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EXPONENTIAL CARMICHAEL FUNCTION

Лелеченко А. В. Експоненціальна функція Кармайкла. Розглянемо експоненціальну функцію Кармайкла $\lambda^{(e)}$, таку, що $\lambda^{(e)}$ мультиплікативна та $\lambda^{(e)}(p^a) = \lambda(a)$, де λ "— звичайна функція Кармайкла. У статті обговорюється значення $\sum \lambda^{(e)}(n)$, де n пробігає деякі підмножини $[1, x]$, та наведені оцінки залишкового члену, побудовані за допомогою аналітичних методів, а надто оцінок $\int_1^T \left|\zeta(\sigma + it)\right|^m dt$.
Ключові слова: експоненціальні дільники, функція Кармайкла, моменти дзета-функції Рімана.

Лелеченко А. В. Экспоненциальная функция Кармайкла. Рассмотрим экспоненциальную функцию Кармайкла $\lambda^{(e)}$, такую, что $\lambda^{(e)}$ мультипликативна и $\lambda^{(e)}(p^a) = \lambda(a)$, где λ — обычная функция Кармайкла. В работе обсуждается величина $\sum \lambda^{(e)}(n)$, где n пробегает некоторые подмножества $[1, x]$, и даны оценки остаточного члена, построенные с помощью аналитических методов и в особенности оценок $\int_1^T \left|\zeta(\sigma+it)\right|^m dt$.
Ключевые слова: экспоненциальные делители, функция Кармайкла, моменты дзета-функции Римана.

Lelechenko A. V. Exponential Carmichael function. Consider exponential Carmichael function $\lambda^{(e)}$ such that $\lambda^{(e)}$ is multiplicative and $\lambda^{(e)}(p^a) = \lambda(a)$, where λ is usual Carmichael function. We discuss the value of $\sum \lambda^{(e)}(n)$, where n runs over certain subsets of $[1, x]$, and provide bounds on the error term, using analytic methods and especially estimates of $\int_1^T \left|\zeta(\sigma + it)\right|^m dt$.
Key words: exponential divisors, Carmichael function, moments of Riemann zeta-function.

Introduction. Consider an operator E over arithmetic functions such that for every f the function Ef is multiplicative and

$$(Ef)(p^a) = f(a), \qquad p \text{ is prime}.$$

For various functions f (such as the divisor function, the sum-of-divisor function, Möbius function, the totient function and so on) the behaviour of Ef was studied by many authors, starting from Subbarao [12]. The bibliography can be found in [10].

The notation for Ef, established by previous authors, is $f^{(e)}$.

Carmichael function λ is an arithmetic function such that

$$\lambda(p^a) = \begin{cases} \phi(p^a), & p > 2 \text{ or } a = 1, 2, \\ \phi(p^a)/2, & p = 2 \text{ and } a > 2, \end{cases}$$

and if $n = p_1^{a_1} \cdots p_m^{a_m}$ is a canonical representation, then

$$\lambda(n) = \operatorname{lcm}\bigl(\lambda(p_1^{a_1}), \ldots, \lambda(p_m^{a_m})\bigr).$$

This function was introduced at the beginning of the XX century in [1], but intense studies started only in 1990-th, e. g. [2]. Carmichael function finds applications in cryptography, e. g. [3].

Consider also the family of multiplicative functions

$$\delta_r(p^a) = \begin{cases} 0, & a < r, \\ 1, & a \geqslant r, \end{cases} \qquad r \text{ is integer.}$$

Function δ_2 is a characteristic function of the set of square-full numbers, δ_3 — of cube-full numbers and so on. Of course, $\delta_1 \equiv 1$.

Denote $\lambda_r^{(e)}$ for the product of δ_r and $\lambda^{(e)}$:

$$\lambda_r^{(e)}(n) = \delta_r(n)\lambda^{(e)}(n).$$

The aim of our paper is to study asymptotic properties of $\lambda^{(e)} \equiv \lambda_1^{(e)}$, $\lambda_2^{(e)}$, $\lambda_3^{(e)}$ and $\lambda_4^{(e)}$.

Notations.

Letter p with or without indexes denotes a prime number.

We write $f \star g$ for Dirichlet convolution

$$(f \star g)(n) = \sum_{d|n} f(d)g(n/d).$$

Denote

$$\tau(a_1, \ldots, a_k; n) := \sum_{d_1^{a_1} \cdots d_k^{a_k} = n} 1.$$

In asymptotic relations we use $\sim$, $\asymp$, Landau symbols O and o, Vinogradov symbols $\ll$ and $\gg$ in their usual meanings. All asymptotic relations are given as an argument (usually x) tends to the infinity.

Everywhere $\varepsilon > 0$ is an arbitrarily small number (not always the same even in one equation).

As usual $\zeta(s)$ is Riemann zeta-function. Real and imaginary components of the complex s are denoted as $\sigma := \Re s$ and $t := \Im s$, so $s = \sigma + it$.

For a fixed $\sigma \in [1/2, 1]$ define

$$m(\sigma) := \sup\biggl\{ m \biggm| \int_1^T \bigl|\zeta(\sigma + it)\bigr|^m dt \ll T^{1+\varepsilon} \biggr\}.$$

and

$$\mu(\sigma) := \limsup_{t\to\infty} \frac{\log\bigl|\zeta(\sigma + it)\bigr|}{\log t}.$$

Below $H_{2005} = (32/205+\varepsilon, 269/410+\varepsilon)$ stands for Huxley's exponent pair from [5].

Preliminary lemmas.

Lemma 1. *Let $F\colon \mathbb{Z} \to \mathbb{C}$ be a multiplicative function such that $F(p^a) = f(a)$, where $f(n) \ll n^\beta$ for some $\beta > 0$. Then*

$$\limsup_{n\to\infty} \frac{\log F(n) \operatorname{llog} n}{\log n} = \sup_{n\geqslant 1} \frac{\log f(n)}{n}.$$

Proof. See [13].

Lemma 2. *Let $f(t) \geqslant 0$. If*

$$\int_1^T f(t)\,dt \ll g(T),$$

where $g(T) = T^\alpha \log^\beta T$, $\alpha \geqslant 1$, then

$$I(T) := \int_1^T \frac{f(t)}{t} dt \ll \begin{cases} \log^{\beta+1} T & \text{if } \alpha = 1, \\ T^{\alpha-1} \log^\beta T & \text{if } \alpha > 1. \end{cases}$$

Proof. Let us divide the interval of integration into parts:

$$I(T) \leqslant \sum_{k=0}^{\log_2 T} \int_{T/2^{k+1}}^{T/2^k} \frac{f(t)}{t} dt < \sum_{k=0}^{\log_2 T} \frac{1}{T/2^{k+1}} \int_1^{T/2^k} f(t) dt \ll \sum_{k=0}^{\log_2 T} \frac{g(T/2^k)}{T/2^{k+1}}.$$

Now the lemma's statement follows from elementary estimates.

Lemma 3. *For $\sigma \geqslant 1/2$ and for any exponent pair (k, l) such that $l - k \geqslant \sigma$ we have*

$$\mu(\sigma) \leqslant \frac{k + l - \sigma}{2} + \varepsilon.$$

Proof. See [6, (7.57)].

A well-known application of Lemma 3 is

$$\mu(1/2) \leqslant 32/205, \tag{1}$$

following from the choice $(k, l) = H_{2005}$. Another (maybe new) application is

$$\mu(3/5) \leqslant 1409/12170, \tag{2}$$

following from

$$(k, l) = \left(\frac{269}{2434}, \frac{1755}{2434} \right) = ABAH_{2005},$$

where A and B stands for usual A- and B-processes [7, Ch. 2].

Lemma 4. *Let $\eta > 0$ be arbitrarily small. Then for growing $|t| \geqslant 3$*

$$\zeta(s) \ll \begin{cases} |t|^{1/2-(1-2\mu(1/2))\sigma}, & \sigma \in [0, 1/2], \\ |t|^{2\mu(1/2)(1-\sigma)}, & \sigma \in [1/2, 1-\eta], \\ |t|^{2\mu(1/2)(1-\sigma)} \log^{2/3} |t|, & \sigma \in [1-\eta, 1], \\ \log^{2/3} |t|, & \sigma \geqslant 1. \end{cases} \tag{3}$$

More exact estimates for $\sigma \in [1/2, 1-\eta]$ are also available, e. g.

$$\mu(\sigma) \ll \begin{cases} 10\big(\mu(3/5) - \mu(1/2)\big)\sigma + \big(6\mu(1/2) - 5\mu(3/5)\big), & \sigma \in [1/2, 3/5], \\ 5\mu(3/5)(1-\sigma)/2, & \sigma \in [3/5, 1-\eta], \end{cases} \tag{4}$$

Proof. Estimates follow from Phragmén—Lindelöf principle, exact and approximate functional equations for $\zeta(s)$ and convexity properties. See [14, Ch. 5] and [6, Ch. 7.5] for details.

Lemma 5. *For any integer r*

$$\max_{n \leqslant x} \lambda_r^{(e)}(n) \ll x^{\varepsilon}.$$

Proof. Surely $\lambda_r^{(e)}(n) \leqslant \lambda^{(e)}(n)$. By Lemma 1 we have

$$\limsup_{n\to\infty} \frac{\log \lambda^{(e)}(n) \log\log n}{\log n} = \sup_m \frac{\log \lambda(m)}{m} = \frac{\log 4}{5} =: c,$$

because $\lambda(m) \leqslant m-1$. It implies

$$\max_{n \leqslant x} \lambda^{(e)}(n) \ll x^{c/\log\log n} \ll x^{\varepsilon}.$$

Lemma 6. *Let $L_r(s)$ be the Dirichlet series for $\lambda_r^{(e)}$:*

$$L_r(s) := \sum_{n=1}^{\infty} \lambda_r^{(e)}(n) n^{-s}.$$

Then for $r = 1, 2, 3, 4$ we have $L_r(s) = Z_r(s) G_r(s)$, where

$$Z_1(s) = \zeta(s)\zeta(3s)\zeta^2(5s), \tag{5}$$

$$Z_2(s) = \zeta(2s)\zeta^2(3s)\zeta(4s)\zeta^2(5s), \tag{6}$$

$$Z_3(s) = \zeta^2(3s)\zeta^2(4s)\zeta^4(5s), \tag{7}$$

$$Z_4(s) = \zeta^2(4s)\zeta^4(5s)\zeta^2(6s)\zeta^6(7s), \tag{8}$$

Dirichlet series $G_1(s)$, $G_2(s)$, $G_3(s)$ converge absolutely for $\sigma > 1/6$ and $G_4(s)$ converges absolutely for $\sigma > 1/8$.

Proof. Follows from the identities

$$1 + \sum_{a \geqslant 1} \lambda^{(e)}(p^a) x^a = 1 + x + x^2 + 2x^3 + 2x^4 + 4x^5 + 2x^6 + 6x^7 + O(x^8)$$

$$= \frac{1 + O(x^8)}{(1-x)(1-x^3)(1-x^5)^2},$$

$$1 + \sum_{a \geqslant 2} \lambda^{(e)}(p^a) x^a = \frac{1 + O(x^6)}{(1-x^2)(1-x^3)^2(1-x^4)(1-x^5)^2},$$

$$1 + \sum_{a \geqslant 3} \lambda^{(e)}(p^a) x^a = \frac{1 + O(x^6)}{(1-x^3)^2(1-x^4)^2(1-x^5)^4},$$

$$1 + \sum_{a \geqslant 4} \lambda^{(e)}(p^a) x^a = \frac{1 + O(x^8)}{(1-x^4)^2(1-x^5)^4(1-x^6)^2(1-x^7)^6}.$$

Lemma 7. *Let $\Delta(x)$ be the error term in the well-known asymptotic formula for $\sum_{n\leqslant x}\tau(a_1,a_2,a_3,a_4;n)$, let $A_4 = a_1+a_2+a_3+a_4$ and let (k,l) be any exponent pair. Suppose that the following conditions are satisfied:*

1. $(k+l+2)a_4 < (k+l)a_1 + A_4$.

2. $2(k+l+1)a_1 \leqslant (2k+1)(a_2+a_3)$.

(3.1) $la_1 \leqslant ka_2$ *and* $(k+l+1)a_1 \geqslant k(a_2+a_3)$

or

(3.2) $la_1 \geqslant ka_2$ *and* $(l-k)(2k+1)a_3 \leqslant (2l-2l-1)(k+l+1)a_1 + \big(2k(k-l+1)+1\big)a_2$.

Proof. This is [8, Th. 3] with $p=4$.

Lemma 8.

$$m(\sigma) \geqslant \begin{cases} 4/(3-4\sigma), & 1/2 \leqslant \sigma \leqslant 5/8,\\ 10/(5-6\sigma), & 5/8 \leqslant \sigma \leqslant 35/54,\\ 19/(6-6\sigma), & 35/54 \leqslant \sigma \leqslant 41/60,\\ 2112/(859-948\sigma), & 41/60 \leqslant \sigma \leqslant 3/4,\\ 12408/(4537-4890\sigma), & 3/4 \leqslant \sigma \leqslant 5/6,\\ 4324/(1031-1044\sigma), & 5/6 \leqslant \sigma \leqslant 7/8,\\ 98/(31-32\sigma), & 7/8 \leqslant \sigma \leqslant 0.91591\ldots,\\ (24\sigma-9)/(4\sigma-1)(1-\sigma), & 0.91591\ldots \leqslant \sigma \leqslant 1-\varepsilon. \end{cases}$$

Proof. See [6, Th. 8.4].

Main Results.

Theorem 1.

$$\sum_{n\leqslant x}\lambda^{(e)}(n) = c_{11}x + c_{13}x^{1/3} + (c'_{15}\log x + c_{15})x^{1/5} + O(x^{1153/6073+\varepsilon}),$$

where c_{11}, c_{13}, c_{15} and c'_{15} are computable constants.

Proof. Lemma 6 and equation (5) implies that $\lambda^{(e)} = \tau(1,3,5,5;\cdot)\star g_1$, where $\sum_{n\leqslant x} g_1(n) \ll x^{1/6+\varepsilon}$. Due to [7]

$$\sum_{n\leqslant x}\tau(1,3,5,5;n) = x\zeta(3)\zeta^2(5)\operatorname*{res}_{s=1}\zeta(s) + 3x^{1/3}\zeta(1/3)\zeta^2(5/3)\operatorname*{res}_{s=1/3}\zeta(3s) +$$
$$+ 5x^{1/5}\zeta(1/5)\zeta(3/5)\operatorname*{res}_{s=1/5}\zeta^2(5s) + R(x).$$

To estimate $R(x)$ we use Lemma 7 with $a_1=1$, $a_2=3$, $a_3=a_4=5$. Exponent pair $(k,l)=H_{2005}$ satisfies conditions 1, 2 and 3.2 and thus

$$R(x) \ll x^{(k+l+2)/(k+l+14)} = x^{1153/6073+\varepsilon}, \qquad 1/6 < 1153/6073 < 1/5.$$

Now the convolution argument completes the proof.

Exponential totient function $\phi^{(e)}$ has similar to $\lambda^{(e)}$ Dirichlet series:

$$\sum_{n=1}^{\infty} \phi^{(e)}(n) = \zeta(s)\zeta(3s)\zeta^2(5s)H(s),$$

where $H(s)$ converges absolutely for $\sigma > 1/6$. Theorem 1 can be extended to this case without any changes, so

$$\sum_{n\leqslant x} \phi^{(e)}(n) = c_{11}x + c_{13}x^{1/3} + (c'_{15}\log x + c_{15})x^{1/5} + O(x^{1153/6073+\varepsilon}).$$

This improves the result of Pétermann [11], who obtained $\sum_{n\leqslant x} \phi^{(e)}(n) = c_{11}x + c_{13}x^{1/3} + O(x^{1/5}\log x)$.

Theorem 2.

$$\sum_{n\leqslant x} \lambda_2^{(e)}(n) = c_{22}x^{1/2} + (c'_{23}\log x + c_{23})x^{1/3} + c_{24}x^{1/4} + O(x^{1153/5586+\varepsilon}),$$

where c_{22}, c_{23}, c'_{23} and c_{24} are computable constants.

Proof. Similar to Theorem 1 with following changes: now by (6)

$$\lambda_2^{(e)} = \tau(2,3,3,4;\cdot) \star g_2,$$

where $\sum_{n\leqslant x} g_2(n) \ll x^{1/6+\varepsilon}$. But

$$\sum_{n\leqslant x} \tau(2,3,3,4;n) = 2x^{1/2}\zeta^2(3/2)\zeta(2) \operatorname*{res}_{s=1/2} \zeta(2s) +$$
$$+ 3x^{1/3}\zeta(2/3)\zeta(4/3) \operatorname*{res}_{s=1/3} \zeta^2(3s) + 4x^{1/4}\zeta(1/2)\zeta^2(3/4) \operatorname*{res}_{s=1/4} \zeta(4s) + R(s).$$

Again by Lemma 7 with $a_1 = 2$, $a_2 = a_3 = 3$, $a_4 = 4$, $(k,l) = H_{2005}$ we get

$$R(x) \ll x^{(k+l+2)/(k+l+12)} = x^{1153/5586+\varepsilon}, \qquad 1/5 < 1153/5586 < 1/4.$$

Theorem 3.

$$\sum_{n\leqslant x} \lambda_3^{(e)}(n) = (c'_{33}\log x + c_{33})x^{1/3} + (c'_{34}\log x + c_{34})x^{1/4} +$$
$$+ P_{35}(\log x)x^{1/5} + O(x^{1/6+\varepsilon}), \quad (9)$$

where c_{33}, c'_{33}, c_{34} and c'_{34} are computable constants, P_{35} is a polynomial of degree 3 with computable coefficients.

Proof. Lemma 6 and equation (7) implies that $\lambda_3^{(e)} = z_3 \star g_3$, where z_3 is defined implicitly by

$$\sum_{n=1}^{\infty} z_3(n)n^{-s} = Z_3(s) = \zeta^2(3s)\zeta^2(4s)\zeta^4(5s),$$

and g_3 is a multiplicative function such that $\sum_{n\leqslant x} g_3(n) \ll x^{1/6+\varepsilon}$.

The main term at the right side of (9) equals to

$$M_3(x) := \left(\operatorname*{res}_{s=1/3} + \operatorname*{res}_{s=1/4} + \operatorname*{res}_{s=1/5} \right) \left(\zeta^2(3s)\zeta^2(4s)\zeta^4(5s)x^s s^{-1} \right).$$

To obtain the desirable error term it is enough to prove that

$$\sum_{n\leqslant x} z_3(n) = M_3(x) + O(x^{1/6+\varepsilon}).$$

By Perron formula for $c := 1/3 + 1/\log x$ we have

$$\sum_{n\leqslant x} z_3(n) = \frac{1}{2\pi i} \int\limits_{c-iT}^{c+iT} Z_3(s)x^s s^{-1} ds + O(x^{1+\varepsilon}T^{-1}).$$

Substituting $T = x$ and moving the contour of the integration till $[1/6 - ix, 1/6 + ix]$ we get

$$\sum_{n\leqslant x} f_3(n) = M_3(x) + O(I_0 + I_- + I_+ + x^\varepsilon),$$

where

$$I_0 := \int\limits_{1/6-ix}^{1/6+ix} Z_3(s)x^s s^{-1} ds, \qquad I_\pm := \int\limits_{1/6\pm ix}^{c\pm ix} Z_3(s)x^s s^{-1} ds.$$

Firstly,

$$I_+ \ll x^{-1} \int\limits_{1/6}^{c} Z_3(\sigma + ix)x^\sigma d\sigma.$$

Let $\alpha(\sigma)$ be a function such that $Z_3(\sigma + ix) \ll x^{\alpha(\sigma)+\varepsilon}$. By (3) we have

$$\alpha(\sigma) \leqslant \begin{cases} (16 - 68\sigma)\mu(1/2) < 4/5, & \sigma \in [1/6, 1/5), \\ (8 - 28\sigma)\mu(1/2) < 3/4, & \sigma \in [1/5, 1/4), \\ (4 - 12\sigma)\mu(1/2) < 2/3, & \sigma \in [1/4, 1/3), \\ 0, & \sigma \in [1/3, c]. \end{cases}$$

This means that $I_+ \ll x^\varepsilon$. Plainly, the same estimate holds for I_-.

Secondly, it remains to prove that $I_0 \ll x^{1/6+\varepsilon}$. Here

$$I_0 \ll x^{1/6} \int\limits_{1}^{x} Z_3(1/6 + it)t^{-1} dt$$

and taking into account Lemma 2 it is enough to show $\int_1^x Z_3(1/6+it)dt \ll x^{1+\varepsilon}$. Applying Cauchy inequality twice we obtain

$$\int_1^x Z_3(1/6+it)dt \ll \left(\int_1^x \left|\zeta^4(1/2+it)\right| dt \right)^{1/2} \times$$
$$\times \left(\int_1^x \left|\zeta^8(2/3+it)\right| dt \right)^{1/4} \left(\int_1^x \left|\zeta^{16}(5/6+it)\right| dt \right)^{1/4} \ll$$
$$\ll x^{(1+\varepsilon)\cdot 1/2} x^{(1+\varepsilon)\cdot 1/4} x^{(1+\varepsilon)\cdot 1/4} \ll x^{1+\varepsilon}$$

since by Lemma 8 $m(1/2) \geqslant 4$, $m(2/3) \geqslant 8$ and $m(5/6) \geqslant 16$.

Theorem 4.

$$\sum_{n\leqslant x} \lambda_4^{(e)}(n) = (c'_{44}\log x + c_{44})x^{1/4} + P_{45}(\log x)x^{1/5} + (c'_{46}\log x + c_{46})x^{1/6} +$$
$$+ P_{47}(\log x)x^{1/7} + O(x^{C_4+\varepsilon}),$$

where c_{44}, c'_{44}, c_{46} *and* c'_{46} *are computable constants,* P_{45} *and* P_{47} *are computable polynomials,* $\deg P_{45} = 3$, $\deg P_{47} = 5$,

$$C_4 = \frac{7863059 - \sqrt{13780693090921}}{85962240} = 0.134656\ldots, \qquad 1/8 < C_4 < 1/7. \tag{10}$$

Proof. We shall follow the outline of Theorem 3. Let us prove that for $c := 1/4 + 1/\log x$ we can estimate

$$I_+ := \int_{C_4+ix}^{c+ix} Z_4(s)x^s s^{-1} ds \ll x^{C_4+\varepsilon}$$

and

$$I_0 := \int_{C_4-ix}^{C_4+ix} Z_4(s)x^s s^{-1} ds \ll x^{C_4+\varepsilon}.$$

We start with $I_+ \ll x^{-1} \int_{C_4}^{c} Z_4(\sigma+ix)x^\sigma d\sigma$. Now let $\alpha(\sigma)$ be a function such that $Z_4(\sigma+ix) \ll x^{\alpha(\sigma)+\varepsilon}$. By (3) and (8) we have

$$\alpha(\sigma) \leqslant \begin{cases} (16-80\sigma)\mu(1/2) < 5/6, & \sigma \in [1/7, 1/6), \\ (12-56\sigma)\mu(1/2) < 4/5, & \sigma \in [1/6, 1/5), \\ (4-16\sigma)\mu(1/2) < 3/4, & \sigma \in [1/5, 1/4), \\ 0, & \sigma \in [1/4, c]. \end{cases}$$

So $\int\limits_{1/7}^{c} Z_4(\sigma+ix)x^{\sigma-1}d\sigma \ll x^{\varepsilon}$ and the only case that requires further investigations is $\sigma \in [C_4, 1/7)$. Instead of (3) we apply (4) together with (1) and (2) to obtain

$$\alpha(\sigma) \leqslant \frac{1045018}{249485} - \frac{2459357}{99794}\sigma, \qquad \sigma \in [1/8, 1/7],$$

which implies $\int\limits_{C_4}^{1/7} x^{\alpha(\sigma)+\sigma-1} d\sigma \ll x^{C_4+\varepsilon}$ as soon as

$$C_4 \geqslant 1591066/12296785 = 0.129388\ldots$$

Our choice of C_4 in (10) is certainly the case.

Let us move on I_0 and prove that $\int_1^x Z_4(C_4+it)\,dt \ll x^{1+\varepsilon}$. For q_1, q_2, q_3, q_4 such that

$$1/q_1 + 1/q_2 + 1/q_3 + 1/q_4 = 1 \qquad \text{and} \qquad q_1, q_2, q_3, q_4 \geqslant 1 \tag{11}$$

by Hölder inequality we have

$$\int\limits_1^x Z_4(C_4+it)\,dt \ll \left(\int\limits_1^x \left|\zeta^{2q_1}(4s+it)\right| dt \right)^{1/q_1} \left(\int\limits_1^x \left|\zeta^{4q_2}(5s+it)\right| dt \right)^{1/q_2} \times$$
$$\times \left(\int\limits_1^x \left|\zeta^{2q_3}(6s+it)\right| dt \right)^{1/q_3} \left(\int\limits_1^x \left|\zeta^{6q_4}(7s+it)\right| dt \right)^{1/q_4}.$$

Choose

$$q_1 = m(4C_4)/2, \quad q_2 = m(5C_4)/4, \quad q_3 = m(6C_4)/2, \quad q_4 = m(7C_4)/6 \tag{12}$$

One can make sure by substituting the value of C_4 from (10) into Lemma 8 that such choice of q_k satisfies (11). Thus we obtain

$$\int\limits_1^x Z_4(C_4+it)\,dt \ll x^{(1+\varepsilon)/q1} x^{(1+\varepsilon)/q2} x^{(1+\varepsilon)/q3} x^{(1+\varepsilon)/q4} \ll x^{1+\varepsilon},$$

which finishes the proof.

Now we obtain lower value of C_4 by improving lower bounds of $m(\sigma)$ from Lemma 8. Estimates below depend on values of

$$\inf_{(k,l)} \frac{ak+bl+c}{dk+el+f}, \tag{13}$$

where (k,l) runs over the set of exponent pairs and satisfies certain linear inequalities. A method to estimate (13) without linear constrains was given by Graham [4]. In the recent paper [9] we have presented an effective algorithm to deal with (13) under a nonempty set of linear constrains.

Let c be an arbitrary function such that $c(\sigma) \geqslant \mu(\sigma)$. Define θ by an implicit equation

$$2c\bigl(\theta(\sigma)\bigr) + 1 + \theta(\sigma) - 2\bigl(1 + c\bigl(\theta(\sigma)\bigr)\bigr)\sigma = 0.$$

Finally, define

$$f(\sigma) = 2\frac{1 + c\bigl(\theta(\sigma)\bigr)}{c\bigl(\theta(\sigma)\bigr)}.$$

Due to Lemma 3 one can take $c(\sigma) = \inf_{l-k \geqslant \sigma} (k + l - \sigma)/2$, where (k, l) runs over the set of exponent pairs. However even rougher choice of c leads to satisfiable values of f such as in [6, (8.71)].

Lemma 9. *Let $\sigma \geqslant 5/8$. Compute*

$$\alpha_1 = \frac{4 - 4\sigma}{1 + 2\sigma}, \qquad \beta_1 = -\frac{12}{1 + 2\sigma}, \qquad m_1 = \frac{1 - \alpha_1}{\mu(\sigma)} - \beta_1,$$

$$\alpha_2(k, l) = \frac{4(1 - \sigma)(k + l)}{(2 + 4l)\sigma - 1 + 2k - 2l}, \qquad \beta_2(k, l) = -\frac{4(1 + 2k + 2l)}{(2 + 4l)\sigma - 1 + 2k - 2l},$$

$$m_2(k, l) = \frac{1 - \alpha_2(k, l)}{\mu(\sigma)} - \beta_2(k, l), \qquad m_2 = \inf_{\alpha_2(k,l) \leqslant 1} m_2(k, l),$$

where (k, l) runs over the set of exponent pairs. Then

$$m(\sigma) \geqslant \min\bigl(m_1, m_2, 2f(\sigma)\bigr).$$

Note that for $\sigma \geqslant 2/3$ the condition $\alpha_2(k, l) \leqslant 1$ is always satisfied.

Proof. Follows from [6, (8.97)] and from $T^\alpha V^\beta \ll T V^{\beta + (\alpha - 1)/\mu(\sigma)}$ for $\alpha < 1$ and $V \ll T^{\mu(\sigma)}$.

Substituting pointwise estimates of $m(\sigma)$ from Lemma 9 instead of segmentwise from Lemma 8 into (12) we obtain following result.

Theorem 5. *The statement of Theorem 4 remains valid for*

$$C_4 = 0.133437785\ldots$$

Conclusion. We have obtained nontrivial error terms in asymptotic estimates of

$$\sum_{n \leqslant x} \lambda_r^{(e)}(n)$$

for $r = 1, 2, 3, 4$. Cases of $r = 1$ and $r = 2$ depend on the method of exponent pairs. Cases of $r = 3$ and $r = 4$ depend on lower bounds of $m(\sigma)$. Note that case of $r = 4$ may be improved under Riemann hypothesis up to $C_4 = 1/8$, because Riemann hypothesis implies $\mu(\sigma) = 0$ and $m(\sigma) = \infty$ for $\sigma \in [1/2, 1]$.

1. **Carmichael R. D.** Note on a new number theory function / R. D. Carmichael // Bull. Amer. Math. Soc. – 1910. – feb. – Vol. 16, no. 5. – P. 232–238.
2. **Erdős P.** Carmichael's lambda function / P. Erdős, C. Pomerance, E. Schmutz // Acta Arith. – 1991. – Vol. 58, no. 4. – P. 363–385.
3. **Friedlander J. B.** Small values of the Carmichael function and cryptographic applications / J. B. Friedlander, C. Pomerance, I. E. Shparlinski // Cryptography and Computational Number Theory. – Basel : Birkhauser Verlag, 2001. – Vol. 20 of Progress in Computer Science and Applied Logic. – P. 25–32.
4. **Graham S. W.** An algorithm for computing optimal exponent pair / S. W. Graham // J. Lond. Math. Soc. – 1986. – Vol. 33, no. 2. – P. 203–218.
5. **Huxley M. N.** Exponential sums and the Riemann zeta function V / M. N. Huxley // Proc. Lond. Math. Soc. – 2005. – Vol. 90, no. 1. – P. 1–41.
6. **Ivić A.** The Riemann zeta-function: Theory and applications / A. Ivić. – Mineola, New York : Dover Publications, 2003. – 562 p.
7. **Krätzel E.** Lattice points / E. Krätzel. – Dordrecht : Kluwer, 1988. – 436 p.
8. **Krätzel E.** Estimates in the general divisor problem / E. Krätzel // Abh. Math. Semin. Univ. Hamb. – 1992. – dec. – Vol. 62, no. 1. – P. 191–206.
9. **Lelechenko A. V.** Linear programming over exponent pairs / A. V. Lelechenko // Acta Univ. Sapientiae, Inform. – 2013. – Vol. 5, no. 2. – (in press).
10. **Lelechenko A. V.** Functions involving exponential divisors: bibliography. – 2014. – URL: https://github.com/Bodigrim/expdiv-bibliography.
11. **Pétermann Y.-F. S.** Arithmetical functions involving exponential divisors: Note on two papers by L. Tóth / Y.-F. S. Pétermann // Ann. Univ. Sci. Budap. Rolando Eőtvős, Sect. Comput. – 2010. – Vol. 32. – P. 143–149.
12. **Subbarao M. V.** On some arithmetic convolutions / M. V. Subbarao // The theory of arithmetical functions: Proceedings of the Conference at Western Michigan University, April 29 – May 1, 1971. – Vol. 251 of Lecture Notes in Mathematics. – Berlin : Springer Verlag, 1972. – P. 247–271.
13. **Suryanarayana D.** On the true maximum order of a class of arithmetic functions / D. Suryanarayana, R. Sita Rama Chandra Rao // Math. J. Okayama Univ. – 1975. – Vol. 17. – P. 95–101.
14. **Titchmarsh E. C.** The theory of the Riemann zeta-function / E. C. Titchmarsh ; Ed. by D. R. Heath-Brown. – 2^{nd}, rev. edition. – New-York : Oxford University Press, 1986. – 418 p.