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SMARANDACHE CEIL FUNCTION OVER $\mathbb{Z}[i]$

Сергеев С. С. Функція Смарандаче над $\mathbb{Z}[i]$. Вивчені арифметичні властивості функції Смарандаче $S_k(\omega)$ та двоїстої функції Смарандаче $\bar{S}_k(\omega)$ над цілими гауссовими числами. Отримані асимптотичні оцінки суматорної функції для функції Смарандаче у секторі та двоїстої функції Смарандаче.

Ключові слова: функція Смарандаче, двоїста функція Смарандаче, асимптотичні оцінки.

Сергеев С. С. Функция Смарандаче над $\mathbb{Z}[i]$. Изучены арифметические свойства функции Смарандаче $S_k(\omega)$ и двойственной функции Смарандаче $\bar{S}_k(\omega)$ над целыми гауссовыми числами. Получены асимптотические оценки сумматорной функции для функции Смарандаче в секторе, а также для двойственной функции Смарандаче.

Ключевые слова: функция Смарандаче, двойственная функция Смарандаче, асимптотические оценки.

Sergeev S. S. Smarandache ceil function over $\mathbb{Z}[i]$. We use analytic method to study the arithmetic properties of the Smarandache ceil function $S_k(\omega)$ and its dual $\bar{S}_k(\omega)$ over the ring of Gaussian integers $\mathbb{Z}[i]$. Constructed asymptotic formula summatory functions for Smarandache ceil function $S_k(\omega)$ in sector and for its dual $\bar{S}_k(\omega)$.

Key words: Smarandache ceil function, dual function to Smarandache ceil function, asymptotic estimates.

INTRODUCTION. For any fixed positive integer $k \geq 2$, the Smarandache ceil function of order k were introduced by F. Smarandache [9] and has the following definition.

$$S_k(n) = \min \{m \in \mathbb{N} : m|m^k\}, \forall n \in \mathbb{N}$$

The dual function of $S_k(n)$ is defined as

$$\bar{S}_k(n) = \max \{m \in \mathbb{N} : m^k|n\}, \forall n \in \mathbb{N}$$

There are many papers on the Smarandache ceil function and its dual. Ding Liping [4] studied the mean value properties of the Smarandache ceil function $S_k(n)$, and obtained a sharp asymptotic formula for it. Xiaoyan Li [11] and P. Varbanets and S. Kirabt [10] estimated an error term in asymptotic formulae for the mean value of the Smarandache dual function $\bar{S}_k(n)$.

In this paper, we use the analytic method to study the arithmetic properties of the Smarandache ceil function and its dual over the ring of Gaussian integers $\mathbb{Z}[i]$.

NOTATION. For $\alpha \in \mathbb{Z}[i]$ we define

$$S_k(\alpha) := \min \left\{ N(\omega), \omega \in \mathbb{Z}[i], 0 \leq \arg \omega \leq \frac{\pi}{2} : \alpha|\omega^k \right\},$$

$$\bar{S}_k(\alpha) := \max \left\{ N(\omega), \omega \in \mathbb{Z}[i], 0 \leq \arg \omega \leq \frac{\pi}{2} : \omega^k | \alpha \right\}.$$

Its easy to check that functions are multiplicative. Also we use some common notations such as $\mathbb{Z}[i]$ for a ring of Gaussian integers, $\alpha \in \mathbb{Z}[i], \alpha = a + bi, N(\alpha) = a^2 + b^2$, $\xi(s)$ — Hecke's zetta-function, $L(s, \chi_4)$ — Dirichlet L -function with non-principal character χ_4 , $\Gamma(z)$ — totient gamma function, O, \ll — Vinogradov's symbol, $\sum^*, (\prod^*)$ — sum (or product) over unassociated Gaussian integers.

AUXILIARY ARGUMENTS. Let $m \in \mathbb{Z}, s \in \mathbb{C}$. Consider Hecke's Z -function defined in half-plane $\text{Res} > 1$ by absolutely convergent series

$$Z_m(s) = \sum_{\omega}^* e^{4mi \arg \omega} N(\omega)^{-s},$$

symbol $*$ means summarizing over not-associated Gaussian integers $\omega \neq 0$.

Lemma 1. *Hecke's Z -function allow analytic extension over all complex s -plane and is integer function, if $m \neq 0$, and on $m = 0$ $Z_0(s)\zeta(s)L(s, \chi_4)$, where $\zeta(s)$ is Riemann zeta-function, $L(s, \chi_4)$ — Dirichlet L -function with non-principal character χ_4 modulo 4. Moreover we have the following functional equation*

$$\pi^{-s} \Gamma(2|m| + s) Z_m(s) = \pi^{-(1-s)} \Gamma(2|m| + 1 - s) Z_{-m}(1 - s).$$

Lemma 2. *We have the following estimates*

- (i) $Z_m(s) \ll (m^2 + t^2)^{\frac{1}{2}-\sigma} \log^4(m^2 + t^2 + 3), s = \sigma + it,$
- (ii) $Z_m(\frac{1}{2} + it) \ll (m^2 + t^2 + 3)^{\frac{1}{6}} \log^4(m^2 + t^2 + 3),$
- (iii) $\int_{-T}^T |Z_m(\frac{1}{2} + \delta + it)|^2 dt \ll (T + |m|) (\log(T + |m|))^a, a > 0 - \text{const}$

if δ is real and $|\delta| < (\log(|t| + |m| + 3))^{-1}$.

Proof. For the proof of (i) we can use functional equation for $Z_m(s)$, apply Stirling formulae for $\Gamma(z)$ and Phragmen-Lindelof principle (using trivial estimation for $Z_m\left(1 + \frac{1}{\log(T+|m|)} + it\right) \ll \log(T + |m|)$ and $Z\left(-\frac{1}{\log(T+|m|)} + it\right) \ll (m^2 + t^2 + 3)^{\frac{1}{2}} \log^4(m^2 + t^2 + 3)$).

Equation (ii) proved by P. Kaufman [7], and estimation (iii) was obtained in the work on M. D. Coleman [3].

Lemma 3. ([2]). *There are absolute constants $c_1 > 0$ and $c_0, 0 < c_0 < 1$ so in area*

$$\text{Res} > 1 - c(\log(t^2 + m^2 + 3))^{-c_0}$$

$Z_m(s) \neq 0$. Moreover in that area we have the estimate

$$(Z_m(s))^{-1} \ll \log^2(m^2 + t^2 + 3).$$

Lemma 4. Let $f(\alpha)$ be an arbitrary function on $(\mathbb{Z})[i]$, and let $\mathcal{F} \subset \mathbb{Z}[i]$ be any set. Then every $\bar{M} \geq 1$ and $0 \leq \varphi_1 < \varphi_2 < \frac{\pi}{2}$ we have

$$\begin{aligned} \sum_{\substack{\alpha \in \mathcal{F} \\ \varphi_1 \leq \arg \alpha < \varphi_2}} f(\alpha) &= (\varphi_2 - \varphi_1) \sum_{\alpha \in \mathcal{F}} f(\alpha) + O\left(\frac{1}{\bar{M}} \sum_{\alpha \in \mathcal{F}} |f(\alpha)|\right) + \\ &+ O\left((\varphi_2 - \varphi_1) \sum_{0 \leq |m| \leq \bar{M}} \left| \sum_{\alpha \in \mathcal{F}} f(\alpha) e^{4mi \arg \alpha} \right|\right). \end{aligned}$$

Proof. This statement is analogue on Vinogradov's lemma. See [1].

MAIN RESULTS. In view of the obvious inequalities

$$S_k(p^m) \leq N(p^m), \bar{S}_k(p^m) \leq N(p^m)^{\frac{1}{k}},$$

where p - Gaussian integer, $m \in \mathbb{N}$, and because of $S_k(\alpha)$ and $\bar{S}_k(\alpha)$ are multiplicative, we easily get in half-plane $\text{Res} > 2$

$$\begin{aligned} \sum_{\omega}^* \frac{S_k(\omega) e^{4mi \arg \omega}}{N(\omega)^s} &:= \\ &:= \prod_p^* \left(1 + \frac{e^{4mi \arg p}}{N(p)^{s-1}} + \frac{e^{4mi \arg p^2}}{N(p)^{2s-1}} + \dots + \frac{e^{4mi \arg p^k}}{N(p)^{ks-1}} + \frac{e^{4mi \arg p^{k+1}}}{N(p)^{(k+1)s-1}} + \dots\right) = \\ &= \prod_p^* \left(1 + \left(1 + \frac{e^{4mi \arg p}}{N(p)^s} + \frac{e^{4mi \arg p^2}}{N(p)^{2s}} + \dots + \frac{e^{4mi \arg p^{k-1}}}{N(p)^{(k-1)s}}\right) \times \right. \\ &\quad \left. \left(\frac{e^{4mi \arg p}}{N(p)^{s-1}} + \frac{e^{4mi \arg p^{k+1}}}{N(p)^{(k+1)s-2}} + \dots\right)\right) = \\ &= \frac{Z_m(s-1)Z_m(ks-1)}{Z_m(2s-2)} G_m^{(k)}(s) = \frac{Z_m(s)Z_m(ks-1)}{Z_m(ks)} H_m^{(k)}(s) \end{aligned} \quad (1)$$

where $G_m^{(k)}$ and $H_m^{(k)}$ are functions defined by Dirichlet series and absolutely convergent in half-plane $\text{Res} \geq \frac{5}{4}$.

$$\sum_{\omega}^* \frac{S_k(\omega) e^{4mi \arg \omega}}{N(\omega)^s} = \frac{Z_m(s)Z_m(ks-1)}{Z_m(ks)}. \quad (2)$$

Relation (1) allows to prove next theorem.

Theorem 1. Let $0 \leq \varphi_1 < \varphi_2 < \frac{\pi}{2}$. Then for all $k = 2, 3, \dots$ we have asymptotic formulae

$$\sum_{\substack{\omega \in \mathbb{Z}[i] \\ \varphi_1 \leq \arg \omega \leq \varphi_2 \\ N(\omega) \leq x}} S_k(\omega) = c_k (\varphi_2 - \varphi_1) x^2 + O\left(x^{\frac{3}{2}} (\log x)^4\right) + O\left((\varphi_2 - \varphi_1) x^2 (\log x)^{-a_1}\right), \quad (3)$$

where $a_1 > 0$, $c_k = \frac{\pi}{4} \cdot \frac{Z_0(2k-1)}{Z_0(2)} \prod_p^* \left(1 + \frac{1 - N(p)^{-2(k-1)}}{(N(p)+1)(N(p^2)-1)}\right)$.

This asymptotic formula is non-trivial if

$$\varphi_2 - \varphi_1 \gg x^{-\frac{1}{2}} (\log x)^b, \quad b > 5.$$

Proof. For $m = 0$ using (1) and Peron's formula we obtain for $c > 2, T > 1$

$$\sum_{N(\omega) \leq x} {}^*S_k(\omega) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{Z_0(s-1)}{Z_0(2(s-1))} H_0^{(k)}(s) \frac{x^s}{s} ds + O\left(\frac{x^c}{T(c-2)}\right). \quad (4)$$

We replace segment $\{Res = c, |Im s| \leq T\}$ with polygon consisting of 2 horizontal parts

$$\mathcal{I}_1 = \left\{ \frac{3}{2} - \frac{c_1}{\log(T^2 + m^2)^{c_0}} \leq Res \leq c, Im s = T \right\},$$

$$\mathcal{I}_2 = \left\{ \frac{3}{2} - \frac{c_1}{\log(T^2 + m^2)^{c_0}} \leq Res \leq c, Im s = -T \right\},$$

and one vertical

$$\mathcal{I}_0 = \left\{ Res = \frac{3}{2} - \frac{c_1}{\log(T^2 + m^2)^{c_0}}, T \leq Im s \leq T \right\},$$

where c_0, c_1 are constants from lemma 5.

Now using residue theorem

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{Z_0(s-1)}{Z_0(2(s-1))} H_0^{(k)}(s) \frac{x^s}{s} ds &= \frac{1}{2\pi i} \left(\int_{\mathcal{I}_1} - \int_{\mathcal{I}_2} + \int_{\mathcal{I}_0} \right) \frac{Z_0(s-1)}{Z_0(2(s-1))} H_0^{(k)}(s) ds + \\ &+ \operatorname{res}_{s=2} \left(\frac{Z_0(s-1)}{Z_0(2(s-1))} H_0^{(k)}(s) \frac{x^s}{s} \right) = I_1 - I_2 + I_0 + \operatorname{res}_{s=2} \left(\frac{Z_0(s-1)}{Z_0(2(s-1))} H_0^{(k)}(s) \frac{x^s}{s} \right). \end{aligned} \quad (5)$$

So from (4)-(5) and using lemmas 2 and 3, we found

$$\sum_{N(\omega) \leq x} {}^*S_k(\omega) = c_k x^2 + O\left(\frac{x^c}{T(c-2)}\right) + O\left(x^{\frac{3}{2}-\delta}(\log T)^{a+1}\right). \quad (6)$$

Setting $c = 2 + \frac{1}{\log x}, T = x^{\frac{1}{2}}, \delta = \frac{c_1}{2 \log T}$ we immediately have

$$\sum_{N(\omega) \leq x} {}^* = c_k x^2 + O\left(x^{\frac{3}{2}}(\log x)^{a_1}\right).$$

For $m \neq 0$ using similar considerations we have that

$$\sum_{N(\omega) \leq x} {}^*S_k(\omega) e^{4mi \arg \omega} = O\left(\frac{x^c}{T(c-2)}\right) + O\left(x^{\frac{3}{2}-\delta}(\log(T+|m|))^{a+1}\right) \quad (7)$$

with $\delta = \frac{c_1}{2 \log(T+|m|)}$

Using lemma 4 and (6) and (7) we finish proof of theorem 1 considering $M = T(\log T)^{-Q_1}, T = x^{\frac{1}{2}}$. ■

Now we are ready to explore function $\bar{S}_k(\omega)$

Theorem 2. Let $x \rightarrow \infty$. Then for $k = 2, 3, \dots$ we have

$$\bar{S}_k(x) := \sum_{N(\omega) \leq x} {}^*\bar{S}_k(\omega) = A_k(x) + \Delta_k(x) \quad (8)$$

where

$$A_k(x) = \begin{cases} \frac{3}{46} (L(2, \chi_4))^{-1} x \log x + \frac{6}{\pi^2 L(2, \chi_4)} \left(1 + \frac{z'_0(2)}{z(2)} + \frac{\pi^2 \cdot \gamma}{6}\right), & k = 2 \\ \frac{\pi Z_0(2)}{Z_0(s)} x + \frac{\pi Z_0(\frac{2}{3})}{12 Z_0(2)} x^{\frac{2}{3}}, & k = 3 \\ \frac{\pi Z_0(3)}{Z_0(4)} x + \frac{\pi Z_0(\frac{1}{2})}{16 Z_0(2)} x^{\frac{1}{2}}, & k = 4 \\ \frac{\pi Z_0(k-1)}{Z_0(2k)} x, & k = 5 \end{cases}$$

$$\Delta_k(x) \ll \begin{cases} x^{\frac{3}{4}} (\log x)^3, & k = 2 \\ x^{\frac{1}{2}} (\log x)^4, & k = 3 \\ x^{\frac{1}{2}} e^{-c_2 (\log x)^{c_3}}, & k \geq 4 \end{cases}$$

$c_2 > 0, 0 < c_3 < 1$ — absolute constants.

Proof Lets use equality (2) with $m = 0$. For $Res > 1$ we have

$$\sum_{\omega}^* \frac{\bar{S}_k(\omega)^s}{N(\omega)} = \frac{Z_0(s) Z_0(ks-1)}{Z_0(ks)} := F_k(s).$$

Function $F_2(s)$ has a pole in $s = 1$ and in zeros of function $Z_0(2s)$. For $k > 2$ singular points will be $s = 1, s = \frac{2}{k}$ and zeros of $Z_0(ks)$. Therefore further we will distinguish 4 cases $k = 2, k = 3, k = 4$ and $k \geq 5$.

For the $k = 2$ case point $s = 1$ is double pole and therefore using Peron's formulae ([8], application, theorem 3.1) we get

$$\sum_{\omega}^* \bar{S}_k(\omega) = \operatorname{res}_{s=1} \left(\frac{Z_0(s) Z_0(2s-1)}{Z_0(2s)} \cdot \frac{x^s}{s} \right) + \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{Z_0(s) Z_0(2s-1)}{Z_0(2s)} \cdot \frac{x^s}{s} ds +$$

$$N(\omega) \leq x + O\left(\frac{x^c}{T^{(c-\frac{3}{2})^2}}\right), \quad (9)$$

$c > \frac{3}{2}, T > 1$.

Lets move path of integration on line $Res = \frac{3}{4}$ and take into account that on segment $|t| \leq T$ of this line

$$Z_0^{-1}(2s) = Z_0^{-1}\left(\frac{3}{2} + 2it\right) \ll 1.$$

Moreover using Cauchy-Schwarz inequality

$$\left| \int_{-T}^T \frac{Z_0(s) Z_0(2s-1)}{Z_0(2s)} \cdot \frac{x^s}{s} ds \right| \ll x^{\frac{3}{4}} \left(\int_1^T \left| Z_0\left(\frac{3}{4} + it\right) \right|^2 \frac{dt}{t} \cdot \int_1^T \left| Z_0\left(\frac{1}{2} + 2it\right) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\ll x^{\frac{3}{4}} \left(\int_1^T \left| \xi\left(\frac{3}{4} + it\right) \right|^4 \frac{dt}{t} \right)^{\frac{1}{4}} \cdot \left(\int_1^T \left| L\left(\frac{3}{4} + it, \chi_4\right) \right|^4 \frac{dt}{t} \right)^{\frac{1}{4}} \cdot \left(\int_1^T \left| \xi\left(\frac{1}{2} + 2it\right) \right|^4 \frac{dt}{t} \right)^{\frac{1}{4}}$$

$$\cdot \left(\int_1^T \left| L\left(\frac{1}{2} + it, \chi_4\right) \right|^4 \frac{dt}{t} \right)^{\frac{1}{4}} \ll x^{\frac{3}{4}} \log^3 T. \quad (10)$$

Further, given that $Z_0(s) = \xi(s)L(s, \chi_4)$ we easily discover

$$\begin{aligned} \operatorname{res}_{s=1} \left(\frac{Z_0(s)Z_0(2s-1)}{Z_0(2s)} \cdot \frac{x^s}{s} \right) &= \frac{3}{46} (L(2, \chi_4))^{-1} x \log x + \\ &+ \frac{6}{\pi^2 L(2, \chi_4)} \left(1 + \frac{Z'_0(2)}{Z(2)} + \frac{\pi^2 \cdot \gamma}{6} \right), \end{aligned} \quad (11)$$

where γ — Euler constant.

From (9)–(11) we get theorem for $k = 2$.

In case $k = 3$ we move path of integration on the line $\operatorname{Res} = \frac{1}{2}$ wherein we move through 2 poles in points $s = 1$ and $s = \frac{2}{3}$.

Reasoning similar to the above gives

$$\begin{aligned} \bar{S}_3(x) &= \operatorname{res}_{s=\frac{2}{3}} \left(\frac{Z_0(s)Z_0(3s-1)}{Z_0(3s)} \cdot \frac{x^s}{s} \right) + \operatorname{res}_{s=1} \left(\frac{Z_0(s)Z_0(3s-1)}{Z_0(3s)} \cdot \frac{x^s}{s} \right) + \\ &+ O \left(\int_1^T \left| \frac{Z_0(\frac{1}{2}+it)Z_0(\frac{3}{2}+3it)}{Z_0(\frac{3}{2}+3it)} \right| \frac{x^{\frac{1}{2}}}{2} dt \right) + \\ &+ O \left(\int_{\frac{1}{2}}^c \left| \frac{Z_0(\sigma+iT)Z_0(3\sigma+iT)}{Z_0(\frac{3}{2}+3iT)} \right| \cdot \frac{x^\sigma}{T} d\sigma \right) + O \left(\frac{x^c}{T^{(c-\frac{3}{2})}} \right) \end{aligned} \quad (12)$$

And so using estimates of four moments of the $\xi(s)$ and $L(s, \chi_4)$ on the half-plane (see [Montgomeri]) leads to the asymptotic formulae

$$\bar{S}_3(x) = \frac{\pi Z_0(2)}{Z_0(3)} x + \frac{\pi Z_0(\frac{2}{3})}{12 Z_0(2)} x^{\frac{4}{3}} + O \left(x^{\frac{1}{2}} (\log x)^4 \right). \quad (13)$$

If $k = 4$ then integrand has 2 simple poles in points $s = 1$ and $s = \frac{1}{2}$ and pole in zeros of $Z_0(2s)$ places to the left of $s = \frac{1}{2}$.

We move path of integration in a region free of zeros $Z_0(2s)$ namely on the line

$$\operatorname{Res} = \frac{1}{2} - \frac{1}{2}c (\log(T^2 + 1))^{c_0}.$$

Then we obtain

$$\bar{S}_4(x) = \frac{\pi Z_0(3)}{Z_0(4)} x + \frac{\pi Z_0(\frac{1}{2})}{16 Z_0(2)} x^{\frac{1}{2}} + O \left(x^{\frac{1}{2}} e^{-c_2 (\log x)^{c_3}} \right). \quad (14)$$

Finally for $k \geq 5$ lets take same path as in case $k = 4$. This leads to asymptotic formulae

$$\bar{S}_k(x) = \frac{\pi Z_0(k-1)}{Z_0(2k)} x + O \left(x^{\frac{1}{2}} e^{-c_2 (\log x)^{c_3}} \right). \quad (15)$$

Relations (13)–(15) proves our theorem. ■

Proof of the Theorem 1 shows that using Theorem 2 we can obtain asymptotic formula for the distribution of values of the function $\bar{S}_k(\omega)$ in narrow sectors $N(\omega) \leq x, \varphi_1 \leq \arg \omega < \varphi_2, \varphi_2 - \varphi_1 \gg x^{-\alpha_k} (\log x)^4$, where $\alpha_2 = \frac{1}{4}, \alpha_k = \frac{1}{2}, k = 3, 4, \dots$

CONCLUSION. In our work we use analytic method to study the arithmetic properties of the Smarandache ceil function $S_k(\omega)$ and its dual $\bar{S}_k(\omega)$ over the ring of Gaussian integers $\mathbb{Z}[i]$. An asymptotic formula for summatory functions for Smarandache ceil function $S_k(\omega)$ in sector and for its dual $\bar{S}_k(\omega)$ is obtained.

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