Mathematical Subject Classification: 34A34, 34A25
UDC 517.926

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ON THE OSCILLATIONS IN THE QUASI-LINEAR SECOND ORDER DIFFERENTIAL SYSTEMS WITH SLOWLY-VARYING PARAMETERS

Щоголев С. А. Про коливання в квазілінійних диференціальних системах другого порядку з повільно змінними параметрами. Для квазілінійної диференціальної системи другого порядку з суто уявними власними значеннями матриці лініної частини отримано умови існування частинного розв'язку, зображуваного у вигляді абсолютно та рівномірно збіжних рядів Фур'є з повільно змінними коефіцієнтами та частотою на асимптотично великому проміжку зміни незалежної змінної.
Ключові слова: диференціальний, повільно змінний, ряди Фур'є.

Щёголев С. А. О колебаних в квазилинейных дифференциальных системах второго порядка с медленно меняющимися параметрами. Для квазилинейной дифференциальной системы второго порядка с чисто мнимыми собственными значениями матрицы линейной части получены условия существования частного решения, представимого в виде абсолютно и равномерно сходящихся рядов Фурье с медленно меняющимися коэффициентами и частотой на асимптотически большом промежутке изменения нзависимой переменной.
Ключевые слова: дифференциальный, медленно меняющийся, ряды Фурье.

Shchogolev S. A. On the oscillations in the quasi-linear second order differential systems with slowly-varying parameters. For the quasi-linear second order differential system with pure imaginary eigenvalues of the matrix of the linear part, the conditions of the existence of the particular solution, representable as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency, are obtained at the asymptotic long interval of the independent variable.
Key words: differential, slowly-varying, Fourier series.

Introduction. In the theory of the differential equations well known the problem of the periodic solutions od the differential equations and its systems [1–3]. However, the strict periodicity of the coefficients of the system and its decisions is some idealization. In real physical systems, the amplitude and frequency of oscillations, generally speaking, are not constant, and represent yourself in a certain sense, slowly varying function of time. An important tool in the study of periodic solutions is a representation of the desired solution in the form of trigonometric Fourier series:

$$x(t) = \sum_{n=-\infty}^{\infty} x_n \mathrm{e}^{in\nu t} \tag{1}$$

(ν – frequency). Sometimes it takes an additional condition

$$\sum_{n=-\infty}^{\infty} |x_n| < +\infty, \tag{2}$$

which guaranteed by $\nu \in \mathbf{R}$ the absolutely and uniformly convergence of series (1). As noted in the [4], there is good reason to replace the study of periodic solutions of the general form by research solutions that can be represented in the form (1) with the additional condition (2). Narrowing of the space of considered solutions a constructive way to their analytical representation, in particular, facilitates the construction of approximate analytical expressions for the solutions in the form of finite trigonometric sums. Similar problems are considered, for example, in [5–7]. In this regard, the following problem arises: to research the similar type solutions of the differential systems with slowly varying parameters, that is to obtain the conditions of the existence of the solutions, which represented as an absolutely and uniformly convergent Fourier-series with slowly varying coefficients and frequency. In this formulation, the problem is substantially different from the problem of periodic solutions of the general form. In some papers of A. V. Kostin and author [8–11] the conditions of existence of solutions of this type are obtained for a quasi-linear differential systems, and researched a systems with different properties of the matrix of the linear part. Considered in these papers systems contained two small parameters μ and ε, first of which characterizes the smallness of nonlinearities, and the second - slow variability coefficients of the systems. The role of these parameters in the study of oscillations differ significantly, and, generally speaking, they do not depend on each other. At the same time in a number of well-known works on the theory of oscillations of quasi-linear systems, these parameters are the same.

Notation. Let $G(\varepsilon_0) = \{t, \varepsilon : \ 0 < \varepsilon < \varepsilon_0, \ -L\varepsilon^{-1} \leqslant t \leqslant L\varepsilon^{-1}, \ 0 < L < +\infty\}$.

Definition 1. We say, that a function $f(t,\varepsilon)$ belong to class $S(m,\varepsilon_0)$ ($m \in \mathbf{N} \cup \{0\}$), if

1) $f : G(\varepsilon_0) \to \mathbf{C}$, 2) $f(t,\varepsilon) \in C^m(G(\varepsilon_0))$ with respect t;

3) $d^k f(t,\varepsilon)/dt^k = \varepsilon^k f_k^*(t,\varepsilon)$ ($0 \leqslant k \leqslant m$),

$$\|f\|_{S(m,\varepsilon_0)} \stackrel{def}{=} \sum_{k=0}^{m} \sup_{G(\varepsilon_0)} |f_k^*(t,\varepsilon)| < +\infty.$$

Under the slowly varying function we mean a function of class $S(m,\varepsilon_0)$.

Definition 2. We say, that a function $f(t,\varepsilon,\theta(t,\varepsilon))$ belong to class $F(m,l,\varepsilon_0,\theta)$ ($m, l \in \mathbf{N} \cup \{0\}$), if this function can be represented as:

$$f(t,\varepsilon,\theta(t,\varepsilon)) = \sum_{n=-\infty}^{\infty} f_n(t,\varepsilon) \exp\left(in\theta(t,\varepsilon)\right),$$

and:

1) $f_n(t,\varepsilon) \in S(m,\varepsilon_0)$;

2) $\|f\|_{F(m,l,\varepsilon_0,\theta)} \stackrel{def}{=} \|f_0\|_{S(m,\varepsilon_0)} + \sum\limits_{n=-\infty}^{\infty} |n|^l \|f_n\|_{S(m,\varepsilon_0)} < +\infty$, in particular

$$\|f\|_{F(m,0,\varepsilon_0,\theta)} = \sum_{n=-\infty}^{\infty} \|f_n\|_{S(m,\varepsilon_0)};$$

3) $\theta(t,\varepsilon)=\int\limits_0^t \varphi(\tau,\varepsilon)d\tau$, $\varphi\in\mathbf{R}^+$, $\varphi\in S(m,\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)}\varphi(t,\varepsilon)=\varphi_0>0$.

Let $f(t,\varepsilon,\theta)\in F(m,l,\varepsilon_0,\theta)$. We denote $\forall\ n\in\mathbf{Z}$:

$$\Gamma_n[f]=\frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)\exp(-in\theta)d\theta,$$

particular

$$\Gamma_0[f]=\frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)d\theta.$$

Some basic properties of functions of the classes $S(m,\varepsilon_0)$ and $F(m,l,\varepsilon_0,\theta)$ properties are stated and proved in [12].

Main Results

1. Statement of the Problem. Consider the following differential system:

$$\frac{dx_j}{dt}=\sum_{k=1}^{2} a_{jk}(t,\varepsilon)x_k+f_j(t,\varepsilon,\theta(t,\varepsilon))+\mu X_j(t,\varepsilon,\theta(t,\varepsilon),x_1,x_2),\ j=1,2,\tag{3}$$

where $\operatorname{colon}(x_1,x_2)\in D\subset\mathbf{R}^2$, $a_{jk}\in S(m,\varepsilon_0)$, $f_j\in F(m,l,\varepsilon_0,\theta)$, X_1,X_2 belongs to class $F(m,l,\varepsilon_0,\theta)$ with respect t,ε,θ and analytic with respect $x_1,x_2\in D$; $\mu\in(0,\mu_0)\in\mathbf{R}^+$. Functions a_{jk}, f_j, X_j $(j,k=1,2)$ are real, and eigenvalues of matrix $(a_{jk}(t,\varepsilon))$ have a form $\pm i\omega(t,\varepsilon)$, where $\omega\in\mathbf{R}^+$.

We study a problem of existence of the particular solutions of the classes $F(m^*,l^*,\varepsilon^*,\theta)$ of the system (3).

The system (3) are considered under the following assumptions:

$$\inf_{G(\varepsilon_0)}|a_{12}(t,\varepsilon)|>0;\tag{4}$$

$$\inf_{G(\varepsilon_0)}|k\omega(t,\varepsilon)-n\varphi(t,\varepsilon)|\geqslant\gamma>0,\ k=1,2;\ n\in\mathbf{Z}\tag{5}$$

(means we study the case of absent of the resonance between frequencies ω and φ in system (3)).

We note, that similar problem are considered by author in paper [13], but in this paper sifficiently using the assumption, that parameters μ and ε are related by $\mu^r\leqslant\varepsilon^2$, where $r\in\mathbf{N}$. This condition, though in some cases performed, yet is sufficiently tough. Therefore in this paper we seek to obtain conditions of the existence of solutions of these classes, which are not supposed to such a relationship between the parameters μ and ε.

2. Auxiliary results. Consider the following system of the differential equations:

$$\varphi(t,\varepsilon)\frac{d\xi_j}{d\theta}=\sum_{k=1}^{2} a_{jk}(t,\varepsilon)\xi_k+f_j(t,\varepsilon,\theta)+\mu X_j(t,\varepsilon,\theta,\xi_1,\xi_2),\ j=1,2,\tag{6}$$

where $(\xi_1,\xi_2)\in D_1\subset\mathbf{R}^2$, a_{jk}, f_j, X_j are the same as in the system (3), but (t,ε) considered as constants.

Lemma. *If condition (5), then* $\exists\ \mu_1 \in (0, \mu_0)$ *such that* $\forall\ \mu \in (0, \mu_1)$ *the system (6) has a particular solution* $\xi_j(t, \varepsilon, \theta, \mu)$ $(j = 1, 2)$*, which belong to class* $F(m, l, \varepsilon_0, \theta)$*.*

Proof. Consider generating system corresponding to system (6):

$$\varphi(t, \varepsilon) \frac{d\xi_{j0}}{d\theta} = \sum_{k=1}^{2} a_{jk}(t, \varepsilon) \xi_{k0} + f_j(t, \varepsilon, \theta), \ j = 1, 2. \tag{7}$$

Easy to show that

$$\Delta_n(t, \varepsilon) = \begin{vmatrix} a_{11}(t, \varepsilon) - in\varphi(t, \varepsilon) & a_{12}(t, \varepsilon) \\ a_{21}(t, \varepsilon) & a_{22}(t, \varepsilon) - in\varphi(t, \varepsilon) \end{vmatrix} =$$
$$= (\omega(t, \varepsilon) - n\varphi(t, \varepsilon))(\omega(t, \varepsilon) + n\varphi(t, \varepsilon)),$$

however, based on the assumption (5):

$$\inf_{G(\varepsilon_0)} |\Delta_n(t, \varepsilon)| \geqslant \gamma^2 > 0 \ \forall\ n \in \mathbf{Z}.$$

Consider the following solution of system (7):

$$\xi_{j0}(t, \varepsilon, \theta) = L_j[f_1, f_2] = \sum_{n=-\infty}^{\infty} \frac{\Delta_{jn}(t, \varepsilon)}{\Delta_n(t, \varepsilon)} \exp(in\theta), \ j = 1, 2,$$

where $\Delta_{jn}(t, \varepsilon)$ are determinants, which obtained from $\Delta_n(t, \varepsilon)$ by replacing in it the j-th column by the $\mathrm{col}(-\Gamma_n[f_1(t, \varepsilon, \theta)],\ -\Gamma_n[f_2(t, \varepsilon, \theta)])$.

Operators L_1, L_2 has a properties:
1) if $f_1, f_2, g_1, g_2 \in F(m, l, \varepsilon_0, \theta)$, then $L_j[f_1, f_2], L_j[g_1, g_2] \in F(m, l, \varepsilon_0, \theta)$, and $L_j[f_1 + g_1, f_2 + g_2] = L_j[f_1, f_2] + L_j[g_1, g_2]$, $L_j[cf_1, cf_2] = cL_j[f_1, f_2]$ $(j = 1, 2)$;
2) $\exists\ K_1 \in (0, +\infty)$ such that

$$\sum_{j=1}^{2} \|L_j[f_1, f_2]\|_{F(m,l,\varepsilon_0,\theta)} \leqslant K_1 \sum_{j=1}^{2} \|f_j\|_{F(m,l,\varepsilon_0,\theta)}.$$

Based on these properties we can state, that $\xi_{10}, \xi_{20} \in F(m, l, \varepsilon_0, \theta)$.

We make in the system (6) the substitution:

$$\xi_j = \xi_{j0}(t, \varepsilon, \theta) + \mu\eta_j, \ j = 1, 2, \tag{8}$$

where η_1, η_2 – new unknown functions. We obtain:

$$\varphi(t, \varepsilon) \frac{d\eta_j}{d\theta} = \sum_{k=1}^{2} a_{jk}(t, \varepsilon) \eta_k + g_j(t, \varepsilon, \theta) + \mu \sum_{k=1}^{2} u_{jk}(t, \varepsilon, \theta) \eta_k +$$
$$+\mu^2 H_j(t, \varepsilon, \theta, \eta_1, \eta_2, \mu), \ j = 1, 2, \tag{9}$$

where $g_j(t, \varepsilon, \theta) = X_j(t, \varepsilon, \theta, \xi_{10}, \xi_{20})$, $u_{jk}(t, \varepsilon, \theta) = \frac{\partial X_j(t, \varepsilon, \theta, \xi_{10}, \xi_{20})}{\partial x_k}$,

$$H_j(t, \varepsilon, \theta, \eta_1, \eta_2, \mu) = \frac{1}{2} \left(\frac{\partial^2 X_j(t, \varepsilon, \theta, \xi_{10} + \nu\mu\eta_1, \xi_{20} + \nu\mu\eta_2)}{\partial x_1^2} \eta_1^2 + \right.$$
$$\left. + 2 \frac{\partial^2 X_j(\cdots)}{\partial x_1 \partial x_2} \eta_1 \eta_2 + \frac{\partial^2 X_j(\cdots)}{\partial x_2^2} \eta_2^2 \right), 0 < \nu < 1.$$

By analyticity of functions X_1, X_2 the functions $g_j, u_{jk} \in F(m, l, \varepsilon_0, \theta)$, the functions H_1, H_2 belongs to class $F(m, l, \varepsilon_0, \theta)$ with respect t, ε, θ and analytic with respect η_1, η_2 in some area of these variables, and g_j, u_{jk}, H_j are real.

Along with the system (9) we consider the linear nonhomogeneous system:

$$\varphi(t,\varepsilon)\frac{d\eta_{j0}}{d\theta} = \sum_{k=1}^{2} a_{jk}(t,\varepsilon)\eta_{k0} + g_j(t,\varepsilon,\theta),\ j = 1,2. \tag{10}$$

This system has a particular solution $\eta_{j0} = L_j[g_1, g_2] \in F(m, l, \varepsilon_0, \theta)$ $(j = 1, 2)$. We seek the solution from class $F(m, l, \varepsilon_0, \theta)$ of system (9) by the method of successive approximations, defining the initial approximation $\eta_{j0}(t, \varepsilon, \theta)$ $(j = 1, 2)$, and the subsequent approximations defining by formulas:

$$\eta_{js} = L_j\left[g_1(t,\varepsilon,\theta) + \mu\sum_{k=1}^{2} u_{1k}(t,\varepsilon,\theta)\eta_{k,s-1} + \mu^2 H_1(t,\varepsilon,\theta,\eta_{1,s-1},\eta_{2,s-1},\mu),\right.$$

$$\left. g_2(t,\varepsilon,\theta) + \mu\sum_{k=1}^{2} u_{2k}(t,\varepsilon,\theta)\eta_{k,s-1} + \mu^2 H_2(t,\varepsilon,\theta,\eta_{1,s-1},\eta_{2,s-1},\mu)\right],$$

$$j = 1,2;\ s = 1,2,\ldots. \tag{11}$$

We denote:

$$\Omega = \left\{\eta_1,\eta_2 \in F(m,l,\varepsilon_0,\theta):\ \sum_{j=1}^{2}\|\eta_j - \eta_{j0}\|_{F(m,l,\varepsilon_0,\theta)} \leqslant d;\ d > 0\right\}.$$

By analyticity of functions H_1, H_2 $\exists M(d), K_2(d) \in (0, +\infty)$ such that $\forall \eta_1^*, \eta_2^*, \eta_1^{**}, \eta_2^{**} \in \Omega$:

$$\sum_{j=1}^{2}\|H_j(t,\varepsilon,\theta,\eta_1^*,\eta_2^*,\mu)\|_{F(m,l,\varepsilon_0,\theta)} \leqslant M(d),$$

$$\sum_{j=1}^{2}\|H_j(t,\varepsilon,\theta,\eta_1^*,\eta_2^*,\mu) - H_j(t,\varepsilon,\theta,\eta_1^{**},\eta_2^{**},\mu)\|_{F(m,l,\varepsilon_0,\theta)} \leqslant$$

$$\leqslant K_2(d)\sum_{j=1}^{2}\|\eta_j^* - \eta_j^{**}\|_{F(m,l,\varepsilon_0,\theta)}.$$

We denote: $u^* = \max\limits_{j,k}\|u_{jk}(t,\varepsilon,\theta)\|_{F(m,l,\varepsilon_0,\theta)}$.

Using techniques contraction mapping principle, it is easy to show, that by condidtion

$$\mu K_1(d)\left(2^{m+1}(2^l+1)u^*\left(K_1(d)\sum_{k=1}^{2}\|g_k\|_{F(m,l,\varepsilon_0,\theta)} + d\right) + \mu M(d)\right) \leqslant d_0 < d$$

all approximations η_{js} $(j = 1, 2;\ s = 0, 1, 2, \ldots)$ remain inside Ω. And by condition

$$\mu K_1(d)(2^{m+1}(2^l+1)u^* + \mu K_2(d)) < 1$$

the process of successive approximations (11) converges to the solution η_1, η_2 from class $F(m, l, \varepsilon_0, \theta)$ of the system (9), and this solution are real.

Lemma are proved.

3. Method of solving the problem. We make in the system (3) the substitution:

$$x_j = \xi_j(t, \varepsilon, \theta, \mu) + y_j,\ \ j = 1, 2, \tag{12}$$

where $\xi_(t, \varepsilon, \theta, \mu)$ $(j = 1, 2)$ – solution from class $F(m, l, \varepsilon_0, \theta)$ of system (6), and y_1, y_2 – new unknown functions. We obtain:

$$\frac{dy_j}{dt} = \sum_{k=1}^{2} a_{jk}(t, \varepsilon) y_k + \varepsilon h_j(t, \varepsilon, \theta, \mu) + \mu \sum_{k=1}^{2} u_{jk}(t, \varepsilon, \theta) y_k +$$

$$+\mu^2 \sum_{k=1}^{2} v_{jk}(t, \varepsilon, \theta, \mu) y_k + \mu Y_j(t, \varepsilon, \theta, y_1, y_2, \mu),\ \ j = 1, 2, \tag{13}$$

where real functions h_j from class $F(m-1, l, \varepsilon_0, \theta)$, real functions

$$v_{jk} = \sum_{s=1}^{2} \frac{\partial^2 X_j(t, \varepsilon, \theta, \xi_{10} + \nu_1 \mu \eta_1, \xi_{20} + \nu_1 \mu \eta_2)}{\partial x_k \partial x_s}\ \eta_s$$

from class $F(m, l, \varepsilon_0, \theta)$, real functions Y_1, Y_2 from class $F(m, l, \varepsilon, \theta)$ with respect t, ε, θ, analytic with respect y_1, y_2 in some area of these variables and contain terms not lower than second order with respect y_1, y_2.

We make in the system (13) the substitution:

$$y_j = \varepsilon y_j^{(0)} + y_j^{(1)},\ \ j = 1, 2,$$

where $y_1^{(1)}, y_2^{(1)}$ – new unknown functions, and $y_1^{(0)}, y_2^{(0)}$ are defined by formulas:

$$y_j^{(0)}(t, \varepsilon, \theta, \mu) = L_j[h_1(t, \varepsilon, \theta, \mu), h_2(t, \varepsilon, \theta, \mu)],\ \ j = 1, 2.$$

As result we obtained:

$$\frac{dy_j^{(1)}}{dt} = \sum_{k=1}^{2} a_{jk}(t, \varepsilon) y_k^{(1)} + \varepsilon^2 h_j(t, \varepsilon, \theta, \mu) + \mu\varepsilon \sigma_j^{(1)}(t, \varepsilon, \theta, \mu) + +\mu \sum_{k=1}^{2} u_{jk}(t, \varepsilon, \theta) y_k^{(1)} +$$

$$+\mu^2 \sum_{k=1}^{2} v_{jk}(t, \varepsilon, \theta, \mu) y_k^{(1)} + \mu\varepsilon \sum_{k=1}^{2} w_{jk}(t, \varepsilon, \theta, \mu) y_k^{(1)} + \mu Y_j^{(1)}(t, \varepsilon, \theta, y_1^{(1)}, y_2^{(1)}, \mu),\ \ j = 1, 2, \tag{14}$$

where $h_j^{(1)} \in F(m-2, l, \varepsilon_0, \theta)$, $\sigma_j^{(1)}, w_{jk} \in F(m-1, l, \varepsilon_0, \theta)$, $Y_j^{(1)}$ belongs to class $F(m-1, l, \varepsilon_0, \theta)$ with respect t, ε, θ, analytic with respect $y_1^{(1)}, y_2^{(2)}$ in some area of these variables and contains terms not lower than second order with respect $y_1^{(1)}, y_2^{(1)}$.

To system (14) we apply the transformation, which reducing its to almost diagonal kind:

$$y_1^{(1)} = a_{12}(t, \varepsilon) y_1^{(2)} + a_{12}(t, \varepsilon) y_2^{(2)},$$

$$y_2^{(1)} = (-i\omega(t,\varepsilon) - a_{11}(t,\varepsilon))y_1^{(2)} + (i\omega(t,\varepsilon) - a_{11}(t,\varepsilon))y_2^{(2)}. \tag{15}$$

Determinant of transformation (15) is equal $2i\omega(t,\varepsilon)a_{12}(t,\varepsilon)$, therefore his non-degeneracy are provided by condidtions (4), (5). As result we obtain:

$$\frac{dy_j^{(2)}}{dt} = (-1)^j i\omega(t,\varepsilon)y_j^{(2)} + \varepsilon\sum_{k=1}^{2}\beta_{jk}(t,\varepsilon)y_k^{(2)} + \varepsilon^2 h_j^{(2)}(t,\varepsilon,\theta,\mu)+$$

$$+\mu\varepsilon\sigma_j^{(2)}(t,\varepsilon,\theta,\mu) + \mu\sum_{k=1}^{2}u_{jk}^{(2)}(t,\varepsilon,\theta)y_k^{(2)} + \mu^2\sum_{k=1}^{2}v_{jk}^{(2)}(t,\varepsilon,\theta,\mu)y_k^{(2)}+$$

$$+\mu\varepsilon\sum_{k=1}^{2}w_{jk}^{(2)}(t,\varepsilon,\theta,\mu)y_k^{(2)} + \mu Y_j^{(2)}(t,\varepsilon,\theta,y_1^{(2)},y_2^{(2)},\mu),\ j=1,2, \tag{16}$$

where $\beta_{jk} \in S(m-1,\varepsilon_0)$, $h_j^{(2)} \in F(m-2,l,\varepsilon_0,\theta)$, $\sigma_{jk}^{(2)} \in F(m-1,l,\varepsilon_0,\theta)$, functions $u_{jk}^{(2)}$, $v_{jk}^{(2)}$ are defined by formulas:

$$u_{jj}^{(2)} = \frac{1}{2}(u_{11}+u_2) + \frac{ia_{11}}{2\omega}(u_{11}-u_2) + (-1)^j\frac{i(\omega^2+a_{11}^2)}{2\omega a_{12}}u_{12} + (-1)^{j-1}\frac{ia_{12}}{2\omega}u_{21},$$

$$u_{jk}^{(2)} = \frac{1}{2}(u_{11}-u_{22}) + (-1)^{j-1}\frac{ia_{11}}{2\omega}(u_{11}-u_{22}) + (-1)^j\frac{i(a_{11}-i\omega)^2}{2\omega a_{12}}u_{12} + (-1)^{j-1}\frac{ia_{12}}{2\omega}u_{21}$$

$(j \neq k)$,

$$v_{jj}^{(2)} = \frac{1}{2}(v_{11}+v_2) + \frac{ia_{11}}{2\omega}(v_{11}-v_2) + (-1)^j\frac{i(\omega^2+a_{11}^2)}{2\omega a_{12}}v_{12} + (-1)^{j-1}\frac{ia_{12}}{2\omega}v_{21},$$

$$v_{jk}^{(2)} = \frac{1}{2}(v_{11}-v_{22}) + (-1)^{j-1}\frac{ia_{11}}{2\omega}(v_{11}-v_{22}) + (-1)^j\frac{i(a_{11}-i\omega)^2}{2\omega a_{12}}v_{12} + (-1)^{j-1}\frac{ia_{12}}{2\omega}v_{21}$$

$(j \neq k)$.

Obviously, that $u_{jk}^{(2)}, v_{jk}^{(2)} \in F(m,l,\varepsilon_0,\theta)$ $(j,k=1,2)$.

Now we increase the order of smallness with respect parameter ε of the off-diagonal elements in matrix of system (16). For this purpose in system (16) we make the substitution:

$$y_1^{(2)} = y_1^{(3)} - \varepsilon\frac{i\beta_{12}(t,\varepsilon)}{2\omega(t,\varepsilon)}\ y_2^{(3)},\ y_2^{(2)} = \varepsilon\frac{i\beta_{21}(t,\varepsilon)}{2\omega(t,\varepsilon)}\ y_1^{(3)} + y_2^{(3)}. \tag{17}$$

Choose $\varepsilon_1 \in (0,\varepsilon_0)$ from condition:

$$\varepsilon_1^2 \sup_{G(\varepsilon_0)}\left|\frac{\beta_{12}(t,\varepsilon)\beta_{21}(t,\varepsilon)}{\omega^2(t,\varepsilon)}\right| < 4.$$

This condition guaranteed the non-degeneracy of transformation (17), and as result its use has been:

$$\frac{dy_j^{(3)}}{dt} = ((-1)^j i\omega(t,\varepsilon) + \varepsilon\beta_{jj}(t,\varepsilon))y_j^{(3)} + \varepsilon^2\sum_{k=1}^{2}\alpha_{jk}(t,\varepsilon)y_k^{(3)} + \varepsilon^2 h_j^{(3)}(t,\varepsilon,\theta,\mu)+$$

$$+\mu\varepsilon\sigma_j^{(3)}(t,\varepsilon,\theta,\mu)+\mu\sum_{k=1}^{2}u_{jk}^{(2)}(t,\varepsilon,\theta)y_k^{(3)}+\mu^2\sum_{k=1}^{2}v_{jk}^{(2)}(t,\varepsilon,\theta,\mu)y_k^{(3)}+$$

$$+\mu\varepsilon\sum_{k=1}^{2}w_{jk}^{(3)}(t,\varepsilon,\theta,\mu)y_k^{(3)}+\mu Y_j^{(3)}(t,\varepsilon,\theta,y_1^{3)},y_2^{(3)},\mu),\ j=1,2, \tag{18}$$

where $\alpha_{jk}\in S(m-2,\varepsilon_1)$, $h_j^{(3)}\in F(m-2,l,\varepsilon_1,\theta)$, $\sigma_j^{(3)},w_{jk}^{(3)}\in F(m-1,l\varepsilon_1,\theta)$, $Y_j^{(3)}$ belongs to class $F(m-1,l,\varepsilon_1,\theta)$ with respect t,ε,θ and contains the terms not lower than second order with respect $y_1^{(3)},y_2^{(3)}$.

With system (18) we consider the linear homogeneous system:

$$\frac{d\widetilde{y}_j}{dt}=((-1)^j i\omega(t,\varepsilon)+\varepsilon\beta_{jj}(t,\varepsilon))\widetilde{y}_j+\mu\sum_{k=1}^{2}u_{jk}^{(2)}(t,\varepsilon,\theta)\widetilde{y}_k+$$

$$+\mu^2\sum_{k=1}^{2}v_{jk}^{(2)}(t,\varepsilon,\theta,\mu)\widetilde{y}_k,\ j=1,2. \tag{19}$$

In paper [12] shawn that by condition (5) $\exists\ \varepsilon_2\in(0,\varepsilon_1)$, $\mu_2\in(0,\mu_1)$ such that $\forall\ \varepsilon\in(0,\varepsilon_2)$, $\forall\ \mu\in(0,\mu_2)$ exists non-degenerating transformation of kind:

$$\widetilde{y}_j=\widetilde{\widetilde{y}}_j+\sum_{k=1}^{2}\psi_{jk}(t,\varepsilon,\theta,\mu)\widetilde{\widetilde{y}}_k,\ j=1,2, \tag{20}$$

where $\psi_{jk}\in F(m-1,l,\varepsilon_2,\theta)$, which reducing the system (19) to kind:

$$\frac{d\widetilde{\widetilde{y}}_j}{dt}=((-1)^j i\omega(t,\varepsilon)+\varepsilon\beta_{jj}(t,\varepsilon)+\mu\lambda_{j0}(t,\varepsilon)+\mu^2\lambda_{j1}(t,\varepsilon,\mu))\widetilde{\widetilde{y}}_j+$$

$$+\mu\varepsilon\sum_{k=1}^{2}d_{jk}(t,\varepsilon,\theta,\mu)\widetilde{\widetilde{y}}_k,\ j=1,2, \tag{21}$$

where $\lambda_{j0}(t,\varepsilon)=\Gamma_0\left[u_{jj}^{(2)}(t,\varepsilon,\theta)\right]$, $\lambda_{j1}(t,\varepsilon,\mu)=\Gamma_0\left[v_{jj}^{(2)}(t,\varepsilon,\theta,\mu)\right]\in S(m,\varepsilon_2)$, $d_{jk}\in F(m-1,l,\varepsilon_2,\theta)$. By using this result through substitution

$$y_j^{(3)}=y_j^{(4)}+\sum_{k=1}^{2}\psi_{jk}(t,\varepsilon,\theta,\mu)y_k^{(4)},\ j=1,2, \tag{22}$$

where ψ_{jk} – the same as that in formula (20), we reduce by $\varepsilon\in(0,\varepsilon_2)$, $\mu\in(0,\mu_2)$ the system (18) to kind:

$$\frac{dy_j^{(4)}}{dt}=((-1)^j i\omega(t,\varepsilon)+\varepsilon\beta_{jj}(t,\varepsilon)+\mu\lambda_{j0}(t,\varepsilon)+\mu^2\lambda_{j1}(t,\varepsilon,\mu))y_j^{(4)}+$$

$$+\varepsilon^2h_j^{(4)}(t,\varepsilon,\theta,\mu)+\mu\varepsilon\sigma_j^{(4)}(t,\varepsilon,\theta,\mu)+\varepsilon^2\sum_{k=1}^{2}\alpha_{jk}^{(4)}(t,\varepsilon,\mu)y_k^{(4)}+$$

$$+\mu\varepsilon\sum_{k=1}^{2} w_{jk}^{(4)}(t,\varepsilon,\theta,\mu)y_k^{(4)} + \mu Y_j^{(4)}(t,\varepsilon,\theta,y_1^{(4)},y_2^{(4)},\mu),\ j=1,2, \tag{23}$$

where $h_j^{(4)} \in F(m-2,l,\varepsilon_2,\theta)$, $\sigma_j^{(4)}, w_{jk}^{(4)} \in F(m-1,l,\varepsilon_2,\theta)$, $\alpha_{jk}^{(4)} \in S(m-2,\varepsilon_2)$, $Y_j^{(4)} \in F(m-1,l,\varepsilon_2,\theta)$ with respect t,ε,θ and and contains the terms not lower than second order with respect $y_1^{(4)}, y_2^{(4)}$.

We denote:

$$\lambda_{j2}(t,\varepsilon,\mu) = (-1)^j i\omega(t,\varepsilon) + \varepsilon\beta(t,\varepsilon) + \mu\lambda_{j0}(t,\varepsilon) + \mu^2\lambda_{j1}(t,\varepsilon,\mu)\ (j=1,2).$$

By condition (5) $\exists\ \varepsilon_3 \in (0,\varepsilon_2)$, $\mu_3 \in (0,\mu_2)$ such that $\forall\ \varepsilon \in (0,\varepsilon_3)$, $\mu \in (0,\mu_3)$ the following inequality is true:

$$\inf_{G(\varepsilon_3)} |\lambda_{j2}(t,\varepsilon,\mu) - in\varphi(t,\varepsilon)| \geqslant \gamma_1 > 0,\ j=1,2,\ n\in\mathbf{Z}. \tag{24}$$

Due to inequality (24) the functions

$$y_j^{(40)}(t,\varepsilon,\theta,\mu) = -\sum_{n=-\infty}^{\infty} \frac{\Gamma_n[\sigma_j^{(4)}(t,\varepsilon,\theta,\mu)]}{\lambda_{j2}(t,\varepsilon,\mu) - in\varphi(t,\varepsilon)} \exp(in\theta),\ j=1,2$$

belongs to class $F(m-1,l,\varepsilon_3,\theta)$. We make in system (3) the substitution:

$$y_j^{(4)} = \mu\varepsilon y_j^{(40)}(t,\varepsilon,\mu) + \varepsilon y_j^{(5)},\ j=1,2, \tag{25}$$

where $y_1^{(5)}$, $y_2^{(5)}$ – new unknown functions. We obtain:

$$\frac{dy_j^{(5)}}{dt} = \lambda_{j2}(t,\varepsilon,\mu)y_j^{(5)} + \varepsilon h_j^{(5)}(t,\varepsilon,\theta,\mu) + \varepsilon^2\sum_{k=1}^{2}\alpha_{jk}^{(5)}(t,\varepsilon,\mu)y_k^{(5)} +$$

$$+\mu\varepsilon\sum_{k=1}^{2} w_{jk}^{(5)}(t,\varepsilon,\theta,\mu)y_k^{(5)} + \mu\varepsilon Y_j^{(5)}(t,\varepsilon,\theta,y_1^{(5},y_2^{(5)},\mu),\ j=1,2. \tag{26}$$

All coefficients of this system belongs to class $F(m-2,l,\varepsilon_3,\theta)$, nonlinearities $Y_1^{(5)}$, $Y_2^{(5)}$ analytic with respect $y_1^{(5)}, y_2^{(5)}$ in some area of these variables.

With system (26) we consider the linear nonhomogeneous and diagonal system:

$$\frac{dy_{j0}^{(5)}}{dt} = \lambda_{j2}(t,\varepsilon,\mu)y_{j0}^{(5)} + \varepsilon h_j^{(5)}(t,\varepsilon,\theta,\mu),\ j=1,2. \tag{27}$$

Suppose one of the following two conditions:

$$\mathrm{Re}\lambda_{j0}(t,\varepsilon) \equiv \mathrm{Re}\lambda_{j1}(t,\varepsilon,\mu) \equiv 0,\ j=1,2. \tag{28}$$

$$\inf_{G(\varepsilon_0)} |\mathrm{Re}\lambda_{j0}(t,\varepsilon)| \geqslant \gamma_2 > 0,\ j=1,2. \tag{29}$$

Then from results of paper [13] follows, that $\exists\ \varepsilon_4 \in (0,\varepsilon_3)$, $\mu_4 \in (0,\mu_3)$ such that $\forall\ \varepsilon \in (0,\varepsilon_4)$, $\forall\ \mu \in (0,\mu_4)$ the system (27) has particular solution $y_{j0}^{(5)}$ $(j=1,2)$ which belong to class $F(m-2,l,\varepsilon_4,\theta)$.

We construct now the process of successive approximations, defining the initial approximation $y_{j0}^{(5)}(t,\varepsilon,\theta,\mu)$ $(j=1,2)$, and and the subsequent approximations defining as solutions of class $F(m-2,l,\varepsilon_4,\theta)$ of the linear nonhomogeneous systems:

$$\frac{dy_{js}^{(5)}}{dt} = \lambda_{j2}(t,\varepsilon,\mu)y_{js}^{(5)} + \varepsilon h_j^{(5)}(t,\varepsilon,\theta,\mu) + \varepsilon^2 \sum_{k=1}^{2} \alpha_{jk}^{(5)}(t,\varepsilon,\mu)y_{k,s-1}^{(5)} +$$

$$+\mu\varepsilon \sum_{k=1}^{2} w_{jk}^{(5)}(t,\varepsilon,\theta,\mu)y_{k,s-1}^{(5)} + \mu\varepsilon Y_j^{(5)}(t,\varepsilon,\theta,y_{1,s-1}^{(5}, y_{2,s-1}^{(5)},\mu),\ j=1,2;\ s=1,2,\ldots. \tag{30}$$

Using techniques contraction mapping principle, it is easy to show, that $\exists\,\varepsilon_5,\mu_5 \in (0,+\infty)$ such that $\forall\,\varepsilon \in (0,\varepsilon_5)$, $\mu \in (0,\mu_5)$ the process (30) converges to the solution $y_j^{(5)}(t,\varepsilon,\theta,\mu)$ $(j=1,2)$ from class $F(m-2,l,\varepsilon_5,\theta)$ of system (26).

4. Principal Result. Thus the following theorem.

Theorem. *Let the system (3) satisfy the conditions (4), (5) and one of the conditions (28), (29). Then* $\exists\,\varepsilon^* \in (0,\varepsilon_0)$, $\mu^* \in (0,\mu_0)$ *such that* $\forall\,\mu \in (0,\mu^*)$ *the system (3) has a particular solution* $x_j(t,\varepsilon,\theta,\mu)$ $(j=1,2)$ *from class* $F(m-2,l,\varepsilon^*,\theta)$.

Consider now the linear nonhomogeneous system:

$$\frac{dx_j}{dt} = \sum_{k=1}^{2} a_{jk}(t,\varepsilon)x_k + f_j(t,\varepsilon,\theta(t,\varepsilon)),\ j=1,2, \tag{31}$$

where a_{jk}, f_j the same as in system (3).

Consequence 1. *Let the system (31) satisfy the conditions (4), (5). Then* $\exists\,\varepsilon_6 \in (0,\varepsilon_0)$ *such that system (31) has a particular solution* $x_j(t,\varepsilon,\theta,\mu)$ $(j=1,2)$ *from class* $F(m-2,l,\varepsilon_6,\theta)$.

5. Examples.

As examples of the application of our results establish the conditions for the existence of solutions from class $F(m-2,l,\varepsilon^*,\theta)$ for systems corresponding to the known in nonlinear mechanics equations of Duffing and Van der Pol.

1. Consider the system of Duffing:

$$\frac{dx_1}{dt} = x_2,\ \frac{dx_2}{dt} = -\omega^2(t,\varepsilon)x_1 + b(t,\varepsilon)\sin\theta(t,\varepsilon) + \mu x_1^3, \tag{32}$$

$\theta(t,\varepsilon) = \int\limits_0^t \varphi(\tau,\varepsilon)d\tau$, $\omega, b, \varphi \in S(m,\varepsilon_0)$; $\omega, b, \varphi \in \mathbf{R}^+$, $\inf\limits_{G(\varepsilon_0)} \varphi > 0$, $\inf\limits_{G(\varepsilon_0)} \omega > 0$.

Obviously the system (32) has a kind (3), where $a_{11} \equiv 0$, $a_{22} \equiv 0$, $a_{12} \equiv 1$, $a_{21} = -\omega^2$, $f_1 \equiv 0$, $f_2 = b\sin\theta$, $X_1 \equiv 0$, $X_2 = x_1^3$.

Assume that the condition (5). Condition (4) holds obviously. Then

$$\xi_{10} = \frac{b}{\omega^2-\varphi^2}\ \sin\theta,\ \xi_{20} = \frac{\varphi b}{\omega^2-\varphi^2}\ \cos\theta,\ u_{11} \equiv 0,\ u_{12} \equiv 0,$$

$$u_{21} = \frac{3b^2}{(\omega^2-\varphi^2)^2}\ \sin^2\theta,\ u_{22} \equiv 0, u_{11}^{(2)} = -u_{22}^{(2)} = \frac{3ib^2}{2\omega(\omega^2-\varphi^2)^2}\ \sin^2\theta,$$

$$\lambda_{10} = -\lambda_{20} = \frac{3ib^2}{4\omega(\omega^2-\varphi^2)^2}, \ \mathrm{Re}\lambda_{j0} \equiv 0 \ (j=1,2),$$

$$v_{11} \equiv 0, \ v_{12} \equiv 0, \ v_{22} \equiv 0, \ v_{21} = 6\left(\frac{b}{\omega^2-\varphi^2} \ \sin\theta + \nu_1\mu\eta_1\right)\eta_1,$$

$$v_{11}^{(2)} = -v_{22}^{(2)} = \frac{3ia_{12}}{\omega}\left(\frac{b}{\omega^2-\varphi^2} \ \sin\theta + \nu_1\mu\eta_1\right)\eta_1,$$

where $0 < \nu_1 < 1$, η_1 – the real function from class $F(m,l,\varepsilon_0,\theta)$.

Hence the functions $v_{11}^{(2)}$, $v_{22}^{(2)}$ - are purely imaginary, therefore $\mathrm{Re}\lambda_{11} \equiv \mathrm{Re}\lambda_{22} \equiv 0$. So true conditions (28). Hence

Consequence 2. *Let system (32) satisfy condition (5). Then* $\exists\, \varepsilon_7 \in (0,\varepsilon_0)$, $\mu_7 \in (0,\mu_0)$ *such that* $\forall\ \mu \in (0,\mu_7)$ *the system (32) has a particular solution* $x_j(t,\varepsilon,\theta,\mu)$ $(j=1,2)$ *from class* $F(m-2,l,\varepsilon_7,\theta)$.

2. Consder the system of Van der Pol:

$$\frac{dx_1}{dt} = x_2, \ \frac{dx_2}{dt} = -\omega^2(t,\varepsilon)x_1 + b(t,\varepsilon)\mathrm{sin}\theta(t,\varepsilon) + \mu(1-x_1^2)x_2, \qquad (33)$$

functions θ, b, ω, a_{jk}, f_j – the same as in system (32), $X_1 \equiv 0$, $X_2 = (1-x_1^2)x_2$.

Assume that the condition (5). Condition (4) holds obviously. Functions $\xi_{j0}(t,\varepsilon,\theta)$ $(j=1,2)$ – the same as in system (32). Then:

$$u_{11} \equiv 0, \ u_{12} \equiv 0, \ u_{21} = -\frac{\varphi b^2}{(\omega^2-\varphi^2)^2} \ \sin 2\theta, \ u_{22} = 1 - \frac{b^2}{(\omega^2-\varphi^2)^2} \ \sin^2\theta,$$

$$u_{11}^{(2)} = 1 - \frac{b^2}{(\omega^2-\varphi^2)^2} \ \sin^2\theta - \frac{i\varphi b^2}{2\omega(\omega^2-\varphi^2)^2} \ \sin 2\theta,$$

$$u_{22}^{(2)} = 1 - \frac{b^2}{(\omega^2-\varphi^2)^2} \ \sin^2\theta + \frac{i\varphi b^2}{2\omega(\omega^2-\varphi^2)^2} \ \sin 2\theta,$$

$$\lambda_{j0} = 1 - \frac{b^2}{2(\omega^2-\varphi^2)^2} \quad (j=1,2).$$

Thus, when the unequality

$$\delta = \sup_{G(\varepsilon_0)}\left|\frac{b(t,\varepsilon)}{\omega^2(t,\varepsilon)-\varphi^2(t,\varepsilon)}\right| < \sqrt{2}, \qquad (34)$$

is true, then condidtions (29) with constant $\gamma_2 = 1-\delta^2/2$. Hence

Consequence 3. *Let system (33) satisfy condidtions (5) and (34). Then* $\exists\, \varepsilon_8 \in (0,\varepsilon_0)$, $\mu_8 \in (0,\mu_0)$ *such that* $\forall\ \mu \in (0,\mu_8)$ *the system (33) has a particular solution* $x_j(t,\varepsilon,\theta,\mu)$ $(j=1,2)$ *from class* $F(m-2,l,\varepsilon_8,\theta)$.

Conclusion. Thus, for the system (3) the sufficient conditions of the existence of the particular solution from class $F(m-2,l,\varepsilon^*,\theta)$ $(0<\varepsilon^*<\varepsilon_0)$ are obtained.