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THE AXISYMMETRICAL PROBLEM ON THE STRESS STATE OF THE TRUNCATED HOLLOW CONE UNDER THE EXTERNAL LOADING

Реут А. В. Вісесиметрична задача про напружений стан порожнистого двічі зрізаного по сферичних поверхнях пружного конуса під впливом нормального навантаження. Розв'язано вісесиметричну задачу про напружений стан порожнистого двічі зрізаного по сферичних поверхнях пружного конуса під впливом нормального навантаження. По конічних поверхнях виконано умови гладкого контакту, по сферичних — задано умови першої основної задачі теорії пружності. За допомогою інтегрального перетворення Попова, що застосовано по кутовій координаті, вихідну крайову задачу зведено у просторі трансформант до одновимірної. Цю задачу подано у вигляді векторної крайової задачі, яка розв'язується точно відносно трансформант переміщень. Застосування оберненого інтегрального перетворення до отриманих виразів трансформант переміщень завершує побудову точного розв'язку поставленої задачі. Проведено дослідження значень нормальних напружень на конічних поверхнях з метою встановити наявність зон розтягуючих напружень.
Ключові слова: порожнистий конус, вісесиметрична задача, зовнішнє навантаження.

Реут А. В. Осесимметричная задача о напряженном состоянии полого дважды усеченного по сферическим поверхностям упругого конуса. Решена осесимметричная задача о напряженном состоянии полого дважды усеченного по сферическим поверхностям упругого конуса, находящегося под действием нормальной нагрузки. На конических поверхностях тела выполнены условия гладкого контакта, на сферических — заданы условия первой основной задачи теории упругости. С помощью интегрального преобразования Попова, применяемого по угловой координате, исходная краевая задача сведена в пространстве трансформант к одномерной. Последняя сформулирована в виде векторной краевой задачи, которая решается точно относительно трансформант смещений. Применение обратного интегрального преобразования к полученным трансформантам смещений завершает построение точного решения поставленной задачи. Проведено исследование значений нормальных напряжений на конических поверхностях конуса с целью установить наличие зон растягивающих напряжений.
Ключевые слова: полый конус, осесимметричная задача, внешняя нагрузка.

Reut A. V. The axisymmetrical problem on the stress state of the truncated hollow cone under the external loading. The axisymmetrical problem on the stress state of a hollow twice truncated by the spherical surfaces elastic cone under action of a normal loading is solved. On the conic surfaces of the body conditions of a smooth contact are satisfied, on the spherical ones — the conditions of a first main problem of elasticity are given. The initial boundary problem is reduced in the transformations' space to the one-dimensional problem with the help of Popov's integral transformation with regard to

the angular coordinate. The one-dimensional problem is formulated in the form of a vector boundary problem, where the unknown vector consists of the unknown displacements' transformations. The vector problem is solved exactly with the apparatus of the matrix differential calculation. The application of the inverse integral transformation to the obtained displacements' transformations finishes the construction of the problem's exact solution. It is carried out the analyses of the normal stress' values on the conic surfaces of a cone with the purpose to establish the presence of the stretching stress' zones.

Key words: hollow cone, axisymmetrical problem, external loading.

Introduction. There are many works devoted to the stress state estimation of the conic form bodies. So, the general solution for the axisymmetric boundary problem for the truncated cone is obtained in [1]. The homogeneous solution for a cone is considered in [2]. A number of the solutions for the boundary problems for cones under various boundary conditions at end faces of a cone and at a conic surface are resulted in [3-5]. In [6, 7] it was supposed, that on a conic surface are executed either conditions of coupling, or a condition of a smooth contact accordingly. The general solution of the axisymmetric boundary problems for the truncated cone is obtained in [8]. The dead weight of a body was not considered in all resulted above problems. In [9] the solution of the axisymmetric boundary problem for a continuous cone with regard of its dead weight is constructed by the fulfilling of the smooth contact conditions on a conic surface. The solution is constructed by the method, offered by G.Ya. Popov. It is based on the application of the new integral transformations [10] directly to the Lame's equations with the subsequent reducing of an initial problem to the vector boundary problem. The last one is solved exactly with the apparatus of the matrix differential calculus. On the basis of the offered approach the elasticity problem for a hollow twice truncated cone which is being under loading of a body weight [10] and the axisymmetric problem of elasticity for a circular cone with an edge with regard of its dead body weight [9] were solved.

Main Results. The elastic (G is the shear module, μ is the Poisson's coefficient) twice truncated hollow cone $a_0 < r < a_1$, $\omega_0 < \theta < \omega_1$, $-\pi < \phi < \pi$ is considered ($r,\ \theta,\ \phi$ are the spherical coordinate system). On the conical surfaces the conditions of the smooth contact are executed

$$u_\theta(r,\ \omega_i) = 0, \tau_{r\theta}(r,\ \omega_i) = 0; i = 0,\ 1; a_0 < r < a_1. \tag{1}$$

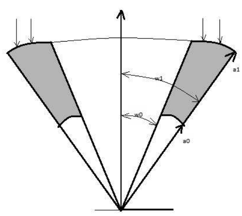

Fig. 1. The elastic twice truncated hollow cone.

On the spherical surfaces the stress are given

$$\sigma_r(a_1,\ \theta)=-p(\theta), \tau_{r\theta}(a_1,\ \theta)=0; \omega_0<\theta<\omega_1 \tag{2}$$

$$\sigma_r(a_0,\ \theta)=0, \tau_{r\theta}(a_0,\ \theta)=0; \omega_0<\theta<\omega_1$$

The displacements $u\equiv u_r(r,\ \theta)$, $v\equiv u_\theta(r,\ \theta)$ satisfy the Lame's equations [7]

$$(r^2u')'-2u-\tfrac{1}{\mu_*}\tfrac{(\sin\theta u^\bullet)^\bullet}{\sin\theta}-\tfrac{\mu_{**}}{\mu_*}\tfrac{(\sin\theta v)^\bullet}{\sin\theta}+\tfrac{\mu_0}{\mu_*}\tfrac{(\sin\theta v')^\bullet}{\sin\theta}=0, \tag{3}$$

$$(r^2v')'+\mu_*\left[\tfrac{(\sin\theta v^\bullet)^\bullet}{\sin\theta}-\tfrac{v}{\sin^2\theta}\right]+\mu_0 ru'^\bullet+2\mu_*u^\bullet=0,$$

where $\mu_0=(1-2\mu)^{-1}$, $\mu_*=\mu_0+1$, $\mu_{**}=\chi\mu_0$, $\chi=3-4\mu$, a stroke above a symbol denotes the first variable derivative, a dote denotes the second variable derivative. One should estimate the cone's stress state.

The problem's reducing to the one dimensional vector boundary problem.

The Popov's integral transformation [11] is applied to the equilibrium's equations by the scheme

$$U_k(r)=\int_{w_0}^{w_1}y_*(\theta,\nu_k)u(r,\theta)d\theta,$$

$$V_k(r)=\int_{w_0}^{w_1}y(\theta,\nu_k)v(r,\theta)d\theta \tag{4}$$

with the inverse formulas

$$u(r,\theta)=-\sum_{k=0}^{\infty}U_k(r)(2\nu_k+1)\times[S_\nu\frac{\partial\Omega_\nu}{\partial\nu}]|_{\nu=\nu_k}^{-1}y_*(\theta,\nu_k),$$

$$v(r,\theta)=-\sum_{k=0}^{\infty}V_k(r)\frac{(2\nu_k+1)}{\nu_k(\nu_k+1)}\times[S_\nu\frac{\partial\Omega_\nu}{\partial\nu}]|_{\nu=\nu_k}^{-1}y(\theta,\nu_k). \tag{5}$$

Here the following designations are taken

$$y(\theta,\nu)=P_\nu^1(\cos\theta)Q_\nu^1(\cos\omega_1)-P_\nu^1(\cos\omega_1)Q_\nu^1(\cos\theta),$$
$$y_*(\theta,\nu)=P_\nu(\cos\theta)Q_\nu^1(\cos\omega_1)-P_\nu^1(\cos\omega_1)Q_\nu(\cos\theta),$$

$\nu=\nu_k, k=0,1,2...$ are the roots of the transcendental equation

$$\Omega_\nu=\Omega(\omega_0,\omega_1)=P_\nu^1(\cos\omega_0)Q_\nu^1(\cos\omega_1)-P_\nu^1(\cos\omega_1)Q_\nu^1(\cos\omega_0)=0. \tag{6}$$

As a result, the Lame's equations (3) in the integral transformations' space take the form

$$(r^2U_k^{'}(r))^{'}-\mu_*^{-1}\mu_0 rV_k^{'}(r)-(2+\mu_*^{-1}N_k)*U_k(r)+\mu_*^{-1}\mu_{**}V_k(r)=0,$$

$$(r^2V_k^{'}(r))^{'}+\mu_0N_krU_k^{'}(r)+2\mu_*N_k\times U_k(r)-\mu_*N_kV_k(r)=0,$$

$$N_k = v_k(\nu_k + 1). \quad (7)$$

The boundary conditions (2) are reformulated in the displacements' designations. Integral transformations (4) are applied to them with the previous variable changing

$$r = a_1\rho, \;\; U_k(a_1\rho) = \tilde{u}_k(\rho), V_k(a_1\rho) = \tilde{v}_k(\rho), \alpha = \frac{a_0}{a_1}, \alpha < \rho < 1$$

$$\begin{array}{ll} (1-\mu)\tilde{u}_k^{'}(\rho) + 2\mu\tilde{u}_k(\rho) - \mu\tilde{v}_k(\rho)|_{\rho=\alpha} = 0 & (1-\mu)\tilde{u}_k^{'}(\rho) + 2\mu\tilde{u}_k(\rho) - \mu\tilde{v}_k(\rho)|_{\rho=1} = p_k \\ N_k\tilde{u}_k(\rho) + \tilde{v}_k^{'}(\rho) - \tilde{v}_k(\rho)|_{\rho=\alpha} = 0 & N_k\tilde{u}_k(\rho) + \tilde{v}_k^{'}(\rho) - \tilde{v}_k(\rho)|_{\rho=1} = 0 \end{array}$$

$$p_k = \int_{w_0}^{w_1} y_*(\theta, \nu_k) p(\theta) d\theta$$

Let's input into consideration the matrixes and the vectors

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, Q = \begin{pmatrix} 0 & -\mu_*^{-1} \\ N & 0 \end{pmatrix},$$

$$P = \begin{pmatrix} -2-\mu_*^{-1} & \mu_*^{-1}\mu_{**} \\ 2\mu_* N & -\mu_* N \end{pmatrix}, \; A = \begin{pmatrix} 2\mu & -\mu \\ N & -1 \end{pmatrix}, \; B = \begin{pmatrix} 1-\mu & 0 \\ 0 & 1 \end{pmatrix}$$

$$\overrightarrow{y}(\rho) = \begin{pmatrix} \tilde{u}_k(\rho) \\ \tilde{v}_k(\rho) \end{pmatrix}, \overrightarrow{f} = \begin{pmatrix} p_k \\ 0 \end{pmatrix}$$

The vector boundary problem is formulated with their help in the form

$$L_2(\overrightarrow{y}(\rho)) = 0, \alpha < \rho < 1$$

$$V[\overrightarrow{y}] = \vec{f} \quad (8)$$

where the differential operator and the boundary functionals are

$$L_2\overrightarrow{y} = I(\rho^2\overrightarrow{y}^{'}(\rho))' + \mu_0 r Q\overrightarrow{y}^{'}(\rho) + P\overrightarrow{y}(\rho),$$

$$V[\overrightarrow{y}] = Ay(a_i) + By'(a_i) = \overrightarrow{f}, \quad a_0 = \alpha, \; a_1 = 1 \quad (9)$$

The solving of the vector one-dimensional boundary problem.
One should construct the solution of the matrix equation

$$L_2 Y(\rho) = 0, \alpha < \rho < 1 \quad (10)$$

before the construction of the vector solution of the equation (8).

With this aim let's substitute the matrix $Y(\rho) = \rho^s I$ (I is the unitary matrix) at the equation (8). It leads to the correlation $L_2 Y(\rho) = \rho^s M(s)$, where $M(s)$ is the 2x2 matrix. The solution of this equation is searched in the form [12]

$$Y(\rho) = \frac{1}{2\pi i}\oint_c \rho^s \frac{\tilde{M}(s)}{\Delta(s)} ds. \quad (11)$$

C is the closed circuit around all poles of the integrated function . These poles are the roots of the transcendental equation $\Delta(s) = 0$, $\Delta(s)$is the matrix's $M(s)$ determinant, $\tilde{M}(s)$is the union matrix [13].

$$\Delta(s) = s^4 + 2s^3 + (2N_k + 1)s^2 - 2(N_k + 1)s + N_k(N_k - 2) = \prod_{i=1}^{4}(s - s_i),$$
$$s_1 = \nu_k + 1,\ s_2 = \nu_k - 1,\ s_3 = -\nu_k,\ s_4 = -\nu_k - 2.$$

Let's input the designations

$$\Omega_k(s) = \frac{1}{2\pi i}\oint_C \frac{s^k \rho^s}{\Delta(s)}ds,\ k = 0, 1. \tag{12}$$

The matrix $Y(\rho)$is written with the help of (12) as

$$Y(\rho) = \begin{pmatrix} \Omega_2(s) - \Omega_1(s) - \mu_* N_k \Omega_0 & \frac{\mu_0}{\mu_*}\Omega_1 - \frac{\mu_{**}}{\mu_*}\Omega_0(s) \\ -\mu_0 N_k \Omega_1 - 2\mu_* N_k \Omega_0 & \Omega_2(s) + \Omega_1(s) - (2 + \mu_*^{-1} N_k)\Omega_0(s) \end{pmatrix}.$$

If one calculate the integrals (12)in the simple poles $s_1 = \nu_k + 1$ and $s_2 = \nu_k - 1$, then one obtain the solution $Y_0(\rho)$, increasing on the infinity, if one take the simple poles $s_3 = -\nu_k, s_4 = -\nu_k - 2$ it will be $Y_1(\rho)$the solution decreasing on the infinity.

$$\begin{aligned} Y_0(\rho) = \rho^{\nu+1} R_{\nu+1} A_+(\nu) - \rho^{\nu-1} R_\nu B_+(\nu) Y_1(\rho) = \\ = \rho^{-\nu} R_\nu A_-(\nu) - \rho^{-\nu-1} R_{\nu+1} B_-(\nu) \end{aligned} \tag{13}$$

Here

$$R_\nu = [2(4\nu^2 - 1)]^{-1},\quad \mu_1 = [2(1 - \mu)]^{-1},\quad \nu = \nu_k\ (k = 1, 2...), \tag{14}$$

$$\begin{aligned} A_+(\nu) &= \begin{pmatrix} 2(\nu + 1) - \mu_0 N_k & \mu_*^{-1}(\mu_0 \nu_k - 2) \\ -\mu_0 N_k(\nu + k + 2) & \mu_1 N_k + 2\nu_k \end{pmatrix}, \\ B_+(\nu) &= \begin{pmatrix} -(\mu_0 N_k + 2\nu) & 2 - \mu_1 \nu \\ \mu_0 N_k(\nu + \kappa) & -(2(\nu + 1) - \mu_1 N_k) \end{pmatrix}, \\ A_-(\nu) &= \begin{pmatrix} -\mu_0 N_k - 2\nu & -\mu_1(\nu + k) \\ \mu_0 N_k(\nu - 4(1 - \mu)) & \mu_1 N - 2(\nu + 1) \end{pmatrix}, \\ B_-(\nu) &= \begin{pmatrix} -(\mu_0 N_k - 2(\nu + 1)) & -\mu_1(\nu + k + 2) \\ N_k(\mu_0 \nu - 2) & \mu_1 N_k + 2\nu \end{pmatrix}. \end{aligned} \tag{15}$$

Let's consider the case $k = 0$. $\nu_0 = 0$ for this case , i.e.ν_0 is the eigenvalue only of the function $P_\nu(\cos\theta)$, hence $P_0(\cos\theta) = 1$, $P_0^1(\cos\theta) = 0$. It leads from it that $u_0(\rho) \neq 0$, when $\nu_0(\rho) = 0$. In this case the one-dimensional problem in the transformations' space is simplified

$$\begin{cases} (\rho^2 u_0'(\rho))' - 2u_0(\rho) = 0, \alpha < \rho < 1, \\ (1 - \mu)u_0'(1) + 2\mu u_0(\alpha) = 0, \\ (1 - \mu)a u_0'(1) + 2\mu u_0(1) = p_0. \end{cases} \tag{16}$$

The unknown constants d_1 and$d_2$of the equation's (16) general solution

$$u_0(\rho) = d_1\rho + \frac{d_2}{\rho^2} \tag{17}$$

one must find from the boundary conditions (16). Finally, the solution for the case $k = 0$ takes the form

$$u_0(\rho) = -\frac{a_1^2 p_0}{2}(\alpha_1 \rho + \alpha_2 \rho^{-2}), \tag{18}$$

where $\alpha_1 = \frac{(\alpha^2+1)(\alpha-1)}{\alpha^2-\alpha+1}, \alpha_2 = \frac{\alpha^3}{\alpha^2-\alpha+1}\frac{1}{(2-5\mu)}$.

Now let's pass to the case $k \geq 1$. The general solution of the vector equation (9) is

$$\overrightarrow{y_k}(\rho) = Y_0(\rho)\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + Y_1(\rho)\begin{pmatrix} C_3 \\ C_4 \end{pmatrix}. \tag{19}$$

The unknown constants $C_i, i = \overline{1,4}$ are found from the satisfying of the boundary conditions (8). The exact solution of the vector one-dimensional boundary problem is constructed in the transformation's space exactly.

The inversing of the obtained solution

For the finally solution of the stated problem construction let's apply the inverse integral transformations (5) to the vector's components (19) correspondently, taking into consideration that in the second formula one should sum the series starting from 1. It is useful during the summation to take the formula for the calculation of the Legendre's function derivative relatively the order

$$\begin{aligned}&\frac{\partial P_\nu^\mu(\cos\theta)}{\partial\nu} = \sqrt{\frac{2}{\pi}}[\Gamma(\tfrac{1}{2}+\mu)\sin\mu\theta]^{-1} \times \int_0^\theta(\cos\phi - \cos\theta)^{\mu-1/2}\frac{\Gamma(\nu-\mu+1)}{\Gamma(\nu+\mu+1)}\times \\ &\times[(\Psi(\nu-\mu+1) - \Psi(\nu+\mu+1)) \times \cos(\phi(\nu+1/2)) - \phi\sin(\phi(\nu+1/2))]d\phi.\end{aligned} \tag{20}$$

For the big values of ν the asymptotic formula is obtained with regard of the formulas for the asymptotic behavior of $\Gamma(z)$ and $\Psi(z)$ functions at the big values of their arguments [14] :

$$\begin{aligned}&\frac{\partial P_\nu^\mu(\cos\theta)}{\partial\nu} \sim \sqrt{\frac{2}{\pi}}[\Gamma(1/2+\mu)\sin^\mu\theta]^{-1} \times \int_0^\theta(\cos\phi - \cos\theta)^{\mu-1/2}\frac{1}{\nu^{2\mu}}\times \\ &\times[\frac{2}{(\nu+1)^2-\mu^2}\cos(\phi(\nu+1/2)) + 4\sin(\phi\sin(\nu+1/2))]d\phi.\end{aligned}$$

The numerical results and discussions

The values of the normal stress $\sigma_\theta(r,\theta)$ on the conical surfaces $\theta = \omega_i,\ i = 0, 1 a_0 < r < a_1$ of the steel cone were investigated. The aim of the investigation is to establish the surfaces' zones of the stretching stress' creation and also the geometric parameters of the cones which lead to such situation.

The results of the numerical investigation show that by the radiuses' ratio a_1/a_0 less than 2, the stretching stress $\sigma_\theta(r,\theta)$ appear on the surfaces at the angles' ratio ω_1/ω_0 less then 1,2. With the increasing of the ratio a_1/a_0 the stretching stress $\sigma_\theta(r,\theta)$ appear at the ratio ω_1/ω_0 value less than 4. This zone of the stretching stress creation is situated near the spherical surface $r = a_1,\ \omega_0 < \theta < \omega_1$. The length of the zone increases with the increasing of the ratio a_1/a_0.

Conclusion.

1. The exact solution of the problem on the stress state of the hollow twice truncated cone is constructed for the case of the smooth contact on the conical surfaces.

2. The investigation of the normal stress on the conical surfaces is worked out with the aim to estimate the condition of the stretching stress' zones creation and conditions of their appearance.

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