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ON A REDUCTION OF A LINEAR HOMOGENEOUS DIFFERENTIAL SYSTEM WITH OSCILLATING COEFFICIENTS TO A SYSTEM WITH SLOWLY-VARYING COEFFICIENTS

Щоголев С. А. Про зведення лінійної однорідної диференціальної системи з коливними коефіцієнтами до системи з повільно змінними коефіцієнтами. Для лінійної однорідної диференціальної системи, коефіцієнти якої зображувані у вигляді абсолютно та рівномірно збіжних рядів Фур'є з повільно змінними коефіцієнтами та частотою, отримано умови існування лінійного перетворення аналогічної структури, що зводить цю систему до системи з повільно змінними коефіцієнтами у нерезонансному випадку на асимптотично великому проміжку зміни незалежної змінної.
Ключові слова: диференціальний, повільно змінний, ряди Фур'є.

Щёголев С. А. О сведении линейной однородной системы с осциллирующими коэффициентами к системе с медленно меняющимися коэффициентами. Для линейной однородной дифференциальной системы, коэффициенты которой представимы в виде абсолютно и равномерно сходящихся рядов Фурье с медленно меняющимися коэффициентами и частотой, получены условия существования линейного преобразования аналогичной структуры, приводящего эту систему к системе с медленно меняющимися коэффициентами в нерезонансном случае на асимптотически большом промежутке изменения независимой переменной.
Ключевые слова: дифференциальный, медленно меняющийся, ряды Фурье.

Shchogolev S. A. On a reduction of a linear homogeneous differential system with oscillating coefficients to a system with slowly-varying coefficients. For the linear homogeneous differential system, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly-varying coefficients and frequency, conditions of existence of the linear transformation with coefficients of similar structure, this system leads to a system with slowly-varying coefficients in a noresonance case in asymptotical large interval of independent variable, are obtained.
Key words: differential, slowly-varying, Fourier series.

Introduction. One of the important problems of the theory of the differential equations is a problem of the reduction of the differential system to a simpler form. In case of linear systems this can be a problem of the reduction to a system with constant coefficients (a reducibility), to a system with triangle (particularly, Jordan or diagonal) matrix of coefficients etc. In case of linear homogeneous system with oscillating coefficients this can be a problem of reduction to a system whose coefficients in some sence slowly-varying. In [1] the author considers the linear homogeneous differential system, whose coefficients are represented as an absolutely and uniformly convergent Fourier-series with slowly-varying coefficients and frequency in asymptotical large, but finite interval of independent variable. The conditions of the existence of the transformation, reducing this system to the system with slowly-varying coefficients are researched. The system which considered contains two small parameters μ and ε, the first of which characterized the smallness of the oscillating terms in the coefficients of the system, and the second is an indicator of the slow variability. In [1] the existence of desired transformation has been proved by the condition $\mu^{r+1} \leqslant \varepsilon^2$, where $r \in \mathbf{N}$. This condition is tough enough, although the study of a some specific systems it holds. Roles of parameters μ and ε are substantially different, and these parameters can be considered independently of each other. Therefore the purpose of this paper is to provide such conditions for the existence of this transformation, which would not require communication between these parameters. This required a significant change in the method of proof used in comparison with the [1].

Notation. Let $G(\varepsilon_0) = \{t, \varepsilon : \ 0 < \varepsilon < \varepsilon_0, \ -L\varepsilon^{-1} \leqslant t \leqslant L\varepsilon^{-1}, \ 0 < L < +\infty\}$.

Definition 1. We say, that a function $f(t,\varepsilon)$ belong to class $S(m,\varepsilon_0)$, $m \in \mathbf{N} \cup \{0\}$, if

1) $f : G(\varepsilon_0) \to \mathbf{C}$,
2) $f(t,\varepsilon) \in C^m(G(\varepsilon_0))$ with respect t,
3) $d^k f(t,\varepsilon)/dt^k = \varepsilon^k f_k^*(t,\varepsilon)$ $(0 \leqslant k \leqslant m)$,

$$\|f\|_{S(m,\varepsilon_0)} \stackrel{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |f_k^*(t,\varepsilon)| < +\infty.$$

Under the slowly-varying function we mean a function of class $S(m,\varepsilon_0)$.

Definition 2. We say, that a function $f(t,\varepsilon,\theta(t,\varepsilon))$ belong to class $F(m,l,\varepsilon_0,\theta)$ $(m,l \in \mathbf{N} \cup \{0\})$, if this function can be represented as

$$f(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t,\varepsilon) \exp\left(in\theta(t,\varepsilon)\right)$$

and:

1) $f_n(t,\varepsilon) \in S(m,\varepsilon_0)$;
2) $\|f\|_{F(m,l,\varepsilon_0,\theta)} \stackrel{def}{=} \|f_0\|_{S(m,\varepsilon_0)} + \sum\limits_{n=-\infty}^{\infty} |n|^l \|f_n\|_{S(m,\varepsilon_0)} < +\infty$, particular

$$\|f\|_{F(m,0,\varepsilon_0,\theta)} = \sum_{n=-\infty}^{\infty} \|f_n\|_{S(m,\varepsilon_0)};$$

3) $\theta(t,\varepsilon) = \int\limits_0^t \varphi(\tau,\varepsilon)d\tau$, $\varphi(t,\varepsilon) \in \mathbf{R}^+$, $\varphi(t,\varepsilon) \in S(m,\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)} = \varphi_0 > 0$.

We denote $(A)_{jk}$ the element a_{jk} of the matrix $A = (a_{jk})_{j,k=\overline{1,n}}$.

We say, that $(n \times n)$-matrix $A(t,\varepsilon,\theta)$ belong to class $F(m,l,\varepsilon,\theta)$, if all elements of this matrix are the functions of the class $F(m,l,\varepsilon,\theta)$. Then we define:

$$\|A\|^*_{F(m,l,\varepsilon_0,\theta)} \stackrel{def}{=} \max_{1 \leqslant j \leqslant n} \sum_{k=1}^{n} \|(A)_{jk}\|_{F(m,l,\varepsilon_0,\theta)}.$$

Let $f(t,\varepsilon,\theta) \in F(m,l,\varepsilon_0,\theta)$. We denote $\forall\ n \in \mathbf{Z}$:

$$\Gamma_n[f] = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)\exp(-in\theta)d\theta,$$

particular

$$\Gamma_0[f] = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)d\theta.$$

State some properties of norms $\|\cdot\|_{S(m,\varepsilon}$, $\|\cdot\|_{F(m,l,\varepsilon_0,\theta)}$, $\|\cdot\|^*_{F(m,l,\varepsilon_0,\theta)}$. Let $c =$ $=$ const, $p(t,\varepsilon), q(t,\varepsilon) \in S(m,\varepsilon_0)$, $u(t,\varepsilon,\theta), v(t,\varepsilon,\theta) \in F(m,l,\varepsilon_0,\theta)$, $(n\times n)$-matrices $A(t,\varepsilon,\theta), B(t,\varepsilon,\theta)$ belongs to class $F(m,l,\varepsilon_0,\theta)$. Then cp, $p\pm q$, pq belongs to class $S(m,\varepsilon_0)$; cu, $u\pm v$, uv belongs to class $F(m,l,\varepsilon_0,\theta)$, matrices cA, $A\pm B$, AB belongs to class $F(m,l,\varepsilon_0,\theta)$, and

1) $\|kp\|_{S(m,\varepsilon_0)} = |k|\cdot\|p\|_{S(m,\varepsilon_0)}$.
2) $\|p+q\|_{S(m,\varepsilon_0)} \leqslant \|p\|_{S(m,\varepsilon_0)} + \|q\|_{S(m,\varepsilon_0)}$.
3) $\|p\|_{S(m,\varepsilon_0)} = \sum\limits_{k=0}^{m} \left\|\frac{1}{\varepsilon^k}\frac{d^kp}{dt^k}\right\|_{S(0,\varepsilon_0)}$.
4) $\|pq\|_{S(m,\varepsilon_0)} \leqslant 2^m\|p\|_{S(m,\varepsilon_0)}\|q\|_{S(m,\varepsilon_0)}$.

We prove this property. Obviously, that

$$\|pq\|_{S(0,\varepsilon_0)} \leqslant \|p\|_{S(0,\varepsilon_0)}\|q\|_{S(0,\varepsilon_0)}.$$

Based on the properties 3) we obtain:

$$\|pq\|_{S(m,\varepsilon_0)} = \sum_{k=0}^{m}\left\|\frac{1}{\varepsilon^k}\frac{d^k(pq)}{dt^k}\right\|_{S(0,\varepsilon_0)} \leq \sum_{k=0}^{m}\frac{1}{\varepsilon^k}\sum_{j=0}^{k}C_k^j\left\|\frac{d^jp}{dt^j}\right\|_{S(0,\varepsilon_0)}\left\|\frac{d^{k-j}p}{dt^{k-j}}\right\|_{S(0,\varepsilon_0)} \leq$$

$$\leq 2^m\left(\sum_{j=0}^{m}\left\|\frac{1}{\varepsilon^j}\frac{d^jp}{dt^j}\right\|_{S(0,\varepsilon_0)}\right)\cdot\left(\sum_{j=0}^{m}\left\|\frac{1}{\varepsilon^j}\frac{d^jq}{dt^j}\right\|_{S(0,\varepsilon_0)}\right) = 2^m\|p\|_{S(m,\varepsilon_0}\cdot\|q\|_{S(m,\varepsilon_0}.$$

5) $\|cu\|_{F(m,l,\varepsilon_0,\theta)} = |c|\cdot\|u\|_{F(m,l,\varepsilon_0,\theta)}$.
6) $\|u+v\|_{F(m,l,\varepsilon_0,\theta)} \leqslant \|u\|_{F(m,l,\varepsilon_0,\theta)} + \|v\|_{F(m,l,\varepsilon_0,\theta)}$.
7) $\|uv\|_{F(m,l,\varepsilon_0,\theta)} \leqslant 2^m(2^l+1)\|u\|_{F(m,l,\varepsilon_0,\theta)}\cdot\|v\|_{F(m,l,\varepsilon_0,\theta)}$.

We prove this property. Based on definition of norm $\|\cdot\|_{F(m,l,\varepsilon_0,\theta)}$ we obtain:

$$\|u\|_{F(m,l,\varepsilon_0,\theta)} = \|u_0\|_{S(m,\varepsilon_0)} + \left\|\frac{\partial^l u}{\partial\theta^l}\right\|_{F(m,0,\varepsilon_0,\theta)}.$$

According to Leibniz formula:

$$\frac{\partial^l(uv)}{\partial\theta^l} = \sum_{\nu=0}^{l}C_l^\nu\frac{\partial^\nu u}{\partial\theta^\nu}\cdot\frac{\partial^{l-\nu}v}{\partial\theta^{l-\nu}}.$$

From the property 4) implies

$$\|uv\|_{F(m,0,\varepsilon_0,\theta)} \leqslant 2^m\|u\|_{F(m,0,\varepsilon_0,\theta)}\cdot\|v\|_{F(m,0,\varepsilon_0,\theta)}.$$

From there we obtain:

$$\|uv\|_{F(m,l,\varepsilon_0,\theta)} \leqslant \|\Gamma_0[uv]\|_{S(m,\varepsilon_0)} + \left\|\frac{\partial^l(uv)}{\partial\theta^l}\right\|_{F(m,0,\varepsilon_0,\theta)} \leq$$

$$\leqslant \|\Gamma_0[uv]\|_{S(m,\varepsilon_0)} + \sum_{\nu=0}^{l} C_l^\nu 2^m \left\|\frac{\partial^\nu u}{\partial\theta^\nu}\right\|_{F(m,0,\varepsilon_0,\theta)} \cdot \left\|\frac{\partial^{l-\nu} v}{\partial\theta^{l-\nu}}\right\|_{F(m,0,\varepsilon_0,\theta)} \leqslant$$

$$\leqslant 2^m\|u\|_{F(m,l,\varepsilon_0,\theta)} \cdot \|v\|_{F(m,l,\varepsilon_0,\theta)} + 2^m 2^l \|u\|_{F(m,l,\varepsilon_0,\theta)} \cdot \|v\|_{F(m,l,\varepsilon_0,\theta)} =$$

$$= 2^m(2^l+1)\|u\|_{F(m,l,\varepsilon_0,\theta)} \cdot \|v\|_{F(m,l,\varepsilon_0,\theta)}.$$

8) $\|cA\|^*_{F(m,l,\varepsilon_0,\theta)} = |c| \cdot \|A\|^*_{F(m,l,\varepsilon_0,\theta)}$.
9) $\|A+B\|^*_{F(m,l,\varepsilon_0,\theta)} \leqslant \|A\|^*_{F(m,l,\varepsilon_0,\theta)} + \|B\|^*_{F(m,l,\varepsilon_0,\theta)}$.
10) $\|AB\|^*_{F(m,l,\varepsilon_0,\theta)} \leq 2^m(2^l+1)\|A\|^*_{F(m,l,\varepsilon_0,\theta)} \cdot \|B\|^*_{F(m,l,\varepsilon_0,\theta)}$.
This property follows directly from definition of norm $\|\cdot\|^*_{F(m,l,\varepsilon_0,\theta)}$ and properties 7).

Main Results

1. Statement of the Problem. Consider the following differential system:

$$\frac{dx}{dt} = (\Lambda(t,\varepsilon) + \varepsilon A(t,\varepsilon) + \mu P(t,\varepsilon,\theta))x, \tag{1}$$

where $x = \mathrm{colon}(x_1,...,x_n)$, $\Lambda(t,\varepsilon) = \mathrm{diag}(\lambda_1(t,\varepsilon),...,\lambda_n(t,\varepsilon))$, $\lambda_j - \lambda_k = i\omega_{jk}(t,\varepsilon)$, $\omega_{jk} \in \mathbf{R}$, $\omega_{jk} \in S(m,\varepsilon_0)$, $A(t,\varepsilon) = (a_{jk}(t,\varepsilon))_{j,k=\overline{1,n}}$, $a_{jk} \in S(m-\ 1,\varepsilon_0)$, $P(t,\varepsilon,\theta) = = (p_{jk}(t,\varepsilon,\theta))_{j,k=\overline{1,n}}$, $p_{jk} \in F(m,l,\varepsilon_0,\theta)$, $\mu \in (0,\mu_0) \subset \mathbf{R}^+$.

We study the problem of the existence of the transformation of the kind:

$$x = (E + \Phi(t,\varepsilon,\theta,\mu))z, \tag{2}$$

where $\Phi \in F(m^*,l,\varepsilon^*,\theta)$ $(m^* \leqslant m,\ \varepsilon^* \leqslant \varepsilon_0)$, reducing the system (1) to kind

$$\frac{dz}{dt} = (\widetilde{\Lambda}(t,\varepsilon,\mu) + \varepsilon^2 H(t,\varepsilon) + \mu\varepsilon B(t,\varepsilon,\mu))z, \tag{3}$$

where $\widetilde{\Lambda} = \mathrm{diag}(\widetilde{\lambda}_1,...,\widetilde{\lambda}_n)$, $H = (h_{jk})_{j,k=\overline{1,n}}$, $B = (b_{jk})_{j,k=\overline{1,n}}$, $\widetilde{\lambda}_j, h_{jk}, b_{jk} \in S(m^*,\varepsilon^*)$. Means coefficients of the system (3) are slowly-varying, while the coefficients of the system (1) are oscillating.

2. Auxiliary results.

Lemma 1. *Suppose that the system (1) satisfies the following condition:*

$$\forall\nu \in \mathbf{Z},\ j,k = \overline{1,n}\ (j \neq k) : \inf_{G(\varepsilon_0)} |\omega_{jk}(t,\varepsilon) - \nu\varphi(t,\varepsilon)| \geq \gamma > 0. \tag{4}$$

Then $\exists\ \mu_1 \in (0,\mu_0)$, $\exists\ \varepsilon_1 \in (0,\varepsilon_0)$ *such that* $\forall\ \mu \in (0,\mu_1)$, $\forall\ \varepsilon \in (0,\varepsilon_1)$ *exists the transformation of kind*

$$x = (E + \widetilde{\Psi}(t,\varepsilon,\theta,\mu))y, \tag{5}$$

where $\widetilde{\Psi} = (\widetilde{\psi}_{jk}(t,\varepsilon,\theta,\mu))_{j,k=\overline{1,n}}$, $\widetilde{\psi}_{jk} \in F(m-1,l,\varepsilon_1,\theta)$, *reducing the system (1) to kind:*

$$\frac{dy}{dt} = (\Lambda(t,\varepsilon) + \varepsilon\Lambda_1(t,\varepsilon) + \mu U(t,\varepsilon,\mu) + \varepsilon^2 H(t,\varepsilon) + \mu\varepsilon V(t,\varepsilon,\theta,\mu))y, \tag{6}$$

where $\Lambda_1 = \mathrm{diag}(a_{11}, \ldots, a_{nn})$, $H = (h_{jk})_{j,k=\overline{1,n}}$, $h_{jk} \in S(m-2, \varepsilon_1)$, $U = \mathrm{diag}(u_1, \ldots, u_n)$, $u_j \in S(m, \varepsilon_1)$, $V = (v_{jk})_{j,k=\overline{1,n}}$, $v_{jk} \in F(m-1, l, \varepsilon_1, \theta)$.

Proof. We increase the first order of smallness with respect parameter ε of the off-diagonal elements in matrix of system (1). For this purpose in system (1) we make the substitution:

$$x = (E - \varepsilon Q(t, \varepsilon))x^1, \tag{7}$$

where $Q = (q_{jk})_{j,k=\overline{1,n}}$, $q_{jj} \equiv 0$, $q_{jk} = a_{jk}/(i\omega_{jk})$ $(j \neq k)$. Obviously that $q_{jk} \in \in S(m-1, \varepsilon_0)$ and $\exists\ \varepsilon_1 \leqslant \varepsilon_0$ such that $\forall \varepsilon \in (0, \varepsilon_1)$ the transformation (7) is non-degenerate. As a result of its application, we obtain:

$$\frac{dx^1}{dt} = (\Lambda(t, \varepsilon) + \varepsilon\Lambda_1(t, \varepsilon) + \varepsilon^2 H(t, \varepsilon) + \mu P(t, \varepsilon, \theta) + \mu\varepsilon\widetilde{P}(t, \varepsilon, \theta))x^1, \tag{8}$$

where Λ_1, H are defined in formulation of the theorem matrices, $\widetilde{P} = Q(E - \varepsilon Q)^{-1}P$. Thus matrix $\widetilde{P}$ belong to class $F(m-1, l, \varepsilon_1, \theta)$.

Consider now the following matrix equation:

$$\varphi(t, \varepsilon)\frac{\partial \Psi}{\partial \theta} = \Lambda(t, \varepsilon)\Psi - \Psi\Lambda(t, \varepsilon) + P(t, \varepsilon, \theta) - U(t, \varepsilon, \mu) +$$

$$+ \mu(P(t, \varepsilon, \theta)\Psi - \Psi U(t, \varepsilon, \mu)), \tag{9}$$

where $(n \times n)$-matrices $U(t, \varepsilon, \mu)$, $\Psi = \Psi(t, \varepsilon, \theta, \mu)$ must be defined. We show, that equation (9) has a solution $\Psi(t, \varepsilon, \theta, \mu) \in F(m-1, l, \varepsilon_1, \theta)$, and matrix $U(t, \varepsilon, \mu)$ will be defined as diagonal with elements from class $S(m, \varepsilon_1)$.

Together with equation (9) consider equation:

$$\varphi(t, \varepsilon)\frac{\partial \Psi_0}{\partial \theta} = \Lambda(t, \varepsilon)\Psi_0 - \Psi_0\Lambda(t, \varepsilon) + P(t, \varepsilon, \theta) - U_0(t, \varepsilon). \tag{10}$$

We show, that equation (10) for some choice of the matrix $U_0(t, \varepsilon)$ has a solution, which belong to class $F(m, l, \varepsilon_1, \theta)$. We set:

$$U_0(t, \varepsilon) = \mathrm{diag}(\Gamma_0[(P(t, \varepsilon, \theta))_{11}], \ldots, \Gamma_0[(P(t, \varepsilon, \theta))_{nn}]), \tag{11}$$

$$(\Psi_0)_{jj} = \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} \frac{\Gamma_\nu[(P(t, \varepsilon, \theta))_{jj}]}{i\nu\varphi(t, \varepsilon)} \exp(i\nu\theta(t, \varepsilon)), \tag{12}$$

$$(\Psi_0)_{jk} = -\sum_{\nu=-\infty}^{\infty} \frac{\Gamma_\nu[(P(t, \varepsilon, \theta))_{jk}]}{i(\omega_{jk}(t, \varepsilon) - \nu\varphi(t, \varepsilon))} \exp(i\nu\theta(t, \varepsilon)) \ (j \neq k). \tag{13}$$

For its choice the matrix $U_0(t, \varepsilon)$ belong to class $S(m, \varepsilon_1)$ and the matrix $\Psi_0(t, \varepsilon, \theta)$ belong to class $F(m, l, \varepsilon_1, \theta)$ and $\exists\ K \in (0, +\infty)$ such that

$$\|U_0\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant K\|P\|^*_{F(m,l,\varepsilon_1,\theta)}, \tag{14}$$

$$\|\Psi_0\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant K\|P\|^*_{F(m,l,\varepsilon_1,\theta)} \tag{15}$$

(here we have that $S(m, \varepsilon_1) \subset F(m, l, \varepsilon_1, \theta)$).

We seek the solution from class $F(m,l,\varepsilon_1,\theta)$ of equation (9) by the method of successive approximations, defining the initial approximation $\Psi_0(t,\varepsilon,\theta)$, and the subsequent approximations defining as solutions from class $F(m,l,\varepsilon_1,\theta)$, of the equations:

$$\varphi(t,\varepsilon)\frac{\partial\Psi_{s+1}}{\partial\theta} = \Lambda(t,\varepsilon)\Psi_{s+1} - \Psi_{s+1}\Lambda(t,\varepsilon) + P(t,\varepsilon,\theta)+$$
$$+\mu(P(t,\varepsilon,\theta)\Psi_s - \Psi_s U_s(t,\varepsilon,\mu)) - U_{s+1}(t,\varepsilon,\mu),\ s = 0,1,2,\ldots, \tag{16}$$

where matrices $U_1, U_2, \ldots$ must be defined also.

We set:

$$U_{s+1} = \mathrm{diag}(\Gamma_0[(P+\mu(P\Psi_s - \Psi_s U_s))_{11}],\ldots,\Gamma_0[(P+\mu(P\Psi_s - \Psi_s U_s))_{nn}], \tag{17}$$

$$(\Psi_{s+1})_{jj} = \sum_{\substack{\nu=-\infty\\(\nu\neq 0)}}^{\infty} \frac{\Gamma_\nu[(P+\mu(P\Psi_s - \Psi_s U_s))_{jj}]}{i\nu\varphi(t,\varepsilon)} \exp(i\nu\theta(t,\varepsilon)), \tag{18}$$

$$(\Psi_{s+1})_{jk} = -\sum_{\nu=-\infty}^{\infty} \frac{\Gamma_\nu[(P+\mu(P\Psi_s - \Psi_s U_s))_{jk}]}{i(\omega_{jk}(t,\varepsilon) - \nu\varphi(t,\varepsilon))} \exp(i\nu\theta(t,\varepsilon))\ (j\neq k). \tag{19}$$

For its choice the matrices U_s $(s = 0,1,2,\ldots)$ belongs to class $S(m,\varepsilon_1)$ and the matrices Ψ_s $(s = 0,1,2,\ldots)$ belongs to class $F(m,l,\varepsilon_1,\theta)$.

We define sets:

$$\Omega_1 = \{\ \Psi \in F(m,l,\varepsilon_1,\theta):\ \|\Psi - \Psi_0\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant d\ \},$$
$$\Omega_2 = \{\ U \in S(m,\varepsilon_1):\ \|U - U_0\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant d\ \},\ d>0.$$

From estimations (14), (15) then follows, that $\forall\ \Psi\in\Omega_1$, $\forall\ U\in\Omega_2$:

$$\|\Psi\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant K\|P\|^*_{F(m,l,\varepsilon_1,\theta)} + d, \tag{20}$$

$$\|U\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant K\|P\|^*_{F(m,l,\varepsilon_1,\theta)} + d. \tag{21}$$

We show, that for sufficiently small values μ all approximations Ψ $(s = 0,1,2,\ldots)$ belongs to Ω_1, and all approximations U_s $(s = 0,1,2,\ldots)$ belongs to Ω_2. Obviously $\Psi_0\in\Omega_1$, $U_s\in\Omega_2$. Suppose by induction, that $\Psi_s\in\Omega_1$, $U_s\in\Omega_2$, and show, that for sufficiently small μ $\Psi_{s+1}\in\Omega_1$, $U_{s+1}\in\Omega_2$. Really, from formulas (11), (12), (13), (17), (18), (19), inequalities (14), (15), (20), (21) and property 10) for norm $\|\cdot\|^*_{F(m,l,\varepsilon_1,\theta)}$ we have:

$$\|\Psi_{s+1} - \Psi_0\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant \mu K\|P\Psi_s - \Psi_s U_s\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant$$
$$\leqslant \mu 2^m(2^l+1)K\left(\|P\|^*_{F(m,l,\varepsilon_1,\theta)}\|\Psi_s\|^*_{F(m,l,\varepsilon_1,\theta)} + \|\Psi_s\|^*_{F(m,l,\varepsilon_1,\theta)}\|\ U_s\|^*_{F(m,l,\varepsilon_1,\theta)}\right) \leq$$
$$\leqslant \mu 2^m(2^l+1)K\left(\|P\|^*_{F(m,l,\varepsilon_1,\theta)}(K\|P\|^*_{F(m,l,\varepsilon_1,\theta)} + d) + (K\|P\|^*_{F(m,l,\varepsilon_1,\theta)} + d)^2\right),$$

and similarly

$$\|U_{s+1} - U_0\|^*_{F(m,l,\varepsilon_1,\theta)} \leqslant$$
$$\leqslant \mu 2^m(2^l+1)K\left(\|P\|^*_{F(m,l,\varepsilon_1,\theta)}(K\|P\|^*_{F(m,l,\varepsilon_1,\theta)} + d) + (K\|P\|^*_{F(m,l,\varepsilon_1,\theta)} + d)^2\right).$$

Require that μ was so small that the inequality:

$$\mu 2^m(2^l+1)K\left(\|P\|^*_{F(m,l,\varepsilon_1,\theta)}(K\|P\|^*_{F(m,l,\varepsilon_1,\theta)}+d)+\right.$$
$$\left.+(K\|P\|^*_{F(m,l,\varepsilon_1,\theta)}+d)^2\right)\le d_0<d. \qquad (22)$$

Then $\Psi_{s+1}\in\Omega_1$, $U_{s+1}\in\Omega_2$ and thus $\Psi_s\in\Omega_1$, $U_s\in\Omega_2$ $(s=0,1,2,\ldots)$.

We prove now convergence of process (17) – (19). From (16) we have:

$$\varphi(t,\varepsilon)\frac{\partial(\Psi_{s+1}-\Psi_s)}{\partial\theta}=\Lambda(t,\varepsilon)(\Psi_{s+1}-\Psi_s)-(\Psi_{s+1}-\Psi_s)\Lambda(t,\varepsilon)+$$
$$+\mu(P(t,\varepsilon,\theta)(\Psi_s-\Psi_{s-1})-(\Psi_sU_s-\Psi_{s-1}U_{s-1}))-(U_{s+1}-U_s).$$

Then from (14), (15), (17), (18), (19) we obtain:

$$\|\Psi_{s+1}-\Psi_s\|^*_{F(m,l,\varepsilon_1,\theta)}\leqslant\mu K\|P(\Psi_s-\Psi_{s-1})-(\Psi_sU_s-\Psi_{s-1}U_{s-1})\|^*_{F(m,l,\varepsilon_1,\theta)}\leqslant$$
$$\leqslant\mu K2^m(2^l+1)\left(\|P\|^*_{F(m,l,\varepsilon_1,\theta)}\|\Psi_s-\Psi_{s-1}\|^*_{F(m,l,\varepsilon_1,\theta)}+\right.$$
$$\left.+\|\Psi_s-\Psi_{s-1}\|^*_{F(m,l,\varepsilon_1,\theta)}\|U_s\|^*_{F(m,l,\varepsilon_1,\theta)}+\|\Psi_s\|^*_{F(m,l,\varepsilon_1,\theta)}\|U_s-U_{s-1}\|^*_{F(m,l,\varepsilon_1,\theta)}\right)\leqslant$$
$$\leqslant\mu K2^m(2^l+1)\left((\|P\|^*_{F(m,l,\varepsilon_1,\theta)}+K\|P\|^*_{F(m,l,\varepsilon_1,\theta)}+d)\|\Psi_s-\Psi_{s-1}\|^*_{F(m,l,\varepsilon_1,\theta)}+\right.$$
$$\left.+(K\|P\|^*_{F(m,l,\varepsilon_1,\theta)}+d)\|U_s-U_{s-1}\|^*_{F(m,l,\varepsilon_1,\theta)}\right).$$

A similar estimate holds for $\|U_{s+1}-U_s\|^*_{F(m,l,\varepsilon_1,\theta)}$. Thus we obtain:

$$\|\Psi_{s+1}-\Psi_s\|^*_{F(m,l,\varepsilon_1,\theta)}+\|U_{s+1}-U_s\|^*_{F(m,l,\varepsilon_1,\theta)}\le 2\mu 2^m(2^l+1)((K+1)\|P\|^*_{F(m,l,\varepsilon_1,\theta)}+d)\times$$
$$\times\left(\|\Psi_s-\Psi_{s-1}\|^*_{F(m,l,\varepsilon_1,\theta)}+\|U_s-U_{s-1}\|^*_{F(m,l,\varepsilon_1,\theta)}\right).$$

Require that μ was so small that the inequality:

$$2\mu 2^m(2^l+1)((K+1)\|P\|^*_{F(m,l,\varepsilon_1,\theta)}+d)<1. \qquad (23)$$

Then desired convergence is guaranteed.

Thus when the inequalities (22), (23), the equation (9) has a solution $\Psi(t,\varepsilon,\theta,\mu)\in$ $\in F(m,l,\varepsilon_1,\theta)$, and diagonal matrix $U(t,\varepsilon,\mu)$ $S(m,\varepsilon_1)$. We make now in the system (8) the substitution:

$$x^1=(E+\mu\Psi(t,\varepsilon,\theta,\mu))y. \qquad (24)$$

As a result we obtain the system of kind (6) in which the matrix $V(t,\varepsilon,\theta,\mu)$ are defined from the equation:

$$(E+\mu\Psi)V=\Lambda_1\Psi-\Psi\Lambda_1+\widetilde{P}((E+\mu\Psi)+\varepsilon(H\Psi-\Psi H)-\frac{1}{\varepsilon}\frac{\partial\Psi}{\partial t}. \qquad (25)$$

Obviously, that for sufficiently small μ equation (25) has a unique solution $V(t,\varepsilon,\theta,\mu)$, and this solution belong to class $F(m-1,l,\varepsilon_1,\theta)$.

Lemma 1 are proved.

Lemma 2. *Let we have the scalar linear non-homogeneous first-order differential equation:*

$$\frac{dx}{dt} = (i\omega(t,\varepsilon) + \varepsilon\alpha(t,\varepsilon) + \mu u(t,\varepsilon))x + \varepsilon v(t,\varepsilon,\theta), \tag{26}$$

where $\omega(t,\varepsilon) \in S(m,\varepsilon_1)$, $\omega(t,\varepsilon) \in \mathbf{R}^+$, $u(t,\varepsilon) \in S(m,\varepsilon_1)$, $\alpha(t,\varepsilon) \in S(m-1,\varepsilon_1)$, $v(t,\varepsilon,\theta) \in F(m-1,l,\varepsilon_1,\theta)$ *and the following conditions:*

1) $\inf\limits_{G(\varepsilon_1)} |\omega(t,\varepsilon) - \nu\varphi(t,\varepsilon)| \geq \gamma > 0 \ \forall \nu \in \mathbf{Z}$;

2) *alternative holds: or* $\mathrm{Re}\, u(t,\varepsilon) \equiv 0$, *or* $\inf\limits_{G(\varepsilon_1)} |\mathrm{Re}\, u(t,\varepsilon)| = \gamma_1 > 0$.

Then $\exists\ \varepsilon_2 \in (0,\varepsilon_1)$, $\mu_2 \in (0,\mu_1)$ such that $\forall\ \mu \in (0,\mu_2)$, $\varepsilon \in (0,\varepsilon_2)$ *the equation (26) has a particular solution* $x(t,\varepsilon,\theta,\mu) \in F(m-1,l,\varepsilon_2,\theta)$, and $\exists K_2 \in (0,+\infty)$ *such that:*

$$\|x(t,\varepsilon,\theta,\mu)\|_{F(m-1,l,\varepsilon_2,\theta)} \leqslant K_2 \|x(t,\varepsilon,\theta,\mu)\|_{F(m-1,l,\varepsilon_2,\theta)}.$$

Proof of this lemma is completely similar to proof of Lemma 2 from paper [2].

Lemma 3. *Let the function*

$$f(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} f_n(t,\varepsilon)\exp(in\theta(t,\varepsilon)) \tag{27}$$

belong to class $F(m-1,l,\varepsilon_1,\theta)$. *Then the function*

$$x(t,\varepsilon,\theta(t,\varepsilon)) = \varepsilon \int\limits_0^t f(\tau,\varepsilon,\theta(\tau,\varepsilon))d\tau$$

belong to class $F(m-1,l,\varepsilon_1,\theta)$ *also, and* $\exists\ K_3 \in (0,+\infty)$ *such that:*

$$\|x\|_{F(m-1,l,\varepsilon_1,\theta)} \leqslant K_3 \|f\|_{F(m-1,l,\varepsilon_1,\theta)}.$$

Proof. We define operators:

$$D_n^0 u = u,\ D_n^1 u = \frac{d}{dt}\left(\frac{u(t,\varepsilon)}{in\varphi(t,\varepsilon)}\right),\ D_n^k u = D_n^1(D_n^{k-1}u).$$

Consider

$$\varepsilon \int\limits_0^t f(\tau,\varepsilon,\theta(\tau,\varepsilon))d\tau = \sum_{\substack{\nu=-\infty \\ (\nu\neq 0)}}^{\infty} \varepsilon \int\limits_0^t f_n(\tau,\varepsilon,\theta(\tau,\varepsilon))e^{in\theta(\tau,\varepsilon)}d\tau$$

(by the uniform convergence of the series (27) term by term integration lawfully). By the $(m-1)$-fold integration by parts we obtain:

$$\varepsilon \int\limits_0^t f_n(\tau,\varepsilon,\theta(\tau,\varepsilon))e^{in\theta(\tau,\varepsilon)}d\tau = a_n(t,\varepsilon)e^{in\theta(t,\varepsilon)} - a_n(0,\varepsilon),$$

where

$$a_n(t,\varepsilon)=\varepsilon\sum_{k=0}^{m-2}(-1)^k\frac{D_n^k(f_n(t,\varepsilon))}{in\varphi(t,\varepsilon)}+(-1)^{m-1}e^{-in\theta(t,\varepsilon)}\int\limits_0^t D_n^{m-1}(f_n(\tau,\varepsilon))e^{in\theta(\tau,\varepsilon)}d\tau.$$

Applicable to the function $a_n(t,\varepsilon)$ operator $D_n^s\left(\frac{d}{dt}\right)$. We obtain:

$$D_n^s\left(\frac{da_n}{dt}\right)=\varepsilon\sum_{k=0}^{m-s-3}(-1)^kD_n^{k+s+1}(f_n(t,\varepsilon))+$$

$$+(-1)^{m+s-2}in\varphi(t,\varepsilon)e^{-in\theta(t,\varepsilon)}\varepsilon\int\limits_0^t D_n^{m-1}(f_n(\tau,\varepsilon))e^{in\theta(\tau,\varepsilon)}d\tau.$$

Obviously, that $D_n^k(f_n(t,\varepsilon))=\varepsilon^kf_{nk}^*(t,\varepsilon)$, and

$$\sum_{n=-\infty}^{\infty}|n|\sup_{G(\varepsilon_1)}|f_{nk}^*(t,\varepsilon)|<+\infty\ (k=\overline{1,m-1}).$$

So:

$$\left|\varepsilon\int\limits_0^t D_n^{m-1}(f_n(\tau,\varepsilon))e^{in\theta(\tau,\varepsilon)}d\tau\right|\leqslant L\varepsilon^{m-1}\sup_{G(\varepsilon_1)}|f_{n,m-1}^*(t,\varepsilon)|.$$

From theese estimations follows, that $\forall\ s=\overline{0,m-2}$: $d^{s+1}a_n(t,\varepsilon)/dt^{s+1}=\varepsilon^{s+1}a_{ns}^*(t,\varepsilon)$, and

$$\sum_{n=-\infty}^{\infty}|n|\sup_{G(\varepsilon_1)}|a_{ns}^*(t,\varepsilon)|<+\infty.$$

Lemma 3 are proved.

3. Principal Results.

Theorem. *Suppose the system (1) such, that:*

1) $\forall\nu\in\mathbf{Z},\ j,k=\overline{1,n}\ (j\neq k):\ \inf\limits_{G(\varepsilon_0)}|\omega_{jk}(t,\varepsilon)-\nu\varphi(t,\varepsilon)|\geq\gamma>0;$ (28)

2) *the elements* $u_j(t,\varepsilon,\mu)$ $(j=\overline{1,n})$ *of the diagonlal matrix* $U(t,\varepsilon,\mu)$*, whish defined in Lemma 1, have the alternative:*
or $\mathrm{Re}(u_j(t,\varepsilon,\mu)-u_k(t,\varepsilon,\mu))\equiv 0$ $(j,k=\overline{1,n},\ j\neq k)$;
or $\inf\limits_{G(\varepsilon_1)}|\mathrm{Re}(u_j(t,\varepsilon,\mu)-u_k(t,\varepsilon,\mu))|\geq\gamma_0>0$, *where* ε_1 *are defined in Lemma 1.*

Then $\exists_3\in(0,\varepsilon_0)$, $\mu_3\in(0,\mu_0)$ *such that* $\forall\ \varepsilon\in(0,\varepsilon_3)$, $\forall\ \mu\in(0,\mu_3)$ *exists the transformation of kind (2), where* $\Phi(t,\varepsilon,\theta,\mu)\in F(m-1,l,\varepsilon_3,\theta)$*, which reducing the system (1) to form (3), where* $H\in S(m-2,\varepsilon_3)$, $B\in S(m-1,\varepsilon_3)$.

Proof. Based on the Lemma 1, we reduce the system (1) by the transformation (5) to kind (6). We construct now the transformation

$$y=(E+\mu X(t,\varepsilon,\theta,\mu))z,\tag{29}$$

reducing the system (6) to form (3). We obtain the follows differential equation with respect matrix X:

$$\frac{dX}{dt}=\widetilde{\Lambda}(t,\varepsilon,\mu)X-X\widetilde{\Lambda}(t,\varepsilon,\mu)+\varepsilon V(t,\varepsilon,\theta,\mu)-\varepsilon B(t,\varepsilon,\mu)+$$
$$+\varepsilon^2(H(t,\varepsilon)X-XH(t,\varepsilon))+\mu\varepsilon(V(t,\varepsilon,\theta,\mu)X-XB(t,\varepsilon,\mu)), \qquad (30)$$

where $\widetilde{\Lambda}=\Lambda(t,\varepsilon)+\varepsilon\Lambda_1(t,\varepsilon)+\mu U(t,\varepsilon,\mu)=\mathrm{diag}(\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_n)$, $\widetilde{\lambda}_j=\lambda_j(t,\varepsilon)+\varepsilon a_{jj}(t,\varepsilon)+$ $+\mu u_j(t,\varepsilon,\mu)\in S(m-1,\varepsilon_1)$ $(j=\overline{1,n})$. The matrix $B(t,\varepsilon,\mu)$ must be defined.

With the equation (30) we consider the truncated equation:

$$\frac{dX_0}{dt}=\widetilde{\Lambda}(t,\varepsilon,\mu)X_0-X_0\widetilde{\Lambda}(t,\varepsilon,\mu)+\varepsilon V(t,\varepsilon,\theta,\mu)-\varepsilon B_0(t,\varepsilon,\mu), \qquad (31)$$

where the matrix $B_0(t,\varepsilon,\mu)$ must be defined. In the component-wise form the equation (31) has the kind:

$$\frac{d((X_0)_{jk})}{dt}=(\widetilde{\lambda}_j(t,\varepsilon,\mu)-\widetilde{\lambda}_k(t,\varepsilon,\mu))(X_0)_{jk}+\varepsilon(V(t,\varepsilon,\theta,\mu)-B_0(t,\varepsilon,\mu))_{jk}, j,k=\overline{1,n}. \qquad (32)$$

Consider the case $j=k$. We have:

$$\frac{d((X_0)_{jj})}{dt}=\varepsilon(V(t,\varepsilon,\theta,\mu))_{jj}-\varepsilon(B_0(t,\varepsilon,\mu))_{jj},\ j=\overline{1,n}. \qquad (33)$$

Assume $(B_0)_{jj}=\Gamma_0[(V)_{jj}]$ $(j=\overline{1,n})$. Then based on Lemma 3 the equation (33) has a particular solution $(X_0)_{jj}$ from class $F(m-1,l,\varepsilon_1,\theta)$, and $\exists\ K_4\in(0,+\infty)$ such that:

$$\|(X_0)_{jj}\|_{F(m-1,l,\varepsilon_1,\theta)}\leqslant K_4\|(V)_{jj}\|_{F(m-1,l,\varepsilon_1,\theta)}.$$

Let now $j\neq k$. Then we have:

$$\frac{d((X_0)_{jk})}{dt}=(i\omega_{jk}(t,\varepsilon)+\varepsilon(a_{jj}(t,\varepsilon)-a_{kk}(t,\varepsilon))+\mu(u_j(t,\varepsilon,\mu)-u_k(t,\varepsilon,\mu)))(X_0)_{jk}+$$
$$+\varepsilon(V(t,\varepsilon,\theta,\mu))_{jk}-\varepsilon(B(t,\varepsilon,\mu))_{jk},\ j,k=\overline{1,n};\ j\neq k. \qquad (34)$$

Assume $(B_0)_{jk}\equiv 0$ $(j\neq k)$. Then based on Lemma 2 by condidtion 2) of the theorem we obtain, that equation (34) has a particular solution $(X_0)_{jk}$ from class $F(m-1,l,\varepsilon_3,\theta)$ $(\varepsilon_1\leqslant\varepsilon_1)$, and $\exists K_5\in(0,+\infty)$ such that:

$$\|(X_0)_{jk}\|_{F(m-1,l,\varepsilon_3,\theta)}\leqslant K_5\|(V)_{jk}\|_{F(m-1,l,\varepsilon_1,\theta)}.$$

It follows that if $B_0=\mathrm{diag}(\Gamma_0[(V)_{11}],\ldots,\Gamma_0[(V)_{nn}])$, then matrix equation (31) has a particular solution $X_0(t,\varepsilon,\theta,\mu)$ from class $F(m-1,l,\varepsilon_4,\theta)$ $(\varepsilon_4\leqslant\varepsilon_1)$, and $\exists K_6\in(0,+\infty)$ such that:

$$\|X_0\|^*_{F(m-1,l,\varepsilon_4,\theta)}\leqslant K_6\|V\|^*_{F(m-1,l,\varepsilon_1,\theta)}.$$

Now pursuing arguments similar to the proof of the Lemma 1, it is easy to show that $\exists\ \varepsilon_5\in(0,\varepsilon_4)$, $\mu_5\in(0,\mu_1)$ such that $\forall\ \varepsilon\in(0,\varepsilon_5)$, $\mu\in(0,\mu_5)$ the equation (30) has a particular solution $X(t,\varepsilon,\theta,\mu)$ from class $F(m-1,l,\varepsilon_5,\theta)$.

The theorem are proved.

Conclusion. Thus, for the system (1) the sufficient conditions of the existence of the transformation, which reducing this system close to a system with slowly-varying coefficients and the algorithm for constructing this transformation are obtained.

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