{e}^{-\frac{x^2}{2\sigma^2}} \quad (x \in \mathbb{R},\ \sigma > 0);$$

2.24.c.
$$\frac{2}{\pi} \int_0^{\infty} \frac{\sin^2 \xi}{\xi^2} \cos(2\xi x)\, d\xi = \begin{cases} 1 - |x|, & |x| \le 1, \\ 0, & |x| > 1. \end{cases}$$

2.24.d. Using exercise 2.24.c, derive the equality

$$\int_0^{\infty} \frac{\sin^2 \xi}{\xi^2}\, d\xi = \frac{\pi}{2}.$$

2.25. Find a continuous and absolutely integrable function f on $[0, \infty)$ such that

$$\int_0^\infty f(x) \cos \xi x \, dx = \frac{1}{1+\xi^2} \quad (\xi \in \mathbb{R}).$$

Is f determined uniquely?

2.26. Find a continuous and absolutely integrable function f on $[0, \infty)$ such that

$$\int_0^\infty f(x) \cos \xi x \, dx = \frac{1-\cos \xi}{\xi^2} \quad (\xi \in \mathbb{R}).$$

Is f determined uniquely?

2.27. Using the Fourier transform of the function Π from Example 2.2 and Plansherel identity (2.33), prove that

$$\int_0^\infty \frac{\sin^2 \xi}{\xi^2} \, d\xi = \frac{\pi}{2}.$$

2.28. Let $\widehat{g}(\xi) = (\xi \cos \xi - \sin \xi)/\xi^2$.

2.28.a. Using the rule of differentiation of Fourier transform, find g.

2.28.b. Evaluate the integral

$$\int_0^\infty \left[\frac{x \cos x - \sin x}{x^2} \right]^2 dx.$$

2.29. Prove the equality

$$\frac{x \sin \pi x}{1-x^2} = \frac{1}{2} \left[\frac{\sin \pi(1-x)}{1-x} + \frac{\sin \pi(1+x)}{1+x} \right] \quad (x \neq \pm 1).$$

Using this equality, evaluate:

2.29.a.
$$\int_0^\infty \frac{x \sin \pi x}{1-x^2} \, dx;$$

2.29.b.
$$\int_0^\infty \left(\frac{x \sin \pi x}{1-x^2} \right)^2 dx.$$

2.30. Let $W(t,\alpha)$ be the Gauss – Weierstrass kernel (its properties see in Lemma 2.16). Denote the right-hand side of equality (2.18) (see Lemma 2.15) by

$$ {}_{\alpha}W * f(x) = \int\limits_{-\infty}^{\infty} f(x-t)W(t,\alpha)\,dt = \int\limits_{-\infty}^{\infty} f(s)W(x-s,\alpha)\,ds. $$

Let a function $f \in \mathcal{A}(\mathbb{R})$ be bounded and piecewise continuous. Prove that ${}_{\alpha}W * f(x)$ uniformly with respect to x in $\mathbb{R}$ tends to zero as $\alpha \to \infty$.

2.31. Assuming that $f \in \mathcal{A}(\mathbb{R})$, prove that

$$ \int\limits_{-\infty}^{\infty} |{}_{\alpha}W * f(x)|\,dx \le \int\limits_{-\infty}^{\infty} |f(x)|\,dx. $$

2.32. Let a function $f \in \mathcal{A}(\mathbb{R})$ be continuous and $m \le f(x) \le M$ $(x \in \mathbb{R})$. Prove that $m \le {}_{\alpha}W * f(x) \le M$ $(x \in \mathbb{R})$.

2.33. Using the definition of a convolution, prove that $\Pi * \Pi(x) = \Lambda(x)$.

2.34. Evaluate the convolution $\left[\frac{1}{a^2+x^2}\right] * \left[\frac{1}{b^2+x^2}\right]$.

2.35. Let $f \in \mathcal{A}(\mathbb{R})$ and let

$$ g(x) = \int\limits_{-\infty}^{\infty} f(x-u)\frac{1-\cos u}{u^2}\,du, \quad x \in \mathbb{R}. $$

Express $\widehat{g}$ in terms of $\widehat{f}$.

2.36. Using Fourier transforms, solve the equations

2.36.a.

$$ \int\limits_{-\infty}^{\infty} f(t)f(x-t)\,dt = \mathrm{e}^{-4\pi x^2}; $$

2.36.b.

$$ \int\limits_{-\infty}^{\infty} f(t)f(x-t)\,dt = \frac{1}{1+x^2}. $$

3. Legendre polynomials

3.1. Definition and recursion formula

The classical *Legendre polynomials* are algebraic polynomials that form an orthogonal system on the interval $[-1,1]$.

First we introduce a general notion of a generating function which will be used to define the Legendre polynomials.

Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of numbers. We consider the power series

$$\sum_{n=0}^{\infty} a_n r^n.$$

If this series has non-zero radius of convergence, then its sum is called the *generating function* for the sequence $\{a_n\}$.

For example, if all $a_n = 1$, then

$$F(r) = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}, \quad |r| < 1.$$

Set

$$F(x,r) = \frac{1}{(1-2rx+r^2)^{1/2}}.$$

For a fixed $x \in [-1,1]$, we consider $F(x,r)$ as a function of r. If $x = 1$, then

$$F(1,r) = \frac{1}{1-r} = \sum_{n-0}^{\infty} r^n.$$

If $x = -1$, then

$$F(-1,r) = \frac{1}{1+r} = \sum_{n=0}^{\infty} (-1)^n r^n.$$

Now we fix an arbitrary $x \in [-1,1]$. The function $r \mapsto F(x,r)$ is infinitely differentiable at $r = 0$. Denote by $P_n(x)$ the nth Taylor coefficient of this function at $r = 0$, that is,

$$P_n(x) = \frac{1}{n!} \cdot \frac{\partial^n F(x,0)}{\partial r^n}. \tag{3.1}$$

In particular,

$$P_0(x) = 1, \quad P_1(x) = x. \tag{3.2}$$

Thus, the Taylor expansion of $F(x,r)$ with respect to r is the power series

$$\sum_{n=0}^{\infty} P_n(x) r^n. \tag{3.3}$$

It will be shown below that this series converges to $F(x,r)$ for $|x| \leq 1$ and $|r| < 1$. For small $|r|$ one can prove it with the use of the Taylor expansion for $(1-z)^{-1/2}$ for $|z| < 1$, taking $z = 2xr - r^2$, and applying the binomial formula to each power of $(2xr - r^2)^n$.

We shall derive the *recursion formula* for $P_n(x)$. We observe that it is not necessary to use the convergence of the series (3.3) since we can operate only with Taylor coefficients.

We have

$$\frac{\partial F}{\partial r}(x,r) = \frac{x-r}{(1-2xr+r^2)^{3/2}}.$$

Thus,

$$\left(1-2xr+r^2\right) \frac{\partial F}{\partial r}(x,r) = (x-r)F(x,r). \tag{3.4}$$

It follows from (3.3) that the Taylor series for $\partial F/\partial r$ is

$$\sum_{n=0}^{\infty} (n+1) P_{n+1}(x) r^n$$

(this series is obtained by a formal term-by-term differentiation of the series (3.3)). Therefore for any $n \geq 1$ the Taylor coefficient of r^n for the function at the left-hand side of (3.4) is

$$(n+1)P_{n+1}(x) - 2nxP_n(x) + (n-1)P_{n-1}(x).$$

On the other hand, for the function at the right-hand side of (3.4) the Taylor coefficient of r^n is

$$xP_n(x) - P_{n-1}(x).$$

Since these coefficients are equal, we get

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0. \tag{3.5}$$

This relation is called a *recursion formula.* It follows immediately by induction and (3.2) that $P_n(x)$ is an algebraic polynomial of degree n. The polynomial $P_n(x)$ is called *Legendre polynomial.* The function $F(x,r)$ is called the *generating function* for the Legendre polynomials.

Proposition 3.1. The system of Legendre polynomials has the following properties:

(i) the leading coefficient of $P_n(x)$ is

$$a_n = \frac{1 \cdot 3 \cdot \dots \cdot (2n-1)}{n!};$$

(ii) $P_{2n}(x)$ contains only *even* powers of x; $P_{2n-1}(x)$ contains only *odd* powers of x. Thus, $P_n(-x) = (-1)^n P_n(x)$;

(iii) $P_n(1) = 1$, $P_n(-1) = (-1)^n$;

(iv) for each $n \in \mathbb{N}$ the power x^n is a linear combination of $P_k(x)$, $k = 0, \dots, n$;

(v) any algebraic polynomial Q of degree m can be represented in the form

$$Q(x) = \sum_{k=0}^{m} a_k P_k(x).$$

Proof. Indeed, for the proof of (i) we observe that by the recursion formula (3.5),

$$(n+1)a_{n+1} = (2n+1)a_n,$$

and thus

$$a_{n+1} = \frac{2n+1}{n+1} a_n.$$

It remains to apply the induction.

Statements (ii) – (iv) also can be easily proved by induction, with the use of the recursion formula (3.5); (v) follows from (iv). $\square$

3.2. Rodrigues formula

The Rodrigues formula gives an explicit expression for Legendre polynomials. Sometimes it is used for the definition of these polynomials.

We shall apply the following *Leibniz's rule.* Let $u = u(x)$, $v = v(x)$ be n times differentiable functions. Then

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(n-k)} v^{(k)}.$$

Theorem 3.2. For any $n \in \mathbb{N}$

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n. \tag{3.6}$$

PROOF. Denote by $R_n(x)$ the right hand side of (3.6). It is obvious that

$$R_0(x) = 1 = P_0(x) \quad \text{and} \quad R_1(x) = x = P_1(x).$$

The theorem will be proved if we show that R_n satisfy the same recursion formula as P_n.

We have

$$\frac{d}{dx}\left(x^2 - 1\right)^{n+1} = 2(n+1)x\left(x^2 - 1\right)^n.$$

Thus,

$$\begin{aligned} R_{n+1}(x) &= \frac{1}{2^{n+1}(n+1)!} \frac{d^{n+1}}{dx^{n+1}} \left(x^2 - 1\right)^{n+1} \\ &= \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[x\left(x^2 - 1\right)^n\right]. \end{aligned} \tag{3.7}$$

By Leibniz's rule, we obtain

$$\begin{aligned} R_{n+1}(x) &= \frac{1}{2^n n!} \left[x \frac{d^n}{dx^n} \left(x^2 - 1\right)^n + n \frac{d^{n-1}}{dx^{n-1}} \left(x^2 - 1\right)^n\right] \\ &= xR_n(x) + \frac{n}{2^n n!} \Phi_n(x), \end{aligned}$$

where

$$\Phi_n(x) = \frac{d^{n-1}}{dx^{n-1}} \left(x^2 - 1\right)^n.$$

Performing one differentiation on the right hand side of (3.7), we have

$$\left(x\left(x^2-1\right)^n\right)' = \left(x^2-1\right)^n + 2nx^2\left(x^2-1\right)^{n-1}$$
$$= (2n+1)\left(x^2-1\right)^n + 2n\left(x^2-1\right)^{n-1}.$$

From here and (3.7),

$$R_{n+1}(x) = \frac{1}{2^n n!}\frac{d^{n-1}}{dx^{n-1}}\left[(2n+1)\left(x^2-1\right)^n + 2n\left(x^2-1\right)^{n-1}\right]$$
$$= R_{n-1}(x) + \frac{2n+1}{2^n n!}\Phi_n(x).$$

Thus,

$$R_{n+1}(x) - R_{n-1}(x) = \frac{2n+1}{2^n n!}\Phi_n(x). \tag{3.8}$$

Before we have already obtained that

$$R_{n+1}(x) - xR_n(x) = \frac{n}{2^n n!}\Phi_n(x).$$

From these equalities it follows that

$$(n+1)R_{n+1}(x) - (2n+1)xR_n(x) + nR_{n-1}(x) = 0,$$

which coincides with the recursion formula for P_n. Thus, $R_n = P_n$. $\square$

From (3.8) we obtain *the second recursion formula* for Legendre polynomials.

Theorem 3.3. For any $n \in \mathbb{N}$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x). \tag{3.9}$$

3.3. Orthogonality

In this section we show that the Legendre polynomials form an orthogonal system on $[-1,1]$.

Theorem 3.4. The Legendre polynomials satisfy the following orthogonality relations

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & m \neq n, \\ 2/(2n+1), & m = n. \end{cases}$$

PROOF. Using Rodrigues formula, we have for any function $\varphi \in C^{(n)}[-1,1]$

$$2^n n! \int_{-1}^{1} \varphi(x) P_n(x)\, dx = \int_{-1}^{1} \varphi(x) \frac{d^n}{dx^n} \left(x^2-1\right)^n \, dx$$

$$= (-1)^n \int_{-1}^{1} \varphi^{(n)}(x) \left(x^2-1\right)^n \, dx.$$

The last equality follows by a successive n-fold integration by parts; all the integrated terms are equal to zero. If $m < n$ and $\varphi(x) = P_m(x)$, then $\varphi^{(n)}(x) = 0$. Thus,

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad (m \neq n). \tag{3.10}$$

Now, we denote

$$C_n = \int_{-1}^{1} P_n^2(x)\, dx.$$

Let a_n be the leading coefficient of P_n. We have

$$P_n(x) = \frac{a_n}{a_{n-1}} x P_{n-1}(x) + Q(x),$$

where $Q(x)$ is a polynomial of degree not greater than $n-1$. By Proposition 3.1 (v) and equality (3.10), polynomials P_n and Q are orthogonal on $[-1,1]$, and hence

$$C_n = \frac{a_n}{a_{n-1}} \int_{-1}^{1} P_n(x) x P_{n-1}(x)\, dx.$$

By the recursion formula (3.5),

$$x P_n(x) = \frac{1}{2n+1} \left[(n+1) P_{n+1}(x) + n P_{n-1}(x)\right].$$

This gives that

$$C_n = \frac{a_n}{a_{n-1}} \frac{n}{2n+1} C_{n-1}.$$

By Proposition 3.1 (i),

$$\frac{a_n}{a_{n-1}} = \frac{2n-1}{n}.$$

Thus,

$$C_n = \frac{2n-1}{2n+1} C_{n-1}.$$

Besides, $C_0 = 2$. Thus, $C_n = 2/(2n+1)$. $\square$

Corollary 3.5. For any $n \in \mathbb{N}$ the polynomial P_n on the interval $[-1, 1]$ is orthogonal to any polynomial Q of the degree not greater than $n - 1$.

Indeed, by Proposition 3.1 (v), any such polynomial Q is represented in the form

$$Q(x) = \sum_{k=0}^{n-1} c_k P_k(x).$$

It remains to apply Theorem 3.4.

To some extent, the converse statement also is true.

Proposition 3.6. Let G be an algebraic polynomial of degree n, $n \in \mathbb{N}$. Assume that the polynomial G on the interval $[-1, 1]$ is orthogonal to any polynomial of degree $\leq n-1$. Then $G(x) = c\, P_n(x)$, where c is some constant.

Proof. By Proposition 3.1 (v), the polynomial G can be represented in the form

$$G(x) = \sum_{k=0}^{n} c_k P_k(x).$$

By Theorem 3.4, we have

$$\int_{-1}^{1} G(x) P_m(x)\, dx = \frac{2}{2m+1} c_m$$

for any $m \leq n - 1$. On the other hand, by our assumption, the integral at the left-hand side is equal to zero. Hence, $c_m = 0$ for any $m \leq n - 1$, and $G(x) = c_n P_n(x)$. $\square$

So, $\{P_n(x)\}_{n=0}^{\infty}$ is an orthogonal system on $[-1, 1]$. We shall consider expansions into Fourier series with respect to this system. Let f be a Riemann integrable function on $[-1,1]$. Taking into account that

$$\|P_n\|_2 = \sqrt{\frac{2}{2n+1}},$$

and applying formula (1.17), we define *the Fourier-Legendre coefficients* of the function f by

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)\,dx \quad (n = 0, 1, ...). \tag{3.11}$$

Example 3.7. Expand the function

$$f(x) = \begin{cases} 1, & 0 < x \leq 1, \\ 0, & -1 \leq x \leq 0 \end{cases}$$

in series of Legendre polynomials.

Solution. By (3.11), we have

$$c_n = \frac{2n+1}{2}\int_{0}^{1} P_n(x)\,dx \quad (n = 0, 1, ...).$$

First, $c_0 = 1/2$. Further, for $n \geq 1$ we apply Rodrigues formula (3.6). This gives

$$c_n = \frac{2n+1}{2^{n+1}n!}\int_0^1 \frac{d^n}{dx^n}\left(x^2-1\right)^n\,dx = \frac{2n+1}{2^{n+1}n!}\frac{d^{n-1}}{dx^{n-1}}\left(x^2-1\right)^n\Big|_0^1.$$

Set $\varphi(x) = (x^2-1)^n$. Clearly, $\varphi^{(n-1)}(1) = 0$. On the other hand, by the binomial formula,

$$\varphi(x) = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}x^{2k}.$$

The derivatives of order $n-1$ of the powers x^{2k} at $x = 0$ are equal to 0 unless $2k = n-1$. For an even n this case cannot hold and thus $c_n = 0$ for even n. Let $n = 2m+1$. Then simple computations give that

$$c_{2m+1} = \frac{4m+3}{(2m+1)!2^{2m+2}}(-1)^m\binom{2m+1}{m}(2m)!$$

$$= (-1)^m \frac{4m+3}{4^{m+1}} \frac{(2m)!}{m!(m+1)!}.$$

Thus,

$$f(x) \sim \frac{1}{2} + \sum_{m=0}^{\infty} (-1)^m \frac{4m+3}{4^{m+1}} \frac{(2m)!}{m!(m+1)!} P_{2m+1}(x). \quad \square$$

3.4. Completeness

In this section we shall prove that the system of Legendre polynomials is complete. According to Definition 1.43, this means that for any function f continuous on $[-1,1]$, its *Fourier-Legendre series*

$$\sum_{n=0}^{\infty} c_n P_n(x) \tag{3.12}$$

(where c_n are defined by (3.11)) mean square converges to f. The proof of this fact is based upon the following *Weierstrass theorem on approximation by algebraic polynomials.*

Theorem 3.8 (Weierstrass). Let f be a continuous function on $[a,b]$. Then for any $\varepsilon > 0$ there exists an algebraic polynomial Q such that

$$|f(x) - Q(x)| < \varepsilon \quad \text{for all} \quad x \in [a,b].$$

PROOF. We shall derive this theorem from the Weierstrass theorem on approximation by trigonometric polynomials (Theorem 1.52). We may assume that $[a,b] = [0,\pi]$ (otherwise we apply the linear change of variable $x = a + t(b-a)/\pi,\ 0 \le t \le \pi$, and consider the function $\varphi(t) = f(a + t(b-a)/\pi)$). Further, we extend the function f to $[-\pi, 0]$ as an even function, and then to the whole real line with the period 2π. Denote the extended function by g. Clearly, g is a continuous even 2π-periodic function, and

$$g(x) = f(x) \quad \text{for all} \quad x \in [0,\pi]. \tag{3.13}$$

Let $\varepsilon > 0$. By Theorem 1.52, there exists a trigonometric polynomial T such that

$$|g(x) - T(x)| < \frac{\varepsilon}{2} \quad \text{for all} \quad x \in \mathbb{R}. \tag{3.14}$$

The Taylor series of the function T converges uniformly to $T(x)$ on every bounded interval (since T is a linear combination of cosines and sines). Let S_m be a partial sum of the Taylor series of T. Then for a sufficiently big m we have that

$$|T(x) - S_m(x)| < \frac{\varepsilon}{2} \quad \text{for all} \quad x \in [0, \pi]. \tag{3.15}$$

Now (3.13), (3.14), and (3.15) imply that

$$|f(x) - S_m(x)| < \varepsilon \quad \text{for all} \quad x \in [0, \pi].$$

Since S_m is an algebraic polynomial (of degree m), this proves the theorem.

□

Now we prove *the completeness of the system of Legendre polynomials.*

Theorem 3.9. The system of Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ is complete on $[-1, 1]$.

PROOF. Let f be a continuous function on $[-1, 1]$. We denote by S_n the nth partial sum of the Legendre expansion of f, that is,

$$S_n(x) = \sum_{k=0}^{n} c_k P_k(x),$$

where c_k are defined by (3.11). We show that

$$\lim_{n\to\infty} \int_{-1}^{1} [f(x) - S_n(x)]^2 dx = 0. \tag{3.16}$$

Let $\varepsilon > 0$. By Weierstrass Theorem 3.8, there exists an algebraic polynomial Q such that

$$|f(x) - Q(x)| < \varepsilon \quad \text{for all} \quad x \in [-1, 1]. \tag{3.17}$$

Let m be the degree of Q. Then

$$Q(x) = \sum_{k=0}^{m} a_k P_k(x).$$

Using the least squares property of partial sums of Fourier series (Theorem 1.24) and applying (3.17), we obtain that

$$\int_{-1}^{1} [f(x) - S_n(x)]^2 dx \le \int_{-1}^{1} [f(x) - Q(x)]^2 dx < 2\varepsilon^2$$

for all $n \geq m$. This implies (3.16). $\square$

By Theorem 1.44, the completeness property can be expressed in the following equivalent form.

Theorem 3.10. Any function f continuous on $[-1,1]$ satisfies Parseval's identity

$$\sum_{n=0}^{\infty} \frac{2}{2n+1} c_n^2 = \int_{-1}^{1} f^2(x)\, dx.$$

3.5. Legendre equation

In this section we show that Legendre polynomials satisfy a second order differential equation.

Theorem 3.11. The nth Legendre polynomial $y = P_n(x)$ satisfies the differential equation

$$\left(1-x^2\right) y'' - 2xy' + n(n+1)y = 0. \tag{3.18}$$

Proof. For $n = 0$ and $n = 1$ the theorem is true (see (3.2)). Let $n \geq 2$. Equation (3.18) can be also written in the form

$$\left[\left(1-x^2\right) y'\right]' + n(n+1)y = 0. \tag{3.19}$$

Let $g(x) = \left[\left(1-x^2\right) P_n'(x)\right]'$. Since P_n' is a polynomial of the degree $n-1$, then $\left(1-x^2\right) P_n'(x)$ is a polynomial of the degree $n+1$, and g is a polynomial of the degree n. We show that the polynomial g is orthogonal to any polynomial of the degree not greater than $n-1$. Let Φ be such a polynomial. Integrating twice by parts, we obtain

$$\int_{-1}^{1} g(x)\Phi(x)\, dx = \int_{-1}^{1} \left[\left(1-x^2\right) P_n'(x)\right]' \Phi(x)\, dx$$

$$= -\int_{-1}^{1} \left(1-x^2\right) P_n'(x)\Phi'(x)\, dx = \int_{-1}^{1} P_n(x) \left[\left(1-x^2\right) \Phi'(x)\right]'\, dx \tag{3.20}$$

(the integrated terms are equal to zero due to the factor $1 - x^2$). Since the degree of the polynomial $\Phi'(x)$ doesn't exceed $n-2$, then $\left[\left(1-x^2\right)\Phi'(x)\right]'$ is a polynomial of the degree not greater than $n-1$. By Corollary 3.5, the integral on the right-hand side of (3.20) is equal to zero. Thus, the polynomial g of the degree n on the interval $[-1,1]$ is orthogonal to any polynomial of the degree not greater than $n-1$. By Proposition 3.6, this implies that $g(x) = c\,P_n(x)$, where c is some constant. Applying Proposition 3.1 (i), we easily obtain that the leading coefficient of the polynomial g is equal to $-n(n+1)a_n$, where a_n is the leading coefficient of the polynomial P_n. Thus, $g(x) = -n(n+1)P_n(x)$. It follows that P_n satisfies equation (3.19). $\square$

Equation (3.18) is called *Legendre's equation*. As it was already observed, it can be written in the form

$$\left(\left(1-x^2\right)y'\right)' + n(n+1)y = 0.$$

3.6. Laplace's integral representation

Along with Rodrigues formula, there are known different integral representations for Legendre polynomials. The following representation is due to *Laplace*.

Theorem 3.12. For each n

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\left[x + \left(x^2-1\right)^{1/2}\cos\varphi\right]^n d\varphi. \tag{3.21}$$

Proof. Denote by $y_n(x)$ the right-hand side of (3.21). We have

$$y_0(x) = 1 = P_0(x),$$

$$y_1(x) = \frac{1}{\pi}\int_0^{\pi}\left[x + \left(x^2-1\right)^{1/2}\cos\varphi\right] d\varphi = x = P_1(x).$$

Our theorem will be proved if we show that functions $y_n(x)$ satisfy the same recursion formula as $P_n(x)$.

Set $Q = x + (x^2 - 1)^{1/2} \cos\varphi$. Then

$$y_{n-1}(x) = \frac{1}{\pi} \int_0^\pi Q^{n-1}\, d\varphi,$$

$$y_n(x) = \frac{1}{\pi} \int_0^\pi \left[x + \left(x^2 - 1\right)^{1/2} \cos\varphi\right] Q^{n-1}\, d\varphi$$

and

$$y_{n+1}(x) = \frac{1}{\pi} \int_0^\pi \left[x + \left(x^2 - 1\right)^{1/2} \cos\varphi\right]^2 Q^{n-1}\, d\varphi.$$

This implies that

$$(n+1)y_{n+1}(x) - (2n+1)xy_n(x) + ny_{n-1}(x) = \frac{1}{\pi} \int_0^\pi W Q^{n-1} d\varphi, \qquad (3.22)$$

where

$$W = (n+1) \left[x^2 + 2x(x^2-1)^{1/2} \cos\varphi + (x^2 - 1)\cos^2\varphi\right]$$

$$-(2n+1)x \left[x + \left(x^2 - 1\right)^{1/2} \cos\varphi\right] + n$$

$$= -n(x^2 - 1)\sin^2\varphi + (x^2 - 1)^{1/2} Q \cos\varphi \equiv U + V.$$

Integrating by parts, we obtain

$$\int_0^\pi V Q^{n-1} d\varphi = (x^2 - 1)^{1/2} \int_0^\pi Q^n \cos\varphi d\varphi$$

$$= (x^2 - 1)^{1/2} \left[(Q^n \sin\varphi)\,|_0^\pi + n(x^2 - 1)^{1/2} \int_0^\pi Q^{n-1} \sin^2\varphi d\varphi \right]$$

$$= n(x^2 - 1) \int_0^\pi Q^{n-1} \sin^2\varphi d\varphi = -\int_0^\pi U Q^{n-1} d\varphi.$$

Thus, the right-hand side of equality (3.22) is equal to 0. This implies that $y_n(x)$ satisfy the recursion formula (3.5) and therefore (3.21) holds for all n.

□

Applying Theorem 3.12, we obtain an upper bound for Legendre polynomials.

Corollary 3.13. We have

$$|P_n(x)| \leq 1 \quad \text{for all} \quad x \in [-1,1] \quad \text{and all} \quad n. \tag{3.23}$$

PROOF. Indeed,

$$\left| x + i\left(1 - x^2\right)^{1/2} \cos\varphi \right|^2 = x^2 + \left(1 - x^2\right)\cos^2\varphi \leq x^2 + 1 - x^2 = 1.$$

In view of (3.21), this implies (3.23). $\square$

Applying (3.23) and Weierstrass M-test, we obtain that the series

$$\sum_{n=0}^{\infty} P_n(x) r^n \tag{3.24}$$

converges for all $x \in [-1,1]$ and all $r \in (-1,1)$.

Furthermore, using (3.23) and recursion formula

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x), \tag{3.25}$$

(see (3.9)), we obtain that

$$|P'_n(x)| \leq \frac{n(n+1)}{2} \quad (x \in [-1,1],\ n \in \mathbb{N}). \tag{3.26}$$

Indeed, denote by A_n the maximum of $|P'_n(x)|$ on $[-1,1]$. It follows from(3.25) and (3.23) that

$$A_{n+1} \leq A_{n-1} + 2n + 1. \tag{3.27}$$

We have also $A_0 = 0$ and $A_1 = 1$. Applying these equalities and (3.27), and using the induction, we easily obtain (3.26).

Let $r \in (-1,1)$ be fixed. By (3.26), the series

$$\sum_{n=0}^{\infty} |P'_n(x)||r|^n$$

converges uniformly for $|x| \leq 1$. Thus, by Theorem 7, we obtain the following result.

Corollary 3.14. We have

$$\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} P_n(x)r^n\right) = \sum_{n=0}^{\infty} P_n'(x)r^n \tag{3.28}$$

for all $x \in [-1,1]$ and all $r \in (-1,1)$.

We have already observed that for sufficiently small $|r|$ the sum of series (3.24) coincides with the generating function

$$F(x,r) = \frac{1}{1-2rx+r^2}.$$

Now we can show that

$$F(x,r) = \sum_{n=0}^{\infty} P_n(x)r^n \tag{3.29}$$

for all $x \in [-1,1]$ and all $r \in (-1,1)$.

Indeed, fix $x_0 \in [-1,1]$. The function $y = F(x_0,r)$ satisfies the differential equation

$$(1+r^2-2rx_0)\frac{dy}{dr} + (r-x_0)y = 0 \tag{3.30}$$

for all $r \in (-1,1)$. On the other hand, set

$$g_{x_0}(r) = \sum_{n=0}^{\infty} P_n(x_0)r^n, \quad |r| < 1.$$

Then g_{x_0} is a continuously differentiable function on $(-1,1)$. Moreover, using the recursion formula (3.5), we obtain that g_{x_0} also satisfies equation (3.30). But this equation is equivalent to

$$(\ln y(r))' = \frac{x_0 - r}{1+r^2-2rx_0}.$$

It is easy to see that its solution is unique up to multiplication by a constant. Since $g_{x_0}(0) = F(x_0,0) = 1$, we have that $g_{x_0}(r) = F(x_0,r)$ for all $r \in (-1,1)$. Since x_0 is arbitrary, we obtain that equality (3.29) holds for all $x \in [-1,1]$ and all $r \in (-1,1)$.

Example 3.15. Expand the function $f(x) = (5-4x)^{-1/2}$ in a series of Legendre polynomials on the interval $[-1,1]$.

We have

$$f(x) = \frac{1}{2(1 - x + 1/4)^{1/2}} = \frac{1}{2} F\left(x, \frac{1}{2}\right).$$

Thus, by (3.29),

$$f(x) = \sum_{n=0}^{\infty} \frac{P_n(x)}{2^{n+1}} \quad (|x| \le 1).$$

Exercises to Section 3

3.1. Prove that each of the systems

$$\{P_{2n-1}\}_{n=1}^{\infty} \quad \text{and} \quad \{P_{2n}\}_{n=0}^{\infty}$$

3.1.a. is orthogonal on $[0, 1]$;

3.1.b. is complete on $[0, 1]$.

3.2. Represent the function $f(x) = (10 - 6x)^{-1/2}$ as a series in Legendre polynomials on the interval $[-1, 1]$.

3.3. Use the recursion formulas (3.5) and (3.9) for Legendre polynomials to solve the following problems.

3.3.a. Show that

$$P_n(0) = -\frac{n-1}{n} P_{n-2}(0) \quad (n \geq 2),$$

and evaluate $P_n(0)$.

3.3.b. Show that $P_n'(0) = nP_{n-1}(0)$.

3.4. Expand each of the following functions in series of Legendre polynomials in two ways:

1) using Rodrigues formula (3.6) and exercise 3.3.a;

2) using recursion formula (3.9) and exercise 3.3.a.

3.4.a.
$$f_1(x) = \begin{cases} 1, & 0 < x \leq 1, \\ 0, & -1 \leq x \leq 0. \end{cases}$$

3.4.b.
$$f_2(x) = \operatorname{sign} x \quad (|x| \leq 1).$$

3.4.c.
$$f_3(x) = |x| \quad (|x| \leq 1).$$

3.5. Show that

$$P_n(x) = \frac{1}{2^n} \sum_{j \leq n/2} \frac{(-1)^j (2n-2j)! x^{n-2j}}{j!(n-j)!(n-2j)!}.$$

Hint: use the Rodrigues formula (3.6) and the expansion of $\left(x^2-1\right)^n$ by the binomial formula.

3.6. Show that

$$\int_{-1}^{1} |P_n(x)|\, dx \le \frac{C}{\sqrt{n}} \quad (n \in \mathbb{N}),$$

where C is some constant.

3.7. Let f be a continuously differentiable function on the interval $[-1,1]$. Prove the following estimate

$$|c_n| \le \frac{A}{\sqrt{n}} \quad (n \in \mathbb{N}),$$

where c_n are the Fourier coefficients of the function f with respect to the Legendre system and A is some constant.

Hint: use the recursion formula (3.9), Proposition 3.1 (iii), and exercise 3.6.

3.8. Show that

$$|P_n''(x)| \le \frac{1}{2} n^4 \quad (|x| \le 1).$$

Hint: use the recursion formula (3.9) and inequalities (3.23), (3.26).

3.9. Prove the equalities

$$\begin{aligned} &\left(1-x^2\right) \frac{d}{dx} P_n(x) \\ &= \frac{n(n+1)}{2n+1} \left[P_{n-1}(x) - P_{n+1}(x)\right] \\ &= -nxP_n(x) + nP_{n-1}(x) \\ &= (n+1)xP_n(x) - (n+1)P_{n+1}(x). \end{aligned}$$

3.10. Using equalities from exercise 3.9, prove that

$$(1-x)\left[P_n'(x) + P_{n+1}'(x)\right] = (n+1)\left[P_n(x) - P_{n+1}(x)\right].$$

3.11. Using the Laplace integral representation (3.21), prove the estimate

$$|P_n(x)| \le \frac{2}{\pi} \int_0^{\pi/2} \left[1 - \left(1-x^2\right)\sin^2\varphi\right]^{n/2} d\varphi \quad (|x| \le 1).$$

3.12*. Prove the estimate

$$|P_n(x)| \le \frac{C}{\sqrt{n\,(1-x^2)}} \quad (|x|<1),$$

where C is some constant.

Hint: use exercise 3.11 and inequalities

$$\mathrm{e}^t \ge 1+t \quad (t\in\mathbb{R});$$

$$\sin\varphi \ge \frac{2}{\pi}\varphi \quad \left(0\le\varphi\le\frac{\pi}{2}\right).$$

Answers to exercises

1.2.a. $a_n = 0 \ (n = 0, 1, \dots), \quad b_n = 2(-1)^{n-1}/n \quad (n = 1, 2, \dots).$

1.2.b. $$a_0 = \frac{2}{3}\pi^2, \ a_n = \frac{4}{n^2}(-1)^n, \ b_n = 0 \quad (n = 1, 2, \dots).$$

1.2.c.

$$a_0 = 1, \ a_n = 0, \ b_n = 0 \ (n = 2k); \ b_n = \frac{2}{\pi n} \ (n = 2k-1), \ (k = 1, 2, \dots).$$

1.2.d. $a_0 = 2\pi$, $a_n = 0$, $b_n = -2/n \ (n = 1, 2, \dots)$.

1.3.a. $$\frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)\pi x.$$

1.3.b. $$\frac{12}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{\pi(2n-1)}{2} x.$$

1.5.a. Even ext. : 1. Odd ext.: $$\frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin \frac{\pi(2n-1)}{a} x.$$

1.5.b. Even ext.: $$\frac{a}{2} - \frac{4a}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{\pi(2n-1)}{a} x.$$

Odd ext.: $$\frac{2a}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \sin \frac{\pi n}{a} x.$$

1.5.c.

Even ext.: $$1 - \cos 1 - 2\sum_{n=1}^{\infty} \frac{1 - (-1)^n \cos 1}{\pi^2 n^2 - 1} \cos \pi n x.$$

$$\text{Odd ext.:}\quad 2\pi\sin 1\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{\pi^2n^2-1}\sin\pi nx.$$

1.5.d. Even ext.: $\dfrac{2}{\pi}-\dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos 2nx.$ Odd ext.: $\sin x.$

1.6.a.

$$\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sin\frac{\pi n}{2}x.$$

1.6.b.

$$\frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin(2n-1)\pi x.$$

1.7.

$$(\cos x)^n=\frac{1}{2^{n-1}}\sum_{k=0}^{\left[\frac{n+1}{2}\right]-1}\binom{n}{k}\cos(n-2k)x+B_n,$$

where

$$B_n=2^{-n}\binom{n}{n/2}\ (n=2k),\ \ B_n=0\ (n=2k-1,k=1,2,\dots).$$

1.10. $T_n(x)=\sum\limits_{k=1}^{n}\frac{1}{k}\sin kx$. **1.11.a.** $\alpha_0=\frac{1}{2}$, $\beta_0=0$, $\gamma_0=0$, $F\left(\frac{1}{2},0,0\right)=1$. **1.11.b.** $\alpha_0=\pi^2/3$, $\beta_0=-4$, $\gamma_0=0,04$, $F(\alpha_0,\beta_0,\gamma_0)=\frac{8}{45}\pi^4-\frac{10001}{625}\approx 1,3$. **1.11.c.** $\alpha_0=\beta_0=\gamma_0=0$, $F(0,0,0)=1$. **1.11.d.** $\alpha_0=1$, $\beta_0=-2$, $\gamma_0=0$, $F(1,-2,0)=0$. **1.11.e.** $\alpha_0=\frac{\pi}{2}$, $\beta_0=-\frac{4}{\pi}$, $\gamma_0=0$, $F\left(\frac{\pi}{2},-\frac{4}{\pi},0\right)=\frac{\pi^2}{6}-\frac{16}{\pi^2}$. **1.11.f.** $\alpha_0=\frac{2}{\pi}$, $\beta_0=0$, $\gamma_0=-\frac{4}{99\pi}$, $F\left(\frac{2}{\pi},0,-\frac{4\pi}{99}\right)=1-\frac{78424}{9801\pi^2}$. **1.12.** $\alpha_0=0$, $\beta_0=0$, $\gamma_0=-1$, $F(0,0,-1)=\frac{2}{3}\pi^2-1$. **1.13.** $f(x)=\alpha+\beta\sin x$ $(\alpha,\beta\in\mathbb{R})$. **1.14.** $f(x)=\alpha+\beta\cos x+\gamma\sin x$ $(\alpha,\beta,\gamma\in\mathbb{R})$. **1.15.a.** $\frac{1}{2}\int\limits_a^b f(x)\,dx$. **1.15.b.** 0. **1.21.a.** 0. **1.21.b.** 0 $(n<100)$, π $(n\geq 100)$. **1.22.** 100,5. **1.23.** $\frac{1}{2}$. **1.24.a.** $a_0=1-\frac{2}{3}\pi^2$, $a_n=\frac{4}{\pi^2}(-1)^{n-1}$, $b_n=0$ $(n=1,2,\dots)$. **1.24.b.** π^2 and 0, resp.

1.25. $\dfrac{1}{2}-\dfrac{1}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\sin 2\pi nx;\quad \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}.$

1.26.b. $-\dfrac{\pi+x}{2}.$ **1.26.e.** $\displaystyle\sum_{k=1}^{\infty}\frac{\sin(2k-1)x}{2k-1}.$ **1.27.** $2\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx.$

1.28.a. $\dfrac{\pi^2}{3}+4\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$. **1.28.b.** $2\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\left(\pi^2-\frac{6}{n^2}\right)\sin nx$.

$$\textbf{1.28.c.}\quad \frac{\pi^4}{5}+8\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\pi^2-\frac{6}{n^2}\right)\cos nx.$$

1.29.a. $\pi^2/6$. **1.29.b.** $-\pi^2/12$. **1.33.b.** Yes.
1.33.e. $\pi^2x/6-\pi^2x^2/4+x^3/12$. **1.36.** $A=2/\pi$, $B=1$. Yes.

$$\textbf{1.38.a.}\quad \frac{1}{\pi}+\frac{1}{2}(\cos x+\sin x)-\frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\left(\cos 2kx+k\sin 2kx\right).$$

1.38.b. -1. **1.39.a.** All, except the fourth. **1.39.b.** 1), 2), 6). **1.42.** $\frac{\pi^4}{90}$. **1.45.** $2\pi(n+1)$. **1.46.** $1+\frac{e^{2\pi}-e^{-2\pi}}{4\pi}$. **1.47.** $f(x)=\alpha\sin x$ ($\alpha\in\mathbb{R}$); $f(x)=\beta\cos x$ ($\beta\in\mathbb{R}$). **1.56.** No.

$$\textbf{2.1.a.}\quad \widehat{f}(0)=2,\quad \widehat{f}(\xi)=\frac{\sin 2\pi\xi}{\pi\xi}\ (\xi\neq 0).$$

$$\textbf{2.1.b.}\quad \widehat{f}(0)=0,\quad \widehat{f}(\xi)=-2i\frac{\sin^2\pi\xi}{\pi\xi}\ (\xi\neq 0).$$

2.1.c. $\dfrac{2}{1+4\pi^2\xi^2}$. **2.1.d.** $\dfrac{1}{1+4\pi^2(\xi+1)^2}+\dfrac{1}{1+4\pi^2(\xi-1)^2}$.

$$\textbf{2.1.e.}\quad \widehat{f}(0)=2+i,\quad \widehat{f}(\xi)=\frac{2}{1+4\pi^2\xi^2}+i\frac{\sin\pi\xi}{\pi\xi}\ (\xi\neq 0)$$

$$\textbf{2.1.f.}\quad \widehat{f}(0)=1+\frac{6i}{9+4\pi^2},$$

$$\widehat{f}(\xi)=\frac{\sin^2\pi\xi}{\pi^2\xi^2}+3i\left(\frac{1}{9+4\pi^2(\xi+1)^2}+\frac{1}{9+4\pi^2(\xi-1)^2}\right)\quad(\xi\neq 0).$$

2.3.a. $\dfrac{2}{1+4\pi^2u^2}$. **2.3.b.** $\widehat{h}_C(0)=4,\quad \widehat{h}_C(u)=\dfrac{\sin 4\pi u}{\pi u}\ (u\neq 0)$.

$$\textbf{2.3.c.}\quad \widehat{k}_C(0)=\frac{1}{2},\quad \widehat{k}_C(u)=2\frac{\sin^2\frac{\pi u}{2}}{\pi^2u^2}\ (u\neq 0).$$

2.4.a. $\widehat{f}(0) = 2a,\ \widehat{f}(\xi) = \dfrac{\sin 2\pi a\xi}{\pi\xi} \quad (\xi \neq 0).$

2.4.b. $\widehat{f}\left(\pm\dfrac{1}{2\pi}\right) = \pi, \quad \widehat{f}(\xi) = \dfrac{4\pi\xi}{1-4\pi^2\xi^2}\sin 2\pi^2\xi \quad \left(\xi \neq \pm\dfrac{1}{2\pi}\right).$

2.4.c. $\widehat{f}\left(\pm\dfrac{1}{2\pi}\right) = \mp\pi i, \quad \widehat{f}(\xi) = \dfrac{-2i}{1-4\pi^2\xi^2}\sin 2\pi^2\xi \quad \left(\xi \neq \pm\dfrac{1}{2\pi}\right).$

2.4.d. $2 \cdot \dfrac{1-4\pi^2\xi^2}{\left(1+4\pi^2\xi^2\right)^2}.$ **2.4.e.** $\dfrac{4\pi i\xi}{1+4\pi^2\xi^2}.$

2.5.a. $\widehat{f}(0) = 1, \quad \widehat{f}(\xi) = \dfrac{1}{2\pi i\xi}\left(1-\mathrm{e}^{-2\pi i\xi}\right) \quad (\xi \neq 0).$

2.5.b. $\widehat{f}(0) = a, \quad \widehat{f}(\xi) = \dfrac{1}{2\pi i\xi}\left(1-\mathrm{e}^{-2\pi ia\xi}\right) \quad (\xi \neq 0).$

2.5.c. $\widehat{f}(0) = 0, \quad \widehat{f}(\xi) = \dfrac{i}{\pi\xi}(\cos\pi\xi - 1) \quad (\xi \neq 0).$

2.5.d. $\dfrac{2c\mathrm{e}^{-2\pi ib\xi}}{c^2+4\pi^2\xi^2}.$ **2.5.e.** $\sqrt{\pi}c\mathrm{e}^{-\pi\left(\pi c^2\xi^2+2ib\xi\right)}.$

2.5.f. $-\dfrac{8\pi bci\xi}{\left(c^2+(2\pi\xi+b)^2\right)\left(c^2+(2\pi\xi-b)^2\right)}.$

2.7.a. $\widehat{f}(0) = c, \quad \widehat{f}(\xi) = \dfrac{\sin\pi c\xi}{\pi\xi} \quad (\xi \neq 0).$

2.7.b. $\widehat{f}\left(\pm\dfrac{1}{2}c\right) = \dfrac{c}{2}, \quad \widehat{f}(\xi) = \dfrac{2c}{\pi}\dfrac{\cos\pi c\xi}{1-4c^2\xi^2} \quad \left(\xi \neq \pm\dfrac{1}{2}c\right).$

2.7.c. $\widehat{f}(0) = \dfrac{c}{2}, \quad \widehat{f}\left(\dfrac{1}{c}\right) = \dfrac{c}{4}, \quad \widehat{f}(\xi) = \dfrac{\sin\pi c\xi}{2\pi\xi\left(1-c^2\xi^2\right)} \quad \left(\xi \neq 0,\ \dfrac{1}{c}\right).$

$$\textbf{2.7.d.} \qquad \widehat{f}(0) = c, \quad \widehat{f}(\xi) = \frac{\sin^2 \pi c\xi}{\pi^2 c\xi^2} \quad (\xi \neq 0).$$

$$\textbf{2.7.e.} \quad \left(\frac{1}{2\pi} - \xi^2\right) \mathrm{e}^{-\pi\xi^2}. \quad \textbf{2.7.f.} \quad \left(1 - 4\pi\xi^2\right) \mathrm{e}^{-\pi\xi^2}.$$

$$\textbf{2.8.} \qquad \widehat{f}(\xi) = \begin{cases} \pi/2, & |\xi| = a/(2\pi), \\ \pi, & |\xi| < a/(2\pi), \\ 0, & |\xi| > a/(2\pi). \end{cases}$$

$$\textbf{2.9.a.} \quad \frac{2a}{a^2 + 4\pi^2\xi^2}. \quad \textbf{2.9.b.} \quad \frac{\sqrt{\pi}}{2} \mathrm{e}^{\pi i\xi - \frac{\pi^2}{4}\xi^2}. \quad \textbf{2.9.c.} \quad -\pi^{\frac{3}{2}} i\xi \mathrm{e}^{-\pi^2\xi^2}.$$

$$\textbf{2.10.a.} \quad \frac{1}{a + 2\pi i\xi}. \quad \textbf{2.10.b.} \quad \frac{a + 2\pi i\xi}{(a + 2\pi i\xi)^2 + b^2}. \quad \textbf{2.10.c.} \quad \frac{b}{(a + 2\pi i\xi)^2 + b^2}.$$

2.11.a. $\widehat{f}(-\xi)$. **2.11.b.** $\mathrm{e}^{-2\pi i\xi x_0}\widehat{f}(\xi)$. **2.11.c.** $\widehat{f}\left(\xi - \xi_0/(2\pi)\right)$.

$$\textbf{2.11.d.} \quad \frac{1}{2i}\left[\widehat{f}\left(\xi - \frac{\xi_0}{2\pi}\right) - \widehat{f}\left(\xi + \frac{\xi_0}{2\pi}\right)\right]. \quad \textbf{2.11.e.} \quad \frac{1}{3}\widehat{f}\left(\frac{2\pi\xi - 1}{6\pi}\right).$$

2.11.f. $\frac{1}{2}\widehat{f}\left(\frac{\xi}{2}\right)$. **2.12.** $\mathrm{e}^{-x^2/2}$.

2.17.

$$f(x) = \frac{\sin x}{x} \quad (x \neq 0), \quad f(0) = 1;$$

$$f(x) = 0, \quad \left(x \leq \frac{15}{8}\right), \quad f(k) = k, \quad f\left(k \pm \frac{1}{k^3}\right) = 0 \quad (k = 2, 3, \dots),$$

and we define f as a linear function between any two neighboring points in which f is already defined.

$$\textbf{2.22.} \quad f(0) = \frac{4}{3}, \quad f(x) = \frac{1}{\pi^2}\left(\frac{\sin 2\pi x}{2\pi x} - \cos 2\pi x\right) \quad (x \neq 0).$$

$$\textbf{2.23.} \quad f(0) = 1, \quad f(x) = \frac{\sin^2 \pi x}{\pi^2 x^2} \quad (x \neq 0).$$

2.25. e^{-x}, determined uniquely. **2.26.** $f(x) = 1 - x$ $(0 \le x \le 1)$, $f(x) = 0$ $(x > 1)$, determined uniquely.

$$\textbf{2.28.a.}\quad g(x) = \begin{cases} -2\pi^2 ix, & |x| \le 1/(2\pi), \\ 0, & |x| > 1/(2\pi). \end{cases}$$

2.28.b. $\pi/6$. **2.29.a.** $\pi/2$. **2.29.b.** $\pi^2/4$. **2.34.** $\dfrac{\pi(a+b)}{ab\,(x^2 + (a+b)^2)}$.

$$\textbf{2.35.}\quad \widehat{g}(\xi) = \begin{cases} \pi(1 - 2\pi|\xi|)\widehat{f}(\xi), & |\xi| \le 1/(2\pi), \\ 0, & |\xi| > 1/(2\pi). \end{cases}$$

$$\textbf{2.36.a.}\quad \pm\, 2\mathrm{e}^{-8\pi x^2}. \quad \textbf{2.36.b.}\quad \pm\,\frac{2}{\sqrt{\pi}}\frac{1}{1+4x^2}.$$

$$\textbf{3.2.}\quad \sum_{n=0}^{\infty} \frac{P_n(x)}{3^{n+1}}. \quad \textbf{3.3.a.}\quad P_{2k-1}(0) = 0,$$

$$P_{2k}(0) = (-1)^k \frac{1 \cdot 3 \cdot \cdots \cdot (2k-1)}{2^k k!} \quad (k = 1, 2, \dots).$$

$$\textbf{3.4.a.}\quad \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(4n+3)\frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^n(n+1)!}P_{2n+1}(x).$$

$$\textbf{3.4.b.}\quad \sum_{n=0}^{\infty}(-1)^n \frac{(4n+3)(2n)!}{(2n+2)2^{2n}\,(n!)^2}P_{2n+1}(x).$$

$$\textbf{3.4.c.}\quad \frac{1}{2} - \sum_{n=1}^{\infty}(-1)^n \frac{4n+1}{2n-1} \cdot \frac{1 \cdot 3 \cdot \cdots \cdot (2n-1)}{2^{n+1}(n+1)!}P_{2n}(x).$$

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Bibliography

1. D. Jackson. Fourier series and orthogonal polynomials. Reprint of the 1941 original. Dover Publications, Inc., Mineola, NY, 2004. – 256 p.

2. А. Я. Дороговцев (A. Dorogovtsev). Математический анализ: Сборник задач. Киев, Вища школа. Головное изд-во, 1987. – 408 с.

3. V. Zorich. Mathematical analysis II. Translated from the 2002 fourth Russian edition. Universitext. Springer-Verlag, Berlin, 2009. – 686 p.

4. G. Szegö. Orthogonal polynomials. Fourth edition. American Mathematical Society, Colloquium Publications, Vol. XXIII. American Mathematical Society, Providence, R.I., 1975. – 432 p.

5. Г. М. Фихтенгольц (G. M. Fichtenholz). Курс дифференциального и интегрального исчисления, т. III. М., Наука, 1969. – 662 с.

6. G. B. Folland. Fourier Analysis and its Applications. Brooks / Cole Pul. Company, 1992. – 433 p.

7. A. Pinkus and S. Zafrany. Fourier Series and Integral Transforms. Cambridge Univ. Press, 1997. – 189 p.

8. J. S. Walker. Fourier Analysis. Oxford Univ. Press, New York, Oxford, 1988. – 440 p.

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