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SQUARE-FREE NUMBERS IN THE SEQUENCE $\{n^2+1\}$

Фугело М. А., Попович П. Безквадратні числа послідовності $\{n^2+1\}$. Нехай $B_2(x)$ є числом безквадратних чисел, що належать послідовності зсунутих квадратів в інтервалі $[1, x)$. Раніше було вивчено розподілення значень деяких арифметичних функцій на даній послідовності. Функція $B_2(x)$ представляє собою узагальнення рахункової функції для безкадратних цілих в інтервалі $[1, x)$. Р. Белман [1] отримав нетривіальну оцінку для $B_2(x)$. В даній роботі ми уточнюємо оцінку Белмана, користуючись поєднанням елементарного та аналітичного методів.
Ключові слова: безквадратні числа, асимптотична формула, рівняння Пела.

Фугело Н. А., Попович П. Бесквадратные числа последовательности $\{n^2+1\}$. Пусть $B_2(x)$ это число бесквадратных чисел, принадлежащих последовательности сдвинутых квадратов в интервале $[1, x)$. Ранее было изучено распределение значений некоторых арифметических функций на данной последовательности. Функция $B_2(x)$ представляет собой обобщение счетной функции для бескадратных целых в интервале $[1, x)$. Р. Беллман [1] получил нетривиальную оценку для $B_2(x)$. В данной работе мы уточняем оценку Беллмана, используя сочетание элементарного и аналитического методов.
Ключевые слова: бесквадратные числа, асимптотическая формула, уравнение Пелла.

Fugelo N. A., Popovich P. Square-free numbers in the sequence $\{n^2+1\}$. Let $B_2(x)$ be the number of square-free numbers belonging to the sequence of shifting square on the interval $[1, x)$. The distribution of values of some arithmetic functions on the relevant sequence has been studied ahead. The function $B_2(x)$ is the generalization of counting function for square-free integers on interval $[1, x)$. R. Bellman [1] found a non-trivial estimation for $B_2(x)$. In this work we extend the Bellman's estimate, using the compatibility of elementary and analytic methods.
Key words: square-free numbers, asymptotic formula, Pell's equation.

Introduction. The sequence of natural numbers of form $\{n^2+1\}$, $n = 1, 2, \ldots$, has the complex structure. It's the talk of such fact that it is unknown the set of prime numbers $p = n^2+1$ are finite or infinite. That is why the study of number-theoretical function on the sequence $\{n^2+1\}$, $n \in \mathbb{N}$ is very interesting problem, but challenging task. Recall two important results:

$$\sum_{n\le x} \frac{\varphi(n^2+1)}{n^2+1} = \frac{x}{2} \prod_{p\equiv 1 \pmod 4} \left(1-\frac{2}{p^2}\right) + O(\log x) \qquad (Schwartz), \tag{1}$$

$$\sum_{n\le x} \tau(n^2+1) = c_1 x \log x + c_2 x + O\left(x^{\frac{2}{3}}\right) \qquad (Motohashi). \tag{2}$$

In present paper we construct an asymptotic formula for the sum

$$B_2(x) = \sum_{n\le x} \mu^2(n^2+1),$$

where $\varphi(n)$, $\tau(n)$, $\mu(n)$ are respectively Euler's function, divisor function, Möbius function.

It is obvious that $B_2(x)$ determines the number of square-free integers among of n^2+1, $n = 1, 2, \dots, [x]$. The function $B_2(x)$ is generalization of the function

$$B_1(x) = \sum_{n\le x} \mu^2(n),$$

which study usually by "elementary method" or method of the Dirichlet generating series. Unfortunately, the study of $B_2(x)$ by method of the Dirichlet generating series does not make sense, because $\mu^2(n^2+1)$ does not a multiplicative function. We will combine "elementary" and analytical methods to study the $B_2(x)$. We proved the following theorem.

Theorem 1. *For $x \to \infty$ we have*

$$\sum_{n\le x} \mu^2(n^2+1) = xO\left(x^{\frac{1}{2}} (\log x)^3\right)$$

with an absolute constant in symbol "O".

Auxiliary arguments. For a fix natural k we consider pair of the equations (as n and d):

$$n - kd^2 = \pm 1. \tag{3}$$

The pair of equations calls the Pell's equation.

Denote by $\mathbb{Q}\left(\sqrt{k}\right)$ a real quadratic extension of $\mathbb{Q}$. Every solution (n, d) of the Pell's equation defines the tetrad of numbers $\pm n \pm d\sqrt{k}$ each of which has a norm $n^2 - kd^2 = \pm 1$ (thereof call unity of field). There exists a number $\varepsilon_0 = n_0 \pm d_0\sqrt{k}$, $\varepsilon_0 > 1$, such that $N(\varepsilon_0) = n_0^2 - kd_0^2 = \pm 1$, and every $\varepsilon = n \pm d\sqrt{k}$ with norm $n^2 - kd^2 = \pm 1$ is a degree of ε_0, $\varepsilon = \varepsilon_0^a$, $a \in \mathbb{N}$. That number calls a fundamental unit. So there is one-one correspondence between the solutions (n, d) and natural numbers a (for given unity). Hence, it follows that if (n_0, d_0) be the solution of the Pell's equation and ε_0 be an associated unity then we have for any solution (n, d):

$$\begin{gathered} n - d\sqrt{k} = (n_0 - d_0\sqrt{k})^2 = \\ \left(n_0^2 + \binom{a}{2} kd_0^2 n_0^{a-2} + \cdots\right) - \left(\binom{a}{1} n_0^{a-1} d_0 + \cdots\right)\sqrt{k}, \\ n = n_0^a + \binom{a}{2} kd_0^2 n_0^{a-2} + \cdots, \\ d = \binom{a}{1} n_0^{a-1} d_0 + \binom{a}{3} n_0^{a-3} d_0^3 k + \cdots . \end{gathered} \tag{4}$$

Lemma 1. *Let* $A_k(x)$ *be the number of solutions of the Pell's equation (3) under the condition* $n \le x$. *Then the following estimation*

$$A_k(x) = O_k(\log x)$$

holds.

This assertion follows from (4).

Denote by $\rho(m)$ the number of solutions of the congruence $u^2 \equiv -1 \pmod m$, $1 \le u \le m$. It is clear for a prime p we have

$$\rho(p^\alpha) = \begin{cases} 2 & if \quad p \equiv 1 \pmod 4,\ \alpha = 1, 2, \dots, \\ 0 & if \quad p \equiv 3 \pmod 4) \text{ or } p = 2,\ \alpha > 1, \\ 1 & if \quad p = 2,\ \alpha = 1. \end{cases}$$

Lemma 2. *For* $x \to \infty$

$$\sum_{n \le x} \rho(n) = x + O\left(x^{\frac{1}{2}} (\log x)^3\right).$$

Proof. We have

$$F(s) = \sum_{n=1}^{\infty} \frac{\rho(n)}{n^s} = \frac{\zeta(s) L(s, \chi_4)}{\zeta(2s)} \cdot \left(1 + \frac{1}{2^s}\right)^{-1}, \ \Re s > 1.$$

The Perron's formula gives

$$\sum_{n \le x} \rho(n) = \frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} F(s) \frac{x^s}{s} ds + O\left(\frac{x^c}{T(c-1)}\right), \ c > 1, T > 1. \tag{5}$$

Therefore, we infer

$$\sum_{n \le x} \rho(n) = \frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} \frac{\zeta(s) L(s, \chi_4)}{\zeta(2s)} \left(1 + \frac{1}{2}\right)^{-1} \frac{x^s}{s} ds +$$

$$+ res \lim_{s=1} \left\{ \frac{\zeta(s) L(s, \chi_4)}{\zeta(2s)(1 + 2^{-s})} \cdot \frac{x^s}{s} ds + O\left(\int \lim_{1/2}^{c} x^\sigma T^{1-\sigma} \log T^8 \cdot \frac{ds}{T} \right) + \right. \tag{6}$$

$$\left. + O\left(\frac{x^c}{T(c-1)}\right) \right\}.$$

By the inequality Cauchy-Bunyakovsky we obtain for the first integral in (6)

$$\int\limits_{1/2-iT}^{1/2+iT} \left| \zeta(s) L(s, \chi_4) \zeta^{-1}(2s) (1 + 2^{-s})^{-1} \right| \frac{x^{\frac{1}{2}}}{|s|} dt =$$

$$= O\left(x^{\frac{1}{2}} \left(\int\limits_{-T}^{T} \left| \zeta(\frac{1}{2} + it) \right|^2 \frac{dt}{t} \cdot \int\limits_{-T}^{T} \left| L\left(\frac{1}{2} + it, \chi_4\right) \right|^2 \frac{dt}{t} \right) \cdot \log T \right) =$$

$$= O\left(x^{\frac{1}{2}} (\log T)^3\right).$$

Next,

$$res \lim_{s=1} \left\{ \frac{\zeta(s)L(s,\chi_4)}{\zeta(2s)(1+2^{-s})} \cdot \frac{x^s}{s} \right\} = x \cdot \frac{\pi}{4} \cdot \frac{6}{\pi^2} \cdot \frac{2}{3} = x,$$

$$\int\limits_{1/2}^{c} \left|\zeta(s)L(s,\chi_4)\zeta^{-1}(2s)(1+2^{-s})^{-1}\right| \frac{x^\sigma}{T} d\sigma = O\left(\int\limits_{1/2}^{c} \left(\frac{x}{T}\right)^\sigma (\log T)^3 d\sigma \right) =$$

$$= O\left(\left(\frac{x}{T}\right)^{\frac{1}{2}} \log^3 T \right) + O\left(\frac{x^c}{T^c} \log^3 T \right).$$

Here, we used the estimations for $\zeta(s)$, $L(s,\chi_4)$ with $\Re s \geq \frac{1}{2}$, $1 \leq |\Im s| \leq T$ and also the estimations of the second moments $\zeta(s)$, $L(s,\chi_4)$ on half line $\Re s = \frac{1}{2}$. Taking $c = 1 + \frac{1}{\log x}$, $T = x^{\frac{1}{2}}$ we obtain our assertion.

Main Results. R. Bellman[1] (pp.146-148) have been obtained the asymptotic formula

$$B_2(x) = cx + O\left(\frac{x}{\log x}\right), \quad c = \prod p \left(1 - \frac{\rho(p)}{p^2}\right). \tag{7}$$

Repeating the argument used by Bellman in the proof of (7) we can make more precise this result:

$$B_2(x) = cx + O\left(\frac{x}{\log x (log \log x)^{A_1}} \right),$$

where A_1 is a large constant.

P. Bellman made an attempt to obtain an error term in form $O\left(x^{\frac{2}{3}} \log x\right)$. However, the assertion of author that the equation $n^2 - kd^2 = -1$ (as to n and k) has $O(\log x)$ solutions $n \leq x$, $k \leq x$ for every fixed $d \leq x^{\frac{2}{3}}$, is fallible (example, for $d = 1$ this equation has $O\left(x^{\frac{1}{2}}\right)$ solutions).

For this reason the Bellman's arguments does not lead to goal. We use other method.

By the equality

$$\mu^2(n) = \sum_{d^2|n} \mu(d)$$

we derive

$$\begin{aligned} \sum_{n \leq x} \mu^2(n^2+1) &= \sum_{n \leq x} \sum_{d^2|(n^2+1)} \mu(d) = \sum_{\substack{k,d \\ 1 \leq kd^2 = n^2+1 \leq x^2+1}} \mu(d) = \\ &= \sum_{k \leq x^{\frac{2}{3}} (\log x)^{-\frac{2}{3}}} + \sum_{x^{\frac{2}{3}} (\log x)^{-\frac{2}{3}} < k \leq x^2+1} = \sum\nolimits_1 + \sum\nolimits_2, \end{aligned} \tag{8}$$

say.

We have

$$\left|\sum\nolimits_1\right| \leq \sum_{n \leq x} \sum_{\substack{k \leq \left(\frac{x}{\log x}\right)^{\frac{2}{3}} \\ n^2 - kd^2 = 1}} = O\left(x^{\frac{2}{3}} (\log x)^{\frac{1}{3}}\right). \tag{9}$$

(We taken into account that by Lemma 1 for every $k \le \left(x\log^{-1}x\right)^{\frac{2}{3}}$ it exists $O(\log x)$ values of n and d, $n \le x$, for which $n^2 - kd^2 = \pm 1$).

Next, by $k > (x\log^{-1}x)^{\frac{2}{3}}$ and $kd^2 \le x^2+1$, we have $d \le x^{\frac{2}{3}}(\log x)^{\frac{1}{3}}$.

Therefore

$$\sum\nolimits_2 = \sum_{k>x^{\frac{2}{3}}\log^{-\frac{2}{3}}x} \sum_{\substack{d\le x^{\frac{2}{3}}\log^{\frac{1}{3}}x\\ kd^2=n^2+1\le x^2+1}} \mu(d) = \sum_{d\le x^{\frac{2}{3}}\log^{\frac{1}{3}}x} \mu(d) \sum_{\substack{n^2+1\equiv 0 \pmod{d^2}\\ x^{\frac{1}{3}}\log^{-\frac{1}{3}}x<n\le x}} 1 = \\ = \sum_{d\le x^{\frac{1}{2}}} + \sum_{x^{\frac{1}{2}}<d\le x^{\frac{2}{3}}\log^{\frac{1}{3}}x} = \sum\nolimits_{21} + \sum\nolimits_{22}. \tag{10}$$

Application Lemma 2, gives

$$\sum\nolimits_{21} = \sum_{d\le x^{\frac{1}{2}}} \mu(d) \sum_{\substack{n^2+1\equiv 0 \pmod{d^2}\\ x^{\frac{1}{3}}\log^{-\frac{1}{3}}x<n\le x}} 1 = \sum_{d\le x^{\frac{1}{2}}} \mu(d)\left\{\frac{x}{d^2}\rho(d^2)\right\} + O\left(\rho(d^2)\right) + \\ + O\left(\frac{x^{\frac{1}{3}}\cdot\rho(d^2)}{(\log x)^{\frac{1}{3}}d^2}\right) = x\sum_{d=1}^{\infty}\frac{\mu(d)\rho(d^2)}{d^2} + O\left(x\sum_{d>x^{\frac{1}{2}}}\frac{\rho(d^2)}{d^2}\right) + O\left(\sum_{d\le x^{\frac{1}{2}}}\rho(d^2)\right) + \\ + O\left(\frac{x^{\frac{1}{3}}}{(\log x)^{\frac{1}{3}}}\sum_{d\le x^{\frac{1}{2}}}\frac{\rho(d^2)}{d^2}\right) = x\prod_{p\equiv 1 \pmod 4}\left(1-\frac{\rho(p^2)}{p^2}\right) + O\left(x^{\frac{1}{2}}\right). \tag{11}$$

Moreover,

$$\sum\nolimits_{22} = O\left(\sum_{x^{\frac{1}{2}}<d\le x^{\frac{2}{3}}\log^{\frac{1}{3}}x} \sum_{\substack{n^2+1\equiv 0 \pmod{d^2}\\ n\le x}} 1\right) = \\ = O\left(\sum_{x^{\frac{1}{2}}<d\le x^{\frac{2}{3}}\log^{\frac{1}{3}}x}\left\{\frac{x}{d^2}\rho(d^2)+\rho(d^2)\right\}\right) = \\ = O\left(x^{\frac{1}{2}}\right) + O\left(x^{\frac{2}{3}}\log^{\frac{1}{3}}x\right) = O\left(x^{\frac{2}{3}}\log^{\frac{1}{3}}x\right). \tag{12}$$

Now from (7)-(11) derive

$$B_2(x) = x\prod_{p\equiv 1 \pmod 4}\left(1-\frac{\rho(p^2)}{p^2}\right) + O\left(x^{\frac{2}{3}}\log^{\frac{1}{3}}x\right).$$

Conclusion. With the similar method it may be obtained the asymptotic formula for the sum

$$\sum_{n\le x}\mu^2(n+a),\ \ a\ne -b^2,\ b\in\mathbb{Z},$$

and hence, taking into account well-known result about the sum $\sum_{n\le x}\mu^2(n)\mu^2(n+k)$ it is reputed that the distribution of square-free numbers over the sequence of values of quadratic polynomial have been studied.

1. **Bellman R.** Analytic number theory – An Introduction [text] / Bellman R. – Addison-Wesley, Reading, Massachusetts, 1980.