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THE ELECTROSTATIC FIELD IN THE TWO-DIMENSIONAL REGION BETWEEN UNEVEN ELECTRODES

The problem of the static distribution of the potential φ of the electric field in vacuum in the two-dimensional region of space between two uneven electrodes is set and resolved. Their surface irregularities are modeled with the help of arbitrary periodic functions. The approximate solution of the Laplace's equation $\Delta\varphi = 0$ satisfying corresponding boundary conditions is found to the first order of smallness with respect to the magnitude of small surface irregularities of electrodes. The "theoretical" coordinate dependence of the potential φ within established accuracy agrees well with the corresponding "experimental" dependence, which is obtained by methods of computational modeling with the help of the program package "*COMSOL* Multiphysics" in the simple particular case of the "rectangular" irregularity. The corresponding distribution of the potential φ is depicted on the contour plot. In practice obtained results can be applied, in particular, when conductive probes of arbitrary form are placed on the even surface of the cathode at equal distances from each other, and, hence, have direct relevance to the area of scanning tunneling microscopy.

INTRODUCTION

It is well known that the electrostatic field in the two-dimensional region of space between two perfectly even infinite parallel electrodes, to which the voltage (the potential difference) U is applied, is homogenous and possesses the strength

$$\mathbf{E}_0 = (0, E_0), \quad E_0 = -\frac{U}{d}, \tag{1}$$

where d is the distance between considered electrodes, and the potential

$$\varphi_0(z) = \frac{U}{d} z, \tag{2}$$

where the coordinate z corresponds to the direction, which is perpendicular to both of electrodes. Its value $z = 0$ corresponds to the cathode and the value $z = d$ — to the anode. The subscript "0" indicates that the case of the perfectly even surface of both of electrodes is considered. Besides, it is supposed that the medium, confined between them, only slightly influences on quantities, describing the electric field, and this influence can be neglected, considering the distribution of the potential φ in vacuum. If desired the influence of the medium can be taken into account, introducing into denominators of fractions in formulas (1) and (2) the quantity ε — the relative permittivity of the medium.

The natural question arises, how formulas (1) and (2) for the strength $\mathbf{E}$ and the potential φ of the considered electric field respectively alter, if surfaces of electrodes are not perfectly even. Irregularities, which are peculiar to any real surface, can be interpreted as small deviations of its shape from the flat one. In the analogous way in many cases one can interpret conductive probes of the arbitrary shape, placed on the even surface of the cathode at equal distances from each other [1]. Changing the shape and geometric sizes of probes, as well as the distance between them, one can achieve optimal relations when using them instead of the only one conductive tip in the scanning tunneling microscope and other analogous instruments with the purpose of the improvement of their work. Let us note that some of the latest articles in the area of scanning tunneling microscopy are devoted to carbon nanotubes [2] and graphene [3—7], that is one of the most perspective orientations of modern physics.

The article is constructed in the following way. At first we consider the case of arbitrary surface irregularities of both of electrodes and find in the first order approximation the explicit expression for the potential φ of the electric field between them. Then as an example we consider the simple particular case of the "rectangular" irregularity of the surface of the cathode, find in the same first order approximation explicit expressions for the potential φ and the component E_z of the strength $\mathbf{E}$ along the direction, which is perpendicular to electrodes, and also graph corresponding contour plots, illustrating obtained formulas. Finally, we compare the "theoretical" dependence with the corresponding "experimental" one, obtained by methods of computational modeling with the help of the program package "*COMSOL* Multiphysics", and draw conclusions.

ARBITRARY IRREGULARITY

At first let us consider the case of the arbitrary irregularity. The potential $\varphi(x, z)$ of the electric field satisfies the two-dimensional Laplace's equation

$$\Delta_2\varphi \equiv \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} = 0, \qquad (3)$$

where $\Delta_2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ is the two-dimensional Laplace's operator, and following boundary conditions:

$$\varphi(x, a(x)) = 0,\ \varphi(x, b(x)) = U, \qquad (4)$$

where $a(x)$ and $b(x)$ are some functions, which define the shape of the surface of the cathode and the anode respectively and are supposed to be known. Thus, it is supposed that the potential, which equals 0, is applied to the cathode and the potential, which equals U, — to the anode. Their difference amounts to U.

The exact solution of the equation (3) with boundary conditions (4) is unknown to us, therefore let us look for its approximate solution in the form

$$\varphi(x, z) \approx \varphi_0(z) + \varphi_1(x, z), \qquad (5)$$

where the function $\varphi_0(z)$ is defined by the formula (2) and the additional function $\varphi_1(x, z)$ takes into account the small deviation of the shape of the surface of both of electrodes from the flat one and is supposed to be the quantity of the first order of smallness with respect to this deviation. Thus, in the zero order approximation the potential $\varphi(x, z)$ is defined by the formula (2), corresponding to the case of perfectly even surfaces of both of electrodes, and in the first order approximation — by the formula (5). In the same approximation for functions $a(x)$ and $b(x)$ we obtain

$$\begin{aligned} a(x) &\approx a_0(x) + a_1(x) = a_1(x), \\ b(x) &\approx b_0(x) + b_1(x) = d + b_1(x), \end{aligned} \qquad (6)$$

where functions $a_0(x) = 0$ and $b_0(x) = d$ define positions of the cathode and the anode respectively in the zero order approximation and additional functions $a_1(x)$ and $b_1(x)$ take into account their small deviations from values $z = 0$ and $z = d$ respectively and are supposed to be quantities of the first order of smallness with respect to these deviations.

Let us establish the explicit form of functions $a_1(x)$ and $b_1(x)$. Let us suppose that they are periodic with different periods $2l_1$ and $2l_2$ respectively, where l_1 and l_2 are arbitrary positive real numbers, then in accordance with they can be expanded into following Fourier series:

$$a_1(x) = \frac{A_0}{2} + \sum_{k=1}^{+\infty}\left[A_k \cos\left(\frac{k\pi}{l_1}x\right) + C_k \sin\left(\frac{k\pi}{l_1}x\right)\right], \qquad (7)$$

$$b_1(x) = \frac{B_0}{2} + \sum_{k=1}^{+\infty}\left[B_k \cos\left(\frac{k\pi}{l_2}x\right) + D_k \sin\left(\frac{k\pi}{l_2}x\right)\right], \qquad (8)$$

which coefficients can be found by corresponding well known formulas. They are quantities of the first order of smallness. If deviations of the shape of the surface of both of electrodes from the flat one were on average symmetric with respect to the coordinate z, then equalities

$$A_0 = 0,\ B_0 = 0 \qquad (9)$$

would hold. However, these deviations generally are not symmetric, since in any case there is the electrode itself on the one side from the surface and the medium, in which it is placed, from the other. In spite of this, by the appropriate choice of positions of electrodes one can always achieve fulfillment of equalities (9).

Substituting (5) into (3) and taking into account (2), we obtain the following Laplace's equation with regard to the function $\varphi_1(x, z)$:

$$\frac{\partial^2\varphi_1}{\partial x^2} + \frac{\partial^2\varphi_1}{\partial z^2} = 0. \qquad (10)$$

Substituting (5) into (4) and taking into account (2), we obtain

$$\begin{aligned} &\frac{U}{d}a(x) + \varphi_1(x, a(x)) \approx 0, \\ &\frac{U}{d}b(x) + \varphi_1(x, b(x)) \approx U. \end{aligned} \qquad (11)$$

From (11) in the same first order approximation we obtain

$$\begin{aligned} &\frac{U}{d}a_1(x) + \varphi_1(x, 0) = 0, \\ &\frac{U}{d}b_1(x) + \varphi_1(x, d) = 0. \end{aligned} \qquad (12)$$

Thus, the function $\varphi_1(x, z)$ satisfies the Laplace's equation (10) and boundary conditions (12). The set problem permits of the exact solution. Finding it and substituting the obtained expression and (2) into (5), in the first order approximation we finally obtain

$$\begin{aligned} \varphi(x, z) \approx \frac{U}{d}z &+ \\ &+ \frac{U}{d}\sum_{k=1}^{+\infty}\frac{1}{\sinh\left(\frac{k\pi}{l_1}d\right)}\left[A_k\cos\left(\frac{k\pi}{l_1}x\right) + C_k\sin\left(\frac{k\pi}{l_1}x\right)\right]\sinh\left[\frac{k\pi}{l_1}(z-d)\right] - \\ &- \frac{U}{d}\sum_{k=1}^{+\infty}\frac{1}{\sinh\left(\frac{k\pi}{l_2}z\right)}\left[B_k\cos\left(\frac{k\pi}{l_2}x\right) + D_k\sin\left(\frac{k\pi}{l_2}x\right)\right]\sinh\left(\frac{k\pi}{l_2}z\right). \end{aligned} \qquad (13)$$

"RECTANGULAR" IRREGULARITY

As an example let us consider the case of the "rectangular" surface irregularity of the cathode. Deviations of the shape of the surface of electrodes from the flat one in this case is described by following functions:

$$a_1(x) = \begin{cases} h, & |x| < r \\ -h', & r < |x| < l_1 \end{cases}, \quad (14)$$

$$b_1(x) = 0, \quad (15)$$

where h, h' and r are positive real numbers, $r < l_1$, h and h' are connected by the relation

$$hr = h'(l_1 - r), \quad h' = \frac{r}{l_1 - r} h, \quad (16)$$

ensuring the fulfillment of the first equality (9). The graph of the function (14) when $x \in [-l_1, l_1]$ and $h = 0.1$ cm, $r = 0.1$ cm, $l_1 = 1$ cm is shown on fig. 1.

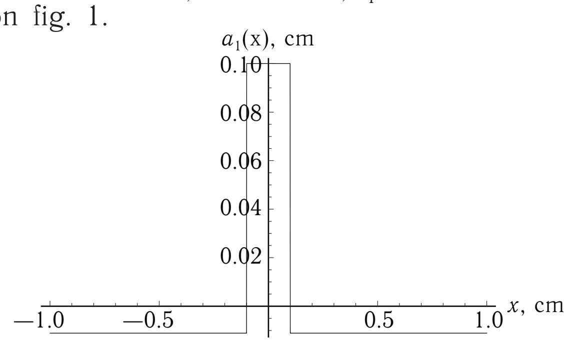

Fig. 1. The graph of the function (14) when $x \in [-l_1, l_1]$ and $h = 0.1$ cm, $r = 0.1$ cm, $l_1 = 1$ cm

Expanding functions (14) and (15) into Fourier series (7) and (8) and substituting coefficients of obtained expansions into (13), we obtain

$$\varphi(x, z) \approx \frac{U}{d} z + \frac{2Uhl_1}{\pi d(l_1 - r)} \times$$
$$\times \sum_{k=1}^{+\infty} \frac{\sin\left(\frac{k\pi}{l_1} r\right)}{k \sinh\left(\frac{k\pi}{l_1} d\right)} \cos\left(\frac{k\pi}{l_1} x\right) \sinh\left[\frac{k\pi}{l_1}(z - d)\right]. \quad (17)$$

Obtained function (17) corresponds to the "rectangular" surface irregularity of the cathode. When $h = 0.1$ cm, $r = 0.1$ cm, $l_1 = 1$ cm, $d = 1$ cm, $U = 1$ V from (17) we obtain

$$\varphi(x, z) \approx z +$$
$$+ \frac{2}{9\pi} \sum_{k=1}^{+\infty} \frac{\sin(0.1k\pi)}{k \sinh(k\pi)} \cos(k\pi x) \sinh[k\pi(z - 1)], \quad (18)$$

where coordinates x and z are measured in centimeters and the potential φ — in volts. The contour plot of the function (18) when $x \in [-0.5, 0.5]$ and $z \in [0.1, 0.5]$ is shown on fig. 2.

From (17) we obtain in the first order approximation the following expression for the component E_z of the strength **E** of the considered electric field along the direction, which is perpendicular to both of electrodes:

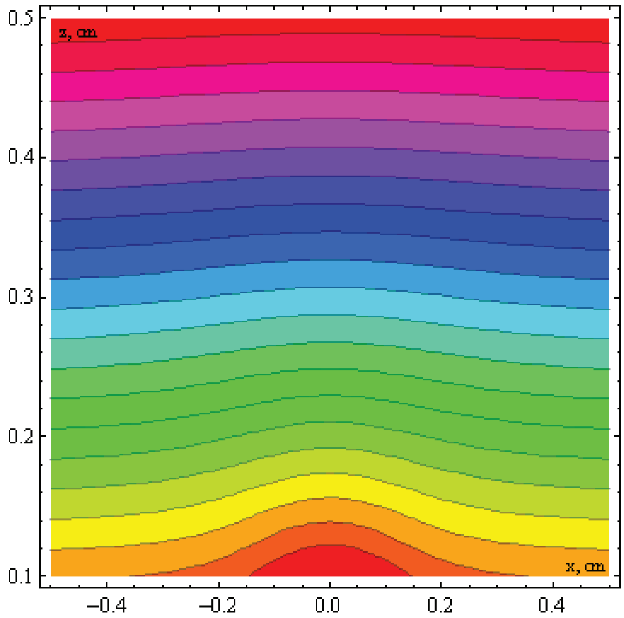

Fig. 2. The contour plot of the function (18) when $x \in [-0.5, 0.5]$ and $z \in [0.1, 0.5]$

$$E_z(x, z) = -\frac{\partial \varphi}{\partial z} \approx -\frac{U}{d} - \frac{2Uh}{d(l_1 - r)} \times$$
$$\times \sum_{k=1}^{+\infty} \frac{\sin\left(\frac{k\pi}{l_1} r\right)}{\sinh\left(\frac{k\pi}{l_1} d\right)} \cos\left(\frac{k\pi}{l_1} x\right) \cosh\left[\frac{k\pi}{l_1}(z - d)\right]. \quad (19)$$

When $h = 0.1$ cm, $r = 0.1$ cm, $l_1 = 1$ cm, $d = 1$ cm, $U = 1$ V from (19) we obtain

$$E_z(x, z) \approx -1 - \frac{2}{9} \times$$
$$\times \sum_{k=1}^{+\infty} \frac{\sin(0.1k\pi)}{\sinh(k\pi)} \cos(k\pi x) \cosh[k\pi(z - 1)], \quad (20)$$

where coordinates x and z are measured in centimeters and the quantity E_z — in volts per centimeter. The contour plot of the function (20) when $x \in [-1, 1]$ and $z \in [0.1, 1]$ is shown on fig. 3.

When $x = 0$ from (20) we obtain

$$E_z(0, z) \approx -1 - \frac{2}{9} \sum_{k=1}^{+\infty} \frac{\sin(0.1k\pi)}{\sinh(k\pi)} \cosh[k\pi(z - 1)]. \quad (21)$$

The graph of the function (21) when $z \in [0.1, 1]$ is shown on fig. 4.

Thus, on fig. 4 the "theoretical" dependence of the function $E_z(0, z)$ on the variable z is depicted. Its comparison with the corresponding "experimental" dependence, which is obtained by methods of computational modeling with the help of the program package "*COMSOL Multiphysics*" and depicted on fig. 5, shows that within established accuracy they agree well with each other.

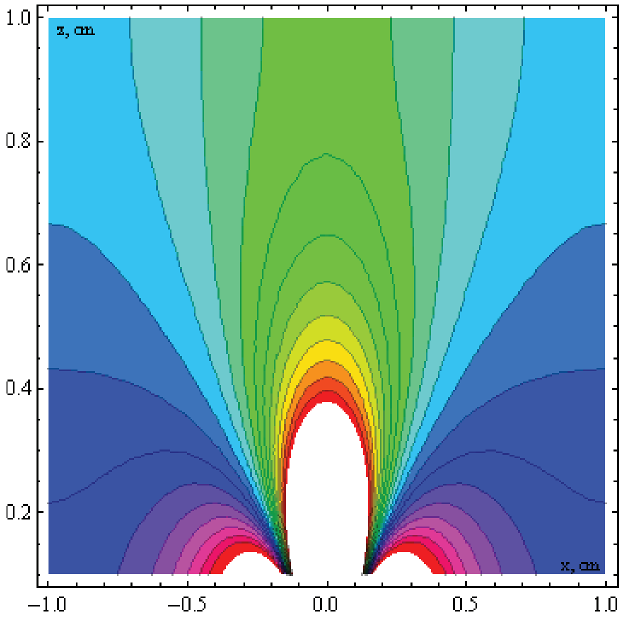

Fig. 3. The contour plot of the function (20) when $x \in [-1, 1]$ and $z \in [0.1, 1]$

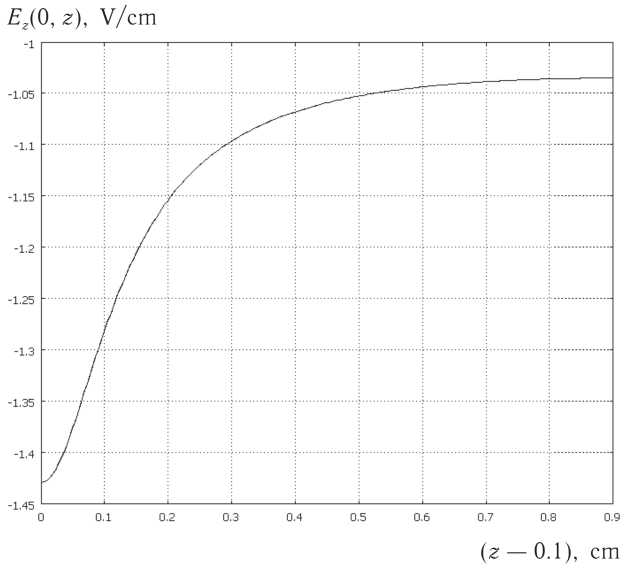

Fig. 5. The graph of the function $E_z(0, z)$ of the variable $(z — 0.1)$ when $z \in [0.1, 1]$, obtained by methods of computational modeling with the help of the program package "*COMSOL* Multiphysics"

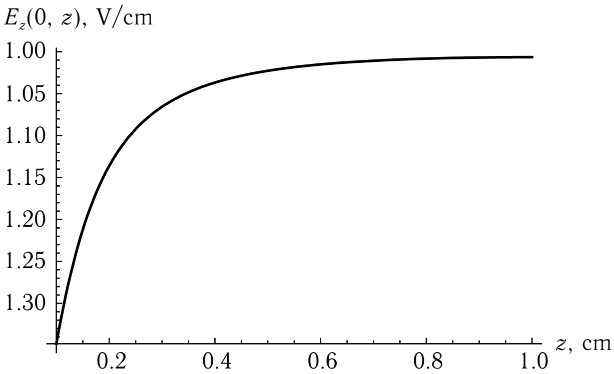

Fig. 4. The graph of the function (21) when $z \in [0.1, 1]$

CONCLUSIONS

Obtained results allow to draw following conclusions:

1. The problem of the static distribution of the potential φ of the electric field in vacuum in the two-dimensional region of space between two uneven electrodes is set and resolved in the first order approximation.

2. The simple particular case of the "rectangular" irregularity of the surface of the cathode is considered and explicit expressions for the potential φ and the component E_z of the strength **E** along the direction, which is perpendicular to electrodes, are found in the first order approximation.

3. Contour plots, illustrating obtained formulas, are graphed and it is shown that the "theoretical" coordinate dependence within established accuracy agrees well with the corresponding "experimental" dependence, which is obtained by methods of computational modeling with the help of the program package "*COMSOL* Multiphysics".

4. In practice obtained results can be applied, in particular, when conductive probes of arbitrary form are placed on the even surface of the cathode at equal distances from each other, and have direct relevance to the area of scanning tunneling microscopy.

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Abstract

The problem of the static distribution of the potential φ of the electric field in vacuum in the two-dimensional region of space between two uneven electrodes is set and resolved. Their surface irregularities are modeled with the help of arbitrary periodic functions. The approximate solution of the Laplace's equation $\Delta\varphi = 0$ satisfying corresponding boundary conditions is found to the first order of smallness with respect to the magnitude of small surface irregularities of electrodes. The "theoretical" coordinate dependence of the potential φ within established accuracy agrees well with the corresponding "experimental" dependence, which is obtained by methods of computational modeling with the help of the program package "*COMSOL* Multiphysics" in the simple particular case of the "rectangular" irregularity. The corresponding distribution of the potential φ is depicted on the contour plot. In practice obtained results can be applied, in particular, when conductive probes of arbitrary form are placed on the even surface of the cathode at equal distances from each other, and, hence, have direct relevance to the area of scanning tunneling microscopy.

Key words: electric field, two-dimensional region, electrode.

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ЭЛЕКТРОСТАТИЧЕСКОЕ ПОЛЕ В ДВУМЕРНОЙ ОБЛАСТИ МЕЖДУ НЕРОВНЫМИ ЭЛЕКТРОДАМИ

Резюме

Поставлена и решена задача о статическом распределении потенциала φ электрического поля в вакууме в двумерной области пространства между двумя неровными электродами. Неровности их поверхностей смоделированы с помощью произвольных периодических функций. Найдено приближенное решение уравнения Лапласа $\Delta\varphi = 0$, удовлетворяющее соответствующим граничным условиям, с точностью до первого порядка малости по величине малых неровностей поверхностей электродов. "Теоретическая" зависимость потенциала φ от координат в пределах установленной точности хорошо согласуется с соответствующей "экспериментальной" зависимостью, которая получена методами численного моделирования с помощью программного пакета "*COMSOL* Multiphysics" в простом частном случае "прямоугольной" неровности. Соответствующее распределение потенциала φ изображено на контурном графике. На практике полученные результаты могут быть применены, в частности, когда на ровной поверхности катода на равных расстояниях друг от друга установлены проводящие зонды произвольной формы, и, следовательно, имеют прямое отношение к области сканирующей туннельной микроскопии.

Ключевые слова: электрическое поле, двумерная область, электрод.

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ЕЛЕКТРОСТАТИЧНЕ ПОЛЕ У ДВОВИМІРНІЙ ОБЛАСТІ МІЖ НЕРІВНИМИ ЕЛЕКТРОДАМИ

Резюме

Поставлена та розв'язана задача про статичний розподіл потенціалу φ електричного поля у вакуумі у двовимірній області простору між двома нерівними електродами. Нерівності їх поверхонь змодельовані за допомогою довільних періодичних функцій. Знайдено наближений розв'язок рівняння Лапласа $\Delta\varphi = 0$, який задовільняє відповідним граничним умовам, з точністю до першого порядку малості по величині малих нерівностей поверхонь електродів. "Теоретична" залежність потенціалу φ від координат у межах встановленої точності добре узгоджується із відповідною "експериментальною" залежністю, яка отримана методами чисельного моделювання за допомогою програмного пакету "*COMSOL* Multiphysics" у простому окремому випадку "прямокутної" нерівності. Відповідний розподіл потенціалу φ зображено на контурному графіку. На практиці отримані результати можуть бути застосовані, зокрема, коли на рівній поверхні катоду на рівних відстанях один від одного встановлені провідні зонди довільної форми і, отже, мають пряме відношення до області скануючої тунельної мікроскопії.

Ключові слова: електричне поле, двовимірна область, електрод.