# Box-like Shells with Longitudinal Cracks 

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#### Abstract

The problem of how to determine the stress state of an infinite boxlike shell of rectangular profile is solved. Two cracks are located on opposite sides of the shell and parallel to its edges. On applying a Fourier transform, the problem can be reduced to a system of two integral equations with respect to jumps at the corner of rotation and normal displacements of the crack edges. The system of integral equations is solved by the method of orthogonal polynomials. Dependence of the stress intensity factor on the length of cracks and the geometrical dimensions of the cross-sections of the shell is demonstrated.


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## 1. Preamble

Thin-walled shells of a rectangular structure are used widely in construction, shipbuilding and mechanical engineering. In order to minimize the tedious details of research into plate construction as force elements of building mechanics, researchers have made various assumptions depending on types of loads and conditions of their fastenings. Among the first papers in this direction, the intense condition of thin-walled cores of open and closed structures, we refer the reader to the related works of Vlasov, Ganilidze, Panovko, Kan, and Reyssner.

Papkovich has applied the methods of plane elasticity theory to the study of box constructions. Thus he assumed that each plate is in a flat intense condition and cooperates with adjoining plates only by tangential efforts. Contrary to Papkovich, in papers of Smotrov and Fleyshman the problems were solved with only the basic assumption that the construction edges do not bend and play a role of rigid support. In a general statement or with the use of a minimum quantity of simplifying assumptions, problems on plate construction were solved by various numerical methods, among which are the following ones: variational-difference
method (method of conjugated gradients), method of finite elements, and the variational method of Kantorovich-Vlasov. The more difficult problems for compound shell constructions, in view of actual conditions of their interaction, were solved by Mossakovsky and his disciples by the homogeneous solutions method. So in a paper of Musiyaki and Poshivalova, the matrix-vector method (based on a method of homogeneous solutions) is offered to analyze the constructions of plate design, and the problem of a folded-plate construction is solved. In a paper of Mossakovsky and Poshivalova the results are given of a comparison of solutions for the problem of a thin-walled bar under constrained torsion to a method of homogeneous solution with Vlasovs results. In a paper of Mossakovsky and Kulikov the method of homogeneous solutions was applied to problems with dynamic loading.

In paper [1] an account of the method of box-like shell constructions was offered, and it reduced the problem to one about the joint planar-bend stress condition for a plate with defects, which role is played by the edges of a shell. The advantage of this method consists in

1) the number of necessary differential equations and conditions of the joint is twice reduced,
and in
2) the solution methods for planar and bending problems for plates with defects now are well developed, and one can find bibliographies in $[5,7]$.
In papers $[2,3]$ the problems of inclusion setting in box-like shells are solved by these methods. In paper [4] the problem of the stress state of a boxed shell with a crack on an shell edge is solved. In the present paper the problem of a longitudinal crack is studied.

## 2. The problem statement

Fig. 1


Let us consider a problem of the stress state of a box-like shell of infinite length and rectangular structure, weakened by a pair of symmetric cracks (Fig. 1). We suppose that all plates of which the shell is made are of one material and have identical thickness $h$, Poisson factor $\nu$, elasticity module $E$, and cylindrical rigidity $D$. Crack edges are loaded by the bending moments $m(y)$ and planar stretching loadings $\sigma(y)$. Loadings that influence the shell are symmetric with the planes
of symmetry of the shell and are such that the crack edges are not closed. By the method stated in [1], the problem is reduced to searching for a differential equations system solution:

$$
\begin{align*}
\Delta^{2} w(x, y) & =0 \\
\Delta^{2} \sigma_{x}(x, y) & =0, \quad-a<x<b, x \neq 0,|y|<\infty \tag{2.1}
\end{align*}
$$

which satisfy the conditions on the shell edge

$$
\begin{align*}
\langle v\rangle & =\left\langle\tau_{x y}\right\rangle=\left\langle\varphi_{x y}\right\rangle=\left\langle M_{x y}\right\rangle=0 \\
\langle u\rangle & =-\left(w_{+}+w_{-}\right) ;\langle w\rangle=u_{+}+u_{-} \\
\left\langle\sigma_{x}\right\rangle & =-h^{-1}\left[\left(V_{x}\right)_{+}+\left(V_{y}\right)_{-}\right]  \tag{2.2}\\
\left\langle V_{x}\right\rangle & =h\left[\left(\sigma_{x}\right)_{+}+\left(\sigma_{y}\right)_{-}\right]
\end{align*}
$$

and boundary conditions

$$
\begin{aligned}
& V_{x}=\tau_{x y}=0 ; M_{x}=m(y) ; \sigma_{x}=\sigma(y) ; x=-a,|y|<c \\
& V_{x}=\varphi_{x}=u=\tau_{x y}=0 ; x=-a,|y|>c \\
& V_{x}=\varphi_{x}=u=\tau_{x y}=0 ; x=b
\end{aligned}
$$

Here $u, v, w$ are the displacements along the axes with respect to $x, y, z$; $\varphi_{x}, M_{x}, V_{x}, \sigma_{x}, \tau_{x y}$ - the angle of turn, bending moment, generalized cross force, normal and tangential stresses. It is convenient to present the boundary conditions as:

$$
\begin{align*}
& V_{x}=\tau_{x y}=0 ; \varphi_{x}=\chi(y) ; u=\mu(y), x=-a \\
& V_{x}=\varphi_{x}=u=\tau_{x y}=0, x=b \tag{2.3}
\end{align*}
$$

where $\chi(y)$ and $\mu(y)$ - unknown functions on an interval $|y|<c$, equal to zero outside this interval- represent by themselves an angle of inclination and normal displacements of the crack edges. Without loss of generality, it is possible to consider that the necessary variable change in $x$ results in both $y$ and $c=1$.After application of a Fourier transformation to elastic unknown values and loadings, similar to the way it was done in [1], and also to unknown functions $\chi(y)$ and $\mu(y)$, we get

$$
\chi_{\alpha}=\int_{-1}^{1} \chi(y) e^{i \alpha y} d y, \mu_{\alpha}=\int_{-1}^{1} \mu(y) e^{i \alpha y} d y
$$

The problem (2.1)-(2.3) is reduced to the system of two integral equations

$$
\frac{1}{\pi} \frac{d^{2}}{d y^{2}} \int_{-1}^{1} \ln |y-\eta|\left[\begin{array}{c}
\mu(\eta)  \tag{2.4}\\
\chi(\eta)
\end{array}\right] d \eta+\int_{-1}^{1}\left[\begin{array}{ll}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{array}\right]\left[\begin{array}{c}
\mu(\eta) \\
\chi(\eta)
\end{array}\right] d \eta=\left[\begin{array}{c}
\sigma_{*}(y) \\
m_{*}(y)
\end{array}\right]
$$

where

$$
\begin{gather*}
m_{*}=2(D \gamma)^{-1} m(y) ; \sigma_{*}=-2 \sigma(y)  \tag{2.5}\\
K_{i j}(y, \eta)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} k_{i j}(\alpha) e^{i \alpha(\eta-y)} d \alpha  \tag{2.6}\\
k_{11}(\alpha)=p_{11}(\alpha)-f_{3 \mu}^{+} \Omega_{0}(-a)+f_{0 \mu}^{+} \Omega_{3}(-a) ; \\
k_{12}(\alpha)=-f_{3 \chi}^{+} \Omega_{0}(-a)+f_{0 \chi}^{+} \Omega_{3}(-a)  \tag{2.7}\\
k_{22}(\alpha)=p_{22}(\alpha)-f_{3 \chi}^{-} \mathrm{M}_{0}(-a)+f_{0 \chi}^{-} \mathrm{M}_{3}(-a) ; \\
k_{21}(\alpha)=-\gamma\left[-f_{3 \mu}^{-} \mathrm{M}_{0}(-a)+f_{3 \mu}^{-} \mathrm{M}_{3}(-a)\right]  \tag{2.8}\\
p_{22}(\alpha) \gamma \exp ^{|\alpha|(a+b)}=-\alpha^{4} G_{\alpha}(-a, b)+\left[0,5 \alpha(a+b)-2(1-\nu)^{-1}\right] \alpha^{2} L G_{\alpha}(-a, b) \\
\Omega_{k}(x)=T_{k}^{+} G_{\alpha}(x, t) ; \mathrm{M}_{k}(x)=R_{2}^{-} T_{k}^{-} G_{\alpha}(x, t) ; \gamma=(3+\nu) /(1-\nu) .
\end{gather*}
$$

Here the following differential operators were used:

$$
\begin{aligned}
R_{k}^{ \pm} f & =\frac{\partial^{k} f}{\partial x^{k}} ; k=0,1 ; R_{2}^{ \pm} f=\left[L+(1 \pm \nu) \alpha^{2}\right] f \\
R_{3}^{ \pm} f & =\frac{\partial}{\partial x}\left[L-(1 \pm \nu) \alpha^{2}\right] f ; L f=\frac{d^{2} f}{d x^{2}}-\alpha^{2} f \\
S f & =\left(T^{-}-T^{+}\right) f ; H f=\left(T^{-}+T^{+}\right) f ; T^{ \pm} f=f( \pm 0) \\
S_{k}^{ \pm} f & =S\left[R_{k}^{ \pm} f\right] ; H_{k}^{ \pm} f=H\left[R_{k}^{ \pm} f\right] ; T_{k}^{ \pm} f=T\left[R_{k}^{ \pm} f\right]
\end{aligned}
$$

and $G(x, \xi)$ - Green function of the boundary problem

$$
L^{2} u(x)=0, \quad x \in(a, b) ; u^{\prime}=u^{\prime \prime \prime}=0, \quad x=-a, b
$$

Vectors $F_{\mu}=\left(f_{0 \mu}^{+}, f_{3 \mu}^{+}, f_{0 \mu}^{-}, f_{3 \mu}^{-}\right) ; F_{\chi}=\left(f_{0 \chi}^{+}, f_{3 \chi}^{+}, f_{0 \chi}^{-}, f_{3 \chi}^{-}\right)$are the solution of the linear algebraic equation system $A F=B$ for the right-hand parts $B=H_{\mu}$ and $B=H_{\chi}$ correspondingly, where

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This solution is obtained by solving the one-dimensional discontinuous boundary problems system

$$
\begin{align*}
L^{2} f_{\alpha}^{ \pm} & =0 ; \quad-a<x<b, x \neq 0  \tag{2.9}\\
R_{3}^{-} f_{\alpha}^{-} & =R_{1}^{+} f_{\alpha}^{+}=0 ; R_{3}^{+} f_{\alpha}^{+}=-\alpha^{4} E \mu_{\alpha} \\
R_{1}^{-} f_{\alpha}^{-} & =\chi_{\alpha}, x=-a  \tag{2.10}\\
R_{3}^{ \pm} f_{\alpha}^{ \pm} & =R_{1}^{ \pm} f_{\alpha}^{ \pm}=0, x=b \\
S_{j}^{ \pm} f_{\alpha}^{ \pm} & =0 \quad j=1,2 \\
S_{3}^{+} f_{\alpha}^{+} & =\alpha^{4} E\left[H_{0}^{-} f_{\alpha}^{-}\right] ; H_{3}^{+} f_{\alpha}^{+}=\alpha^{4} E\left[S_{0}^{-} f_{\alpha}^{-}\right]  \tag{2.11}\\
S_{0}^{+} f_{\alpha}^{+} & =D h^{-1}\left[H_{3}^{-} f_{\alpha}^{-}\right] ; H_{0}^{+} f_{\alpha}^{+}=-D h^{-1}\left[S_{3}^{-} f_{\alpha}^{-}\right]
\end{align*}
$$

Then we can write the solution of problem (2.9)-(2.11) in the form

$$
\begin{align*}
& f_{\alpha}^{-}(x)=\chi_{\alpha} R_{1}^{-} G_{\alpha}(x,-a)+f_{q}^{-}-f_{3}^{-} T_{0}^{-} G_{\alpha}+f_{0}^{-} T_{3}^{-} G_{\alpha} \\
& f_{\alpha}^{+}(x)=\mu\left(-\alpha^{4} E\right) G_{\alpha}(x,-a)+f_{q}^{+}-f_{3}^{+} T_{0}^{+} G_{\alpha}+f_{0}^{+} T_{3}^{+} G_{\alpha} \tag{2.12}
\end{align*}
$$

So, the stated problem is reduced to a system of integral equations (2.4) containing unknown functions $\chi(y)$ and $\mu(y)$, which represent the angle of inclination and normal displacements of the crack edges.

## 3. Construction of the approximate solution of the integral equations system

Let us take advantage of the method of orthogonal polynomials [5] and search for a solution as the expansion of unknown functions into a series about some Chebychev polynomials of the second kind $U_{k}(\eta)$ with the unknown coefficients

$$
\begin{equation*}
\binom{\mu(\eta)}{\chi(\eta)}=\sqrt{1-\eta^{2}} \sum_{k=0}^{\infty}\binom{\mu_{k}}{\chi_{k}} U_{k}(\eta),|\eta|<1 \tag{3.1}
\end{equation*}
$$

Let us substitute (3.1) into (2.4), and multiply each equation of this system by $\sqrt{1-y^{2}} U_{n}(y)$ and integrate by $y$ on the interval $(-1,1)$. We take into consideration the spectral correspondence [5]:

$$
\frac{1}{\pi} \frac{d^{2}}{d x^{2}} \int_{-1}^{1} \ln \frac{1}{|y-x|} \sqrt{1-y^{2}} U_{n}(y) d y=-(n+1) U_{n}(x)
$$

and orthogonal correspondence [6]

$$
\int_{-1}^{1} \sqrt{1-y^{2}} U_{m}(y) U_{n}(y) d y=\frac{\pi}{2} \delta_{m n}
$$

and formulas [6]:

$$
\int_{-1}^{1} \sqrt{\left(1-x^{2}\right)}\left\{\begin{array}{cc}
\sin \alpha x & U_{2 n+1}(x) \\
\cos \alpha x & U_{2 n}(x)
\end{array}\right\} d x=(-1)^{n} \frac{\pi(2 n+3 / 2 \pm 1 / 2}{\alpha}\left\{\begin{array}{l}
J_{2 n+2}(\alpha) \\
J_{2 n+1}(\alpha)
\end{array}\right\}
$$

. Then after simple transformations and permutation of the integration order in which we get expressions for $K_{i j}(y, \eta)$, we pass to an infinite system of linear algebraic equations of the second kind which, by Poincaré-Koch, are normal with respect to the coefficients of expansion:

$$
\begin{equation*}
(n+1)\binom{\mu_{n}}{\chi_{n}}+\sum_{k=0}^{\infty} A^{(k, n)}\binom{\mu_{k}}{\chi_{k}}=\binom{\sigma_{n}}{m_{n}}, \quad n=\overline{0, \infty} \tag{3.2}
\end{equation*}
$$

where components of a matrix $A^{(k, n)}$ and coefficients of the right-hand parts are

$$
\begin{gathered}
A_{i j}^{(2 k, 2 n)}=4(-1)^{n+k}(2 n+1)(2 k+1) \times \int_{0}^{\infty} k_{i j}(\alpha) \alpha^{-2} J_{2 n+1}(\alpha) J_{2 k+1}(\alpha) d \alpha \\
A_{i j}^{(2 k+1,2 n+1)}=4(-1)^{n+k+1}(2 n+2)(2 k+2) \times \int_{0}^{\infty} k_{i j}(\alpha) \alpha^{-2} J_{2 n+2}(\alpha) J_{2 k+2}(\alpha) d \alpha \\
A_{i j}^{(2 k+1,2 n)}=A_{i j}^{(2 k, 2 n+1)}=0 ; \quad i, j=1,2 \\
\binom{\sigma_{n}}{m_{n}}=\frac{2}{\pi} \int_{-1}^{1}\binom{\sigma_{*}(y)}{m_{*}(y)} \sqrt{1-y^{2}} U_{n}(y) d y
\end{gathered}
$$

Thus $k_{i j}(\alpha), \sigma_{*}(y), m_{*}(y)$ are determined in (2.5)-(2.8) and $J_{k}(\alpha)$ is a Bessel function. Let us note that the procedure using the components of matrices $A^{(k, n)}$ is simpler in essential ways owing to block symmetry, which is easily seen by replacing $n$ with $k$ or vice versa. The calculation of integrals with respect to $\alpha$ is also simpler owing to an exponential decrease of the function under integration. The solution of infinite algebraic system (3.2) allows us to determine all elastic unknown values, using the solution of a problem in transformations (2.9)-(2.11) in the form (2.12) and the convolution theorem, and also to estimate the intensity factor of plane $k_{+}$and bend $k_{-}$stresses. Following [7], we shall understand the stress intensity factor $k_{ \pm}$to be the factor through which the main parts of stresses near the crack ends are expressed. By the main parts of stresses we mean the coefficients of the singularities for stresses near to the crack ends. To obtain these main parts formulas it is enough to find a limit with $y \rightarrow \pm 1(|y|>1)$ of the integrals

$$
\binom{\varphi_{+}}{\varphi_{-}}=\frac{1}{\pi} \lim _{y \rightarrow \pm 1}\left(\sqrt{y^{2}-1} \frac{d^{2}}{d y^{2}} \int_{-1}^{1}\binom{\mu(\eta)}{\chi(\eta)} \ln |y-\eta| d \eta\right)
$$

and to use the correspondence [5]

$$
\begin{aligned}
& \frac{1}{\pi} \frac{d^{2}}{d y^{2}} \int_{-1}^{1} \ln \frac{1}{|y-\eta|} \frac{U_{n}(\eta) d \eta}{\sqrt{1-y^{2}}} \\
& \quad=\frac{|y| U_{n}(y)}{\sqrt{y^{2}-1}}+\sqrt{y^{2}-1} \cdot U_{n}^{\prime}(y) \operatorname{sgn} y-\frac{1}{2}(n+1) U_{n}(y),|y|>1
\end{aligned}
$$

As a result, after obtaining the integral main parts in the form

$$
\binom{\varphi_{+}}{\varphi_{-}}=\sum_{k=0}^{\infty}\binom{\mu_{k}}{\chi_{k}} \cdot U_{k}( \pm 1), \quad|y|>1
$$

we can find the stress intensity factors values and the main parts of the elastic values.

The numerical solution of the stated problem (2.1)-(2.3), which was reduced to an infinite system of linear algebraic equations (3.2), was obtained by a reduction method that eliminated four members of expansion for $\mu(y)$ and $\chi(y)$ in (3.1). And the loading, which influences the shell, undertook the role Of the bending moment of intensity $m=$ const and planar normal stresses of intensity $\sigma=$ const that applied to crack edges. Thus the stress intensity factors in both crack vertexes have identical values $k_{ \pm}( \pm 1)=k_{ \pm}$and are connected with dimensionless coefficients $k_{ \pm}^{m}, k_{ \pm}^{\sigma}$, which were calculated, by the following correspondences

$$
\begin{equation*}
k_{ \pm}=k_{ \pm}^{m} \frac{6 m \sqrt{c}}{h^{2}} ; \quad k_{ \pm}=k_{ \pm}^{\sigma} \sigma \sqrt{c} \tag{3.3}
\end{equation*}
$$

In Table 1 the values of stress intensity factors of plane and bend stresses (3.3) with $a / b=0,5$ for a different ratio $c / a$ are shown.

Table 1

| b/a |  | c/a |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 0.1 | 0.4 | 0.8 | 1 | 1.2 | 1.5 | 1.8 | 2 |
| 2 | $k_{+}^{\sigma}$ | 1.010 | 1.130 | 1.270 | 1.610 | 1.800 | 2.090 | 2.390 | 2.590 |
|  | $k_{-}^{m}$ | 0.999 | 0.991 | 0.980 | 0.953 | 0.937 | 0.913 | 0.889 | 0.874 |
|  | $k_{-}^{\sigma} \cdot 10^{3}$ | 0.004 | 0.0703 | 0.155 | 0.343 | 0.408 | 0.443 | 0.398 | 0.324 |
|  | $k_{+}^{m} \cdot 10^{3}$ | 0.021 | 0.346 | 0.759 | 1.710 | 2.120 | 2.590 | 2.900 | 3.050 |
| 1 | $k_{+}^{\sigma}$ | 1.010 | 1.130 | 1.270 | 1.610 | 1.790 | 2.080 | 2.380 | 2.580 |
|  | $k_{-}^{m}$ | 0.992 | 0.988 | 0.974 | 0.935 | 0.912 | 0.875 | 0.838 | 0.814 |
|  | $k_{-}^{\sigma} \cdot 10^{3}$ | 0.003 | 0.050 | 0.109 | 0.219 | 0.243 | 0.230 | 0.163 | 0.092 |
|  | $k_{+}^{m} \cdot 10^{3}$ | 0.015 | 0.246 | 0.533 | 1.130 | 1.340 | 1.560 | 1.680 | 1.730 |
| 0.5 | $k_{+}^{\sigma}$ | 1.010 | 1.130 | 1.270 | 1.620 | 1.820 | 2.120 | 2.450 | 2.670 |
|  | $k_{-}^{m}$ | 0.999 | 0.983 | 0.962 | 0.906 | 0.873 | 0.823 | 0.776 | 0.746 |
|  | $k_{-}^{\sigma} \cdot 10^{3}$ | 0.018 | 0.307 | 0.655 | 1.220 | 1.300 | 1.130 | 0.667 | 0.200 |
|  | $k_{+}^{m} \cdot 10^{3}$ | 0.089 | 0.153 | 0.322 | 0.650 | 0.763 | 0.877 | 0.942 | 0.970 |

The results of calculations show that, under the action of bending loadings, the intensity factors $k_{-}^{m}$ of bend stresses of some orders exceed the intensity factors of plane stresses $k_{-}^{\sigma}$; and under the action of plane loadings the intensity factors $k_{+}^{m}$ of bend stresses on some orders are lower than the intensity factors $k_{+}^{\sigma}$ of plane stresses.

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